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A formalization of Od(Ω)

by

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In our previous work [6], we have developed a theory of self-iterating schemes of ordinal diagrams, which was symbolically denoted by ***Od***(Ω), where Ω is a new scale, a system of stage indicators.

In order to investigate mathematical strength of our theory, we must determine a formal system in which our theory can be formalized: a system in which the accessibility proof of the theory can be carried out. We are here to present a system of second order arithmetic with the Π_1^1-induction, the Π_1^1-comprehension and a kind of uniform inductive definition, which does the work, presuming that the transfinite induction along Ω applied to the Π_1^1-formulas hold.

§1. System *AU*.

The system ***AU*** (arithmetic with a uniform inductive definition) is defined in a manner similar to the system with extended inductive definitions (cf. Chapter 4 of [2] or §28 of [3]). It is Π_1^1-arithmetic augmented by a designated predicate symbol, say H, and a new axiom, the uniform inductive definition (abbreviated to ***UID***) with regards to H.

We do not specify the formulation of the first order part of the system: it is assumed to be formulated in a manner convenient to develop elementary properties of the theory ***Od***(Ω) in it. (See next section.) Although we employ the Gentzen-style sequential formalism, we may not distinguish between free variables and bound variables in the symbolism. In such a case, it is without saying assumed there be no "clash" among variables.

Definition 1.1. Let H be a new predicate symbol with three first ordet arguments. Let ***C*** be a second order, conservative extension of Peano arithmetic with H as an atomic predicate, the detail of which will not be elaborated. ***C*** will be the basis for ***AU***.

The formulas of the system ***AU*** are those of ***C***; the additional axioms (or rules of inference) are the isolated (essentially-Π_1^1) mathematical induction, the isolated comprehension and the new axiom ***UID***, which is described below.

Let Ω be a primitive recursive set with a primitive recursive well-order $<_\Omega$. Let O and S be primitive recursive sets of pairs whose first entries are elements of Ω and which determine primitive recursive linear orders $<^r$ and $<_r$ respectively (uniformly in r) for the second entries of the pairs belonging to O and S respectively whose first entries are r. The requisite for O and S are as follows.

$\{a; (s, a) \in O\} \subset \{a; (r, a) \in O\}$ if $s <_\Omega r$ for some s and r elements of Ω.

$\{i; (r, i) \in S\} = \bigcup_{s <_\Omega r} \{a; (s, a) \in O\}$.

$<^s$ forms an initial segment of $<^r$ if $s <_\Omega r$.

$<_r = \bigcup_{s <_\Omega r} <^s$.

The second entry of any pair in O (hence in S) has a successor. Let us assume that O, Ω, and S can be expressed in the first order part of $\boldsymbol{C}$, and we shall use the same letters O, Ω, S to denote the formal representation of those sets.

Let G be an isolated formula without H, with three first order arguments and one second order argument. Let $<(r; y, x)$ be a formal representation of $y <_r x$. Define $Ac(S; r)$ by

$$\forall z(S(r; z) \supset \forall \phi(\forall x(S(r; x) \wedge \forall y(S(r; y) \wedge <(r; y, x) \supset \phi[y]) \supset \phi[x]) \supset \phi[z])) .$$

The intended meaning of this formula is of course that $<_r$ is a well-order or is accessible.

Now the uniform inductive definition (***UID***) along $<_r$ is formalized in two axioms.

Let $<(\Omega; s, r)$ be a formal representation of $s <_\Omega r$.

UID

1) $\Omega(r)$, $O(r; a)$, $S(r; i)$, $Ac(S; r)$, $H(r; i, a)$ $\rightarrow G(r; i, a, \{s, j, x\}(H(s; j, x) \wedge O(s; x) \wedge S(s; j) \wedge (<(\Omega; s, r) \vee (s = r \wedge <(r; j, i))))).$

2) $\Omega(r)$, $O(r; a)$, $S(r; i)$, $Ac(S; r)$, $G(r; i, a, \{s, j, x\}(H(s; j, x) \wedge O(s; x) \wedge S(s; j) \wedge (<(\Omega; s, r) \vee (s = r \wedge <(r; j, i))))) \rightarrow H(r; i, a)$.

Here $\{x\}H(x)$ denotes the abstract of the formula $H(x)$.

PROPOSITION 1.1. *The course-of-values-induction for the isolated formulas* ***AU*** *can be proved in* ***AU*** *without applications of the comprehension and the* ***UID***.

§2. Arithmetization of $Od(\Omega)$.

Here we refer to the notations in §1 of [6] and in §26 of [3], and to the formalization of the theory of ordinal diagrams in [1]. We shall adopt, however, a rather sloppy formulation, not necessarily distinguishing intentional notations from formalism. Our symbolism is subsequently listed.

Let Ω be a primitive recursive set and let $<_\Omega$ be its well-order. The same letter will be used for a formal representation (in the first order part of $\boldsymbol{C}$: Definition 1.1) of Ω. Ω will be fixed throughout.

1) Letters for variables.

$r, s, t, \cdots$ denote the variables ranging over (Gödel numbers of) the elements of Ω.

$a, b, c, \cdots, x, y, z, \cdots, u, v, w, \cdots$ denote the variables ranging over (Gödel numbers of) the elements of $\boldsymbol{Od}(\Omega)$.

$i, j, k, \cdots, p, q, \cdots$ denote the variables ranging over (Gödel numbers of) the elements of $\boldsymbol{J}_r$ when r is supposed to be fixed.

Let X be any of those letters. $\boldsymbol{X}$ will denote the actual object whose Gödel number is X.

$m, n, l, \cdots$ denote the variables ranging over natural numbers (in the absolute sense).

$\phi, \psi, \cdots, \alpha, \beta, \gamma, \cdots$ denote the second order variables.

2) Relations

$\Omega(r)$: $\boldsymbol{r} \in \Omega$.

$Od(\Omega; a)$: $\boldsymbol{a} \in \boldsymbol{Od}(\Omega)$.

$Od(r; a)$: $\boldsymbol{a} \in \boldsymbol{Od}_r$.

$J(r; a)$: $\boldsymbol{a} \in \boldsymbol{J}_r$.

$m=n$: the natural equality.

$m<n$: the natural inequality.

$r=s$: the equality in Ω.

$<(\Omega; r, s)$: $\boldsymbol{r} <_\Omega \boldsymbol{s}$.

$a \equiv b$: $\boldsymbol{a}$ and $\boldsymbol{b}$ are equal in the theory of $\boldsymbol{Od}(\Omega)$.

$\subset(r; i, b, a)$: $\boldsymbol{b}$ is a $(\boldsymbol{r}, \boldsymbol{i})$-section of $\boldsymbol{a}$.

$<(r; i, b, a)$: $\boldsymbol{b} <_{r,i} \boldsymbol{a}$ in $\boldsymbol{Od}(\Omega)$.

$\leqq(r; i, b, a)$: $b \equiv a \vee <(r; i, b, a)$.

Lim (i): $\boldsymbol{i}$ has no component 0.

$i \equiv j^*$: $\boldsymbol{i} = \boldsymbol{j} \# 0$.

$i \equiv \infty$: $\boldsymbol{i} = \infty$.

$<(i; a, b)$: $\boldsymbol{a} <_i \boldsymbol{b}$ in $\boldsymbol{Od}(\Omega)$.

Con $(r; a)$: $\boldsymbol{a} \in \boldsymbol{J}_r$ or $\boldsymbol{a}$ is $\boldsymbol{r}$-connected.

3) Functions

$l(a)$: the complexity of $\boldsymbol{a}$.

$l(r; a)$: the complexity of $\boldsymbol{a}$ relative to $\boldsymbol{J}_r$; namely the $\boldsymbol{r}$-atoms are regarded as atomic.

$lh(a)$: the length of $\boldsymbol{a}$, or the number of components in $\boldsymbol{a}$.

$lh(r; a)$: the length of $\boldsymbol{a}$ relative to $\boldsymbol{J}_r$; namely, the $\boldsymbol{r}$-atomic part is regarded as one component.

comp (n, a): the nth component of $\boldsymbol{a}$ in its formal expression.

comp $(r; i, n, a)$: the nth greatest component of $\boldsymbol{a}$ relative to the order $<_{r,i}$.

$a\#b$: $\boldsymbol{a\#b}$.
part $(r; i, n, a)$: comp $(r; i, 1, a)\# \cdots \#$
comp $(r; i, n, a)$.

4) About 0

The symbol 0 will be used for the least elements of various well-ordered sets appearing in this article: the natural number 0, the initial element of Ω, the least element of $\boldsymbol{Od}(\Omega)$, etc., as well as the corresponding Gödel numbers. We are certainly aware that this is an unlawful practice. Nevertheless, accurate distinction of those objects would only amount to complex and confusing notations.

PROPOSITION 2.1. *The elementary properties of* $\boldsymbol{Od}(\Omega)$*-diagrams can be proved in the first order part of* $\boldsymbol{AU}$ *(hence of* $\boldsymbol{C}$*).*

See §1 of [6] for elementary properties of $\boldsymbol{Od}(\Omega)$.

We shall henceforth assume Propositions 1.1 and 2.1 without reference each time.

3. Conclusion.

Definition 3.1. 1) Given here are abbreviations of formulas of $\boldsymbol{AU}$ which concern various notions of accessibility. Notice that they are all isolated formulas without non-arithmetical predicates.

1°. The schema of transfinite induction along $<_\Omega$ applied to α, $TI(\Omega; \alpha)$:

$$\forall r(\Omega(r) \wedge (\forall s(\Omega(s) \wedge \forall t(\Omega(t) \wedge <(\Omega; t, s) \supset \alpha[t]) \supset \alpha[s]) \supset \alpha[r]) ,$$

where the condition $\Omega(t)$ may be omitted due to the fact that $\Omega(s) \wedge <(\Omega; t, s)$ implies $\Omega(t)$. Owing to our notational convention, this can be expressed also as

$$\forall r(\forall s(\forall t(<(\Omega; t, s) \supset \alpha[t]) \supset \alpha[s]) \supset \alpha[r]) .$$

2°. The accessibility of the order $<_i$, $Ac(i, a)$:

$$\forall \phi(\forall x(Od(\Omega; x) \wedge \forall y(<(i, y, x) \supset \phi[y]) \supset \phi[x]) \supset \phi[a]) ,$$

or

$$\forall \phi(\forall x(\forall y(<(i, y, x) \supset \phi[y]) \supset \phi[x]) \supset \phi[a]) .$$

3°. The accessibility of the order $<_{r,i}$, $Ac(r; i, a)$:

$$\forall \phi(\forall x(Od(r; x) \wedge \forall y(Od(r; y) \wedge <(r; i, y, x) \supset \phi[y]) \supset \phi[x]) \supset \phi[a]) ,$$

where $Od(r; y)$ may be dropped.

4°. The $<_{r,0}$-accessibility of $\boldsymbol{J}_r$, $Ac(r)$:

$$\forall i(J(r; i) \supset \forall \phi(\forall j(J(r; j) \wedge \forall k(J(r; k) \wedge <(r; 0, k, j) \supset \phi[k]) \supset \phi[j]) \supset \phi[i])) ,$$

where the 0 in $<(r; 0, k, j)$ denotes Gödel number of the initial element of $\boldsymbol{Od}(\Omega)$ (cf. 4) of §2.

2) BASIC ASSUMPTION

$$BA(r; i, a): \Omega(r) \wedge Od(r; a) \wedge (J(r: i) \vee i \equiv \infty) .$$

Definition 3.2. 1) Matrix of ***UID***. We specify an isolated formula $G(r; i, a, \alpha)$ in order to determine the ***UID*** of our system.

$$\begin{aligned}
&G(r; i, a, \alpha):\ (i \equiv 0 \wedge Od(r; a)) \\
&\vee \exists j(i \equiv j^* \wedge \alpha[(r; j, a)] \wedge \forall x(\subset(r; j, x, a) \supset (\alpha[(r; j, x)] \wedge \forall \phi(\forall y(\alpha[(r; j, y)] \\
&\wedge \forall z(\alpha[(r; j, z)] \wedge <(r; j, z, y) \supset \phi[z]) \supset \phi[y]) \supset \phi[x])) \\
&\vee (\mathrm{Lim}\,(i) \wedge \forall k(<(r; 0, k, i) \supset \alpha[(r; k, a)]) \\
&\vee (i \equiv \infty \wedge \forall k(J(r; k) \supset \alpha[(r; k, a)])).
\end{aligned}$$

G will be called the matrix of the ***UID*** to be defined.

2) Let F be a predicate symbol with three first order arguments. The intended meaning of $F(r; i, a)$ is that "***a*** is a (***r***, ***i***)-fan."

3) $A(r; i, a)$ abbreviates

$$\begin{aligned}
&F(r; i, a) \wedge \forall \phi(\forall x(F(r; i, x) \wedge \forall y(F(r; i, y) \wedge \\
&\quad <(r; i, y, x) \supset \phi[y]) \supset \phi[x]) \supset \phi[a]) .
\end{aligned}$$

The intended meaning of $A(r; i, a)$ is that "***a*** is $\prec_{r,i}$-accessible in (***r***, ***i***)-fans."

Definition 3.3. System ***ASU***. Let Ω, S and $\prec_r$ in the definition of ***AU*** be here our Ω, $J(r; i)$ and $\prec_{r,0}$ respectively, regarding ∞ as the maximal element with respect to $\prec_{r,0}$ confined to $\{i\}J(r; i)$. Let F be the H in ***UID*** and let G be our matrix defined above. ***AU*** specified this way will be called ***ASU*** (arithmetic with a specified uniform inductive definition). Thus our ***UID*** looks like this.

1) $BA(r; i, a),\ Ac(r),\ F(r; i, a) \rightarrow G(r; i, a, \{j, x\}(F(r; j, x) \wedge <(r; 0, j, i)))$.

2) $BA(r; i, a),\ Ac(r),\ G(r; i, a, \{j, x\}(F(r; j, x) \wedge <(r; 0, j, i))) \rightarrow F(r; i, a)$.

We shall often omit $BA(r; i, a)$ in a sequent. This should not cause ambiguity by virtue of notational convention in §2.

Notice that our specified ***UID*** is a simple case of the one in §1.

We are to prove the

THEOREM. *The following sequent is provable in* ***ASU***.

(1) $TI(\Omega, \{r\}Ac(r)) \rightarrow \forall r \forall i \forall x(BA(r; i, x) \supset Ac(r; i, x))$.

From the theorem, we can draw our

CONCLUSION. *The following sequent is provable in* ***ASU***.

(2) $TI(\Omega, \{r\}Ac(r)) \rightarrow \forall i \forall x(Od(\Omega; x) \wedge Od(\Omega; i) \supset Ac(i, x))$.

This proves the accessibility of $(\boldsymbol{Od}(\Omega), \prec_i)$ for every $\boldsymbol{i}$ in $\boldsymbol{Od}(\Omega)$

in the system ASU augmented by the Π_1^1-transfinite induction along Ω.

Proof of the Conclusion, assuming the Theorem.

The following sequents are C-provable.

(3) $Od(\Omega; a),\ Od(\Omega; i) \to \exists r(r = \max(stg(a), stg(i)+1))$,

where $s+1$ denotes the successor of s in Ω and $\max(s, t)$ is the greater of s and t in the order of $<_\Omega$.

(4) $Od(\Omega; a),\ Od(\Omega; i),\ \Omega(r),\ r = \max(stg(a), stg(i)+1) \to BA(r; i, a)$.

(5) $BA(r; i, a),\ Ac(r; i, a) \to Ac(i, a)$,

since

$$Od(r; b),\ <(i, c, b) \to <(r; i, c, b)$$

is C-probable.

(1) and (3)~(5) immediately yield (2), the Conclusion. The theorem follows from the

MAIN PROPOSITION.

(6) $Ac(r) \to \forall x \forall i(BA(r; i, x) \supset Ac(r; i, x))$

is prevable in ASU.

Proof of the Theorem, assuming the Main Proposition.

The following lemmas are provable in ASU.

LEMMA 1. *Put*

$$A(r):\ \forall x \forall i(Od(r; x) \wedge (J(r; i) \vee i \equiv \infty)) \supset Ac(r; i, x)).$$

(7) $\Omega(r) \to A(r) \equiv Ac(r)$,

where $A \equiv B$ *is an abbreviation of* $(A \supset B) \wedge (B \supset A)$.

Proof. $Ac(r) \supset A(r)$ is the Main Proposition. The opposite direction is trivial.

LEMMA 2.

(8) $\forall s(<(\Omega; s, r) \supset A(s)) \to A(r)$.

Proof. Due to (7), it suffices to show

(9) $\forall s(<(\Omega; s, r) \supset A(s)) \to Ac(r)$.

The following are C-provable in succession.

(10) $Od(s; b),\ \forall y(<(s; 0, y, b) \supset \alpha[y]),\ <(\Omega; s, r) \to \exists t(<(\Omega; t, r) \wedge Od(t; b)) \wedge \forall y(<(r; 0, y, b) \supset \alpha[y])$,

since

$$Od(s; b),\ <(\Omega; s, r),\ <(r; 0, y, b) \to <(s; 0, y, b).$$

(11) $<(\Omega; s, r),\ \forall x(\exists t(<(\Omega; t, r) \wedge Od(t; x)) \wedge \forall y(<(r; 0, y, x) \supset \alpha[y]) \supset \alpha[x]) \to \forall x(Od(s; x) \wedge \forall y(<(s; 0, y, x) \supset \alpha[y]) \supset \alpha[x])$.

(12) $Od(s; a),\ <(\Omega; s, r),\ \forall x(\exists t(<(\Omega; t, r) \wedge Od(t; x)) \wedge \forall y(<(r; 0, y, x) \supset \alpha[y]) \supset \alpha[x]),\ \forall x(Od(s; x) \wedge \forall y(<(s; 0, y, x) \supset \alpha[y]) \supset \alpha[x]) \supset \alpha[a] \to \alpha[a]$.

From (12), by quantification over α, we obtain (8).

LEMMA 3.

(13) $TI(\Omega, \{r\}Ac(r)) \rightarrow TI(\Omega, \{r\}A(r))$.

Proof. From Lemma 1.

Lemmas 2 and 3 imply the Theorem.

It has thus been confirmed that our task is to establish our Main Proposition. It can be carried out through some auxiliary propositions, which we shall list in the next section.

Note. 1) As will be seen, we may in fact restrict all the formulas involved in the proofs to isolated ones.

2) Quite often the conditional clauses such as $\Omega(r)$, $J(r; i)$, $Od(r; a)$ and $BA(r; i, a)$ will be missing in formulas and sequents. This practice will not endanger the accuracy, due to the predetermined notational conventions (cf. §2).

§4. Auxiliary propositions and the proof of Main Proposition.

PROPOSITION 0. *When* $\boldsymbol{r}=0$, $\boldsymbol{Od}_r$ *is isomorphic to* ε_0 *(both for* $\lessdot_0$ *and* $\lessdot_\infty$*). (See Proposition* 1.1 *in* [6].) *Therefore it is known that the accessibility proof of* $\boldsymbol{Od}_0$ *can be established in the isolated arithmetic.*

The following sequents are ASU-provable.

PROPOSITION 1.

$BA(r; i, a),\ F(r; i, a),\ \forall x(<(r; i, x, a) \wedge F(r; i, x) \supset A(r; i, x))$
$\rightarrow A(r; i, a).$

PROPOSITION 2.

$BA(r; i, a),\ Ac(r; i, a) \rightarrow \forall x(<(r; i, x, a) \supset Ac(r; i, x))$.

PROPOSITION 3.

$BA(r; i, a),\ A(r; i, a) \rightarrow \forall x(<(r; i, x, a) \wedge F(r; i, x) \supset A(r; i, x)).$

PROPOSITION 4.

$Ac(r),\ BA(r; i, a),\ \forall n(1 \leqq n \leqq lh(a) \supset A(r; i, \mathrm{comp}\,(n, a)))$
$\rightarrow A(r; i, a).$

PROPOSITION 5.

$Ac(r),\ BA(r; i, a),\ A(r; i^*, a) \rightarrow A(r; i, a),$

where $i^* \equiv i \# 0.$

PROPOSITION 6.

$Ac(r),\ BA(r; i, a),\ \mathrm{Lim}(i),\ D(r; i),$
$A(r; i, a) \rightarrow \forall p(<(r; 0, p, i) \supset A(r; p, a)),$

where $D(r; i)$ *stands for:*

$\forall k \forall j(J(r; k) \wedge J(r; j) \wedge <(r; 0, k, j) \wedge <(r; 0, j, i)$
$\supset \forall x(A(r; j, x) \wedge F(r; i, x) \supset A(r; k, x))).$

PROPOSITION 7.
$Ac(r),\ BA(r; i, a),\ J(r; i),\ D(r; i),\ A(r; i, a)$
$\rightarrow \forall p(<(r; 0, p, i) \supset A(r; p, a))$.

PROPOSITION 8.
$Ac(r),\ C(r),\ \Omega(r),\ Od(r; a),\ A(r; \infty, a) \rightarrow \forall p(J(r; p) \supset A(r; p, a))$,
where $C(r)$ *stands for*:

$\forall i \forall j(J(r; i) \wedge J(r; j) \wedge <(r; 0, i, j) \supset \forall x(A(r; j, x)$
$\wedge F(r; \infty, x) \supset A(r; i, x)))$.

PROPOSITION 9.
$Ac(r),\ J(r; i) \rightarrow D(r; i)$.

PROPOSITION 10.
$Ac(r) \rightarrow C(r)$.

PROPOSITION 11.
$Ac(r),\ Od(r; a),\ A(r; \infty, a) \rightarrow \forall p(J(r; p) \supset A(r; p, a))$.

PROPOSITION 12.
$Ac(r),\ Od(r; a),\ F(r; \infty, a) \rightarrow A(r; \infty, a)$.

PROPOSITION 13.
$Ac(r),\ Od(r; a),\ F(r; \infty, a) \rightarrow \forall p(J(r; p) \supset A(r; p, a))$.

PROPOSITION 14.
$Ac(r),\ Od(r; a),\ F(r; 0, a) \rightarrow F(r; \infty, a)$.

PROPOSITION 15.
$Ac(r),\ BA(r; i, a),\ F(r; 0, a) \rightarrow A(r; i, a)$.

PROPOSITION 16.
$Ac(r),\ BA(r; i, a),\ A(r; i, a) \rightarrow Ac(r; i, a)$.

In the next section, which is the major part of this article, we shall establish the derivability of those properties in ***ASU***.

Since much of technicalities in [1] can be borrowed, we shall avoid altogether any repetition of such; instead we shall outline the process of proofs, pointing out all the applications of axioms (rules of inference) of higher order, namely second order induction, comprehension and ***UID***.

Expressional convention. 1) For an abstract $\{i\}V(i)$, we say that $\{i\}V(i)$ is progressive (in i) when

$$\forall j(<(j, i) \supset V(j)) \rightarrow V(i)$$

is ***ASU***-provable, where $<(j, i)$ is a linear order.

2) Borrowing an expression from [1], we say that a sequent S follows from S' straight when S is derivable from S' in ***C***.

Proof of Main Proposition.

Assuming those auxiliary propositions, we conclude the accessibility proof; namely:

the Main Proposition, (6), in §3 is an immediate consequence of ***UID*** and Propositions 15 and 16.

What is left now is the proof of the auxiliary propositions.

The following lemma will underlie the proofs of various propositions.

BASIC LEMMA. *Let* $\{i\}V(i)$ *be an isolated abstract. Then*

$$\Omega(r),\ Ac(r),\ \forall i(J(r;i)\wedge\forall j(<(r;0,j,i)\supset V(j))\supset V(i))$$
$$\rightarrow\forall i(J(r;i)\supset V(i))$$

is ***ASU****-provable by comprehension on* $\{i\}V(i)$.

§5. Proof of auxiliary propositions.

Convention. 1) We shall omit the conditional formula $BA(r;i,a)$ coherently throughout this section.

2) In the applications of the induction, the comprehension and the ***UID*** in the subsequent proofs, we do not point out each time that the abstract under question is isolated. The reader should examine that it is indeed isolated.

PROPOSITION 1.

$F(r;i,a),\ \forall x(<(r;i,x,a)\wedge F(r;i,x)\supset A(r;i,x))\rightarrow A(r;i,a)$.

Proof. Straight from the definition of A (cf. 3) of Definition 3.2).

PROPOSITION 2.

$Ac(r;i,a)\rightarrow\forall x(<(r;i,x,a)\supset Ac(r;i,x))$.

Proof.

Abbreviations.

$D(r;i,d,\alpha)$: $\forall x(Od(r;x)\wedge<(r;i,x,d)\supset\alpha[x])$.

$E(r;i,\alpha)$: $\forall y(Od(r;y)\wedge D(r;i,y,\alpha)\supset\alpha[y])$.

(1) $E(r;i,\{y\}D(r;i,y,\alpha))\supset D(r;i,a,\alpha),\ Od(r;c),\ <(r;i,c,a)$
$\rightarrow E(r;i,\alpha)\supset\alpha[c]$.

From (1) by comprehension on $\{y\}D(r;i,y,\alpha)$, follows

(2) $\forall\phi(E(r;i,\phi)\supset\phi[a]),\ <(r;i,c,a)\rightarrow E(r;i,c,\alpha)\supset\alpha[c]$.

The proposition now follows straight.

PROPOSITION 3.

$A(r;i,a)\rightarrow\forall x(<(r;i,x,a)\wedge F(r;i,x)\supset A(r;i,x))$.

Proof.

Abbreviations.

$B(r;i,a,\alpha)$: $\forall x(F(r;i,x)\wedge<(r;i,x,a)\supset\alpha[x])$.

$C(r; i, \alpha)$: $\forall y(F(r; i, y) \wedge B(r; i, y, \alpha) \supset \alpha[y])$.

(1) $C(r; i, \{y\}B(r; i, y, \alpha)) \supset B(r; i, a, \alpha),\ F(r; i, c),\ <(r; i, c, a) \rightarrow C(r; i, \alpha) \supset \alpha[c]$

straight. From (1) by comprehension on $\{y\}B(r; i, y, \alpha)$, we obtain

(2) $F(r; i, c),\ <(r; i, c, a),\ \forall\phi(C(r; i, \phi) \supset \phi[a]) \rightarrow C(r; i, \alpha) \supset \alpha[c]$,

from which follows the proposition.

PROPOSITION 4.

$Ac(r),\ \forall n(1 \leqq n \leqq lh(a) \supset A(r; i, \text{comp}\,(n, a))) \rightarrow A(r; i, a)$.

We shall first show how to deduce the proposition from Lemmas 4.1 and 4.2 stated below.

LEMMA 4.1.

(1) $Ac(r),\ a \equiv b,\ A(r; i, a) \rightarrow A(r; i, b)$.

(2) $Ac(r),\ a \equiv b,\ F(r; i, a) \rightarrow F(r; i, b)$.

LEMMA 4.2.

(3) $Ac(r),\ Od(r; b),\ A(r; i, a),\ A(r; i, b) \rightarrow A(r; i, a \# b)$.

Proof of Proposition 4.

Abbreviations.

$Ab(r; i, d)$: $A(r; i, d) \supset \forall y(A(r; i, y) \supset A(r; i, d \# y))$.

$L(r; i, a)$: $\forall n(1 \leqq n \leqq lh(a) \supset A(r: i, \text{comp}\,(n, a))) \supset A(r; i, a)$.

(4) $r=0 \rightarrow A(r; i, a)$

by Proposition 0.

(5) $\neg r=0,\ Od(r; e),\ A(r; i, e),\ \forall x(Od(r; x) \supset Ab(r; i, x)),$
$A(r: i, \text{comp}\,(lh(a), a)) \rightarrow A(r; i, e \# \text{comp}\,(lh(a), a))$

straight, from which follows

(6) $Ac(r),\ \neg r=0,\ Od(r; e),\ A(r; i, e),\ \forall x(Od(r; x) \supset Ab(r; i, x)),$
$A(r; i, \text{comp}\,(lh(a), a)),\ a \equiv e \# \text{comp}\,(lh(a), a) \rightarrow A(r; i, a)$

by virtue of (1) of Lemma 4.1. Using (6) we obtain

(7) $Ac(r),\ \neg r=0,\ \forall x(Od(r; x) \supset Ab(r; i, x)) \rightarrow (lh(a)=1 \supset L(r; i, a))$
$\wedge (lh(a)>1 \wedge \forall u(Od(r; u) \wedge lh(u)$
$= lh(a)-1 \supset L(r; i, u)) \supset L(r; i, a))$.

Applying induction on $\{m\}(lh(a)=m \supset L(r; i, a))$ to (7), we obtain

(8) $Ac(r),\ \neg r=0,\ \forall x(Od(r; x) \supset Ab(r; i, x)) \rightarrow L(r; i, a)$.

(8) and (3) (Lemma 4.2) imply the proposition.

We are thus to prove Lemmas 4.1 and 4.2.

Proof of Lemma 4.1.

Abbreviations.

$\mathfrak{F}(r; i)$: $\forall x \forall y(x \equiv y \supset (F(r; i, x) \supset F(r; i, y)))$.

$\mathfrak{G}(r; i)$: $\forall x \forall y(x \equiv y \supset (A(r; i, x) \supset A(r; i, y)))$.

$\mathfrak{F}(r; i) \rightarrow \mathfrak{G}(r; i)$

by virtue of Propositions 1 and 3. Using this fact, we can establish straight that $\{i\}(\mathfrak{F}(r; i) \wedge \mathfrak{G}(r; i))$ is progressive. Thus, by the Basic

Lemma in §4, we obtain (1) and (2) simultaneously.

For Lemma 4.2, we need some preliminary properties.

PRE-LEMMA 4.2.1.

(9) $Od(r; b),\ BA(r; i, a) \to BA(r; i, a \# b)$.

PRE-LEMMA 4.2.2.

(10) $Ac(r),\ F(r; i, a),\ F(r; i, b) \to F(r; i, a \# b)$.

Proof.

Abbreviation.

$G(r; i)$: $\forall x \forall y (F(r; i, x) \wedge F(r; i, y) \supset F(r; i, x \# y))$,

or more precisely,

$\forall x \forall y (BA(r; i, x) \wedge Od(r; y) \wedge F(r; i, x) \wedge F(r; i, y) \supset F(r; i, x \# y))$.

Since

$i \equiv 0 \to F(r; i, a) \equiv Od(r; a)$,

(9) implies

(11) $i \equiv 0 \to G(r; i)$.

(12) $i \equiv j^*,\ F(r; i, a),\ Ac(r) \to F(r; j, a) \wedge \forall x (\subset (r; j, x, a) \supset A(r; j, x))$

by ***UID***. Similarly with b.

(13) $G(r; j),\ F(r; j, a) \wedge \forall x (\subset (r; j, x, a) \supset A(r; j, x))$,
$F(r; j, b) \wedge \forall x (\subset (r; j, x, a) \supset A(r; j, x)) \to$
$\forall x (\subset (r; j, x, a \# b) \supset A(r; j, x)) \wedge F(r; j, a \# b)$,

hence by (12),

$Ac(r),\ i \equiv j^*,\ G(r; j),\ F(r; i, a),\ F(r; i, b) \to F(r; i, a \# b)$,

from which follows

(14) $i \equiv j^*,\ G(r; j) \to G(r; i)$.

(15) $\mathrm{Lim}\,(i),\ \forall j (<(r; 0, j, i) \supset G(r; j)) \to G(r; i)$

is immediate. (11), (14), (15) and the Basic Lemma on $\{i\}G(r; i)$ yield

$Ac(r) \to \forall i (J(r; i) \supset G(r; i))$,

which is essentially (10).

PRE-LEMMA 4.2.3.

(16) $Ac(r),\ 1 \leqq m \leqq lh(a),\ F(r; i, a) \to F(r; i, \mathrm{comp}\,(m, a))$.

Proof. $\{i\} \forall a \forall m (1 \leqq m \leqq lh(a) \wedge F(r; i, a) \supset F(r; i, \mathrm{comp}\,(m, a)))$ is progressive in i by ***UID***, hence follows (16) by virtue of the Basic Lemma.

PRE-LEMMA 4.2.4.

(17) $Ac(r),\ 1 \leqq m \leqq lh(a),\ A(r; i, a) \to A(r; i, \mathrm{comp}\,(m, a))$.

Proof. $Ac(r),\ <(r; i, \mathrm{comp}\,(m, a), a),\ A(r; i, a) \to A(r; i, \mathrm{comp}\,(m, a))$ by Proposition 3 and Pre-Lemma 4.2.3, hence

$Ac(r),\ lh(a) > 1,\ 1 \leqq m \leqq lh(a),\ A(r; i, a) \to A(r; i, \mathrm{comp}\,(m, a))$.

The case where $lh(a) = 1$ is immediate.

PRE-LEMMA 4.2.5.

(18) $Ac(r),\ F(r;\,i,\,a\#b)\rightarrow F(r;\,i,\,b)$.

Proof. We can easily establish that $\{i\}\forall x\forall y(F(r;\,i,\,x\#y)\supset F(r{:}\,i,\,x))$ is progressive in i under the premise of (18); so the Basic Lemma yields (18).

PRE-LEMMA 4.2.6.

(19) $Ac(r),\ F(r;\,i,\,a),\ <(r;\,i,\,a,\,b\#c),\ \leqq(r;\,i,\,c,\,\text{comp}\,(r;\,i,\,lh(b),\,b))$
$\rightarrow\leqq(r;\,i,\,a,\,b)\vee\exists x(F(r;\,i,\,x)\wedge a\equiv b\#x\wedge<(r;\,i,\,x,\,c))$.

Proof. From Pre-Lemma 4.2.5 and Lemma 4.1.

Proof of Lemma 4.2.

Abbreviations.

$L(r;\,i,\,b,\,a)$: $\leqq(r;\,i,\,b,\,\text{comp}\,(r{:}\,i,\,lh(a),\,a))$.

$B(r;\,i,\,a,\,\alpha)$: $\forall x(F(r;\,i,\,x)\wedge<(r;\,i,\,x,\,a)\supset\alpha[x])$.

$C(r;\,i,\,\alpha)$: $\forall y(F(r;\,i,\,y)\wedge B(r;\,i,\,y,\,\alpha)\supset\alpha[y])$.

Pre-Lemmas 4.2.1, 4.2.2, 4.2.5, and 4.2.6, and Propositions 1 and 3 establish

(20) $Ac(r),\ A(r;\,i,\,a),\ Od(r;\,b),\ C(r;\,i,\,\{x\}(L(r;\,i,\,x,\,a)\supset A(r;\,i,\,a\#x)))$
$\supset(L(r;\,i,\,b,\,a)\supset A(r;\,i,\,a\#b)),\ L(r;\,i,\,b,\,a)\rightarrow A(r;\,i,\,a\#b)$.

Applying comprehension on

$\{z\}(L(r;\,i,\,z,\,a)\supset A(r;\,i,\,a\#z))$

to (20), we obtain

(21) $Ac(r),\ Od(r;\,b),\ A(r;\,i,\,a),\ A(r;\,i,\,b),\ L(r;\,i,\,b,\,a)\rightarrow A(r;\,i,\,a\#b)$.

(21), Lemma 4.1 and Proposition 3 imply

(22) $Ac(r),\ Od(r;\,b),\ m+1\leqq lh(a\#b),\ A(r;\,i,\,a)$,
$A(r;\,i,\,b)\rightarrow A(r;\,i,\,\text{part}\,(r;\,i,\,m,\,a\#b))$.

By Pre-Lemma 4.2.4,

(23) $Ac(r), A(r;\,i,\,a), Od(r;\,b), A(r;\,i,\,b)\rightarrow A(r;\,i,\,\text{part}\,(r;\,i,\,1,\,a\#b))$.

By (22), (23) and induction on

$\{m\}(m\leqq lh(a\#b)\supset A(r;\,i,\,\text{part}\,(r;\,i,\,m,\,a\#b)))$,

we obtain (3).

This completes the proof of Proposition 4.

PROPOSITION 5.

$Ac(r),\ A(r;\,i^*,\,a)\rightarrow A(r;\,i,\,a)$.

LEMMA 5.1.

(1) $Ac(r),\ F(r;\,i,\,b),\ \subset(r;\,i,\,c,\,b)\rightarrow\forall k(\leqq(r;\,0,\,k,\,i)\supset F(r;\,k,\,c))$.

Proof. It can be easily established that $\{i\}\forall k(\leqq(r;\,0,\,k,\,i)\supset F(r;\,k,\,c))$ is progressive in i under the premises of (1). Thus the conclusion follows by the Basic Lemma applied to the abstract as above.

LEMMA 5.2.

(2) $Ac(r),\ \forall y(A(r;\,i^*y)\wedge<(r;\,i^*,\,y,\,a)\supset A(r;\,i,\,y)),\ A(r;\,i^*,\,a)$
$\rightarrow\forall y(F(r;\,i,\,y)\wedge<(r;\,i,\,y,\,a)\supset A(r;\,i,\,y)\wedge F(r;\,i^*,\,y))$.

Proof.

Abbreviations.

$H(m)$: $\forall y(l(y)=m\supset(F(r;i,y)\wedge<(r;i,y,a)\supset A(r;i,y)\wedge F(r;i^*,y)))$.

Γ: the antecedent of (2).

By (1) (Lemma 5.1),

(3) $Ac(r),\ l(b)=m,\ F(r;i,b),\ \subset(r;i,c,b),\ <(r;i,c,a)$
$\rightarrow<(r;i,c,a)\wedge F(r;i,c)\wedge l(c)<m$.

From (3),

(4) $Ac(r),\ \forall n(n<m\supset H(n)),\ l(b)=m,\ F(r;i,b)\rightarrow F(r;i^*,b)$.

(4) and Proposition 3 prove

(5) $\forall n(n<m\supset H(n)),\ \Gamma,\ l(b)=m,\ F(r;i,b)$,
$<(r;i^*,b,a)\rightarrow A(r;i,b)$.

(6) $\forall n(n<m\supset H(n)),\ \Gamma,\ l(b)=m,\ F(r;i,b),\ b\equiv d$,
$\subset(r;i,d,a)\rightarrow A(r;i,b)\wedge F(r;i^*,b)$

by ***UID*** and Lemma 4.1. (6) and Proposition 3 imply

(7) $\forall n(n<m\supset H(n)),\ \Gamma,\ l(b)=m,\ \exists z(\subset(r;i,z,a)\wedge\leqq(r;i,b,z))$,
$F(r;i,b)\rightarrow A(r;i,b)\wedge F(r;i^*,b)$.

(5) and (7) together with the course-of-values induction on $\{m\}H(m)$ establish (2), since

$<(r;i,b,a),\ \neg<(r;i^*,b,a)\rightarrow\exists z(\subset(r;i,z,a)\wedge\leqq(r;i,b,z))$.

Proof of Proposition 5. It can easily be established that

(8) $A(r;i^*,a)\rightarrow\forall\phi(\forall x(A(r;i^*,x)\wedge\forall y(A(r;i^*,y)\wedge$
$<(r;i^*,y,x)\supset\phi[y])\supset\phi[x])\supset\phi[a])$.

Applying comprehension on $\{x\}A(r;i,x)$ to (8), we obtain

(9) $A(r;i^*,a)\rightarrow\forall x(A(r;i^*,x)\wedge\forall y(A(r;i^*,y)\wedge<(r;i^*,y,x)$
$\supset A(r;i,y))\supset A(r;i,x))\supset A(r;i,a)$.

By Lemma 5.2, ***UID*** and Proposition 1, we obtain

(10) $Ac(r),\ A(r;i^*,a),\ \forall y(A(r;i^*,y)\wedge<(r;i^*,y,a)\supset A(r;i,y))$
$\rightarrow A(r;i,a)$.

(9) and (10) conclude the proposition.

PROPOSITION 6.

$Ac(r)$, Lim (i), $D(r;i)$, $A(r;i,a)\rightarrow\forall p(<(r;0,p,i)\supset A(r;p,a))$,

where $D(r;i)$ stands for

$\forall k\forall j(<(r;0,k,j)\wedge<(r;0,j,i)\supset\forall x(A(r;j,x)\wedge F(r;i,x)$
$\supset A(r;k,x)))$.

Let $\kappa\equiv\kappa(r;i,a)$ denote (Gödel number of) the greatest $\boldsymbol{r}$-index of $\boldsymbol{a}$ below $\boldsymbol{i}$ with regards to $\boldsymbol{<}_{r,0}$ if such exists.

LEMMA 6.1.

(1) $Ac(r)$, Lim (i), $D(r;i)$, $A(r;i,a)\rightarrow\forall p(\leqq(r;0,\kappa^*,p)\wedge$
$\leqq(r;0,p,i)\supset A(r;p,a))$.

For the time being, we assume (1). Abbreviate the antecedent of (1) to Γ and the succedent of (1) to B.

Proof of Proposition 6.

(2) $Ac(r),\ B,\ \Gamma,\ <(r;0,p,\kappa^*)\rightarrow A(r;p,a)$

follows from (1) and the definition of A. Now (1) and (2) imply the proposition.

Abbreviations.

$M(r;i,\gamma,a)$: $\forall y(A(r;i,y)\wedge\forall z(A(r;i,z)\wedge$
$<(r;i,z,y)\supset\gamma[z])\supset\gamma[y])\supset\gamma[a]$.

$M(r;i,a)$: $\forall\phi M(r;i,\phi,a)$

$N(r;i,a)$: $\forall p(\leqq(r;0,\kappa^*,p)\wedge\leqq(r;0,p,i)\supset A(r;p,a))$.

$\tilde{\alpha}[b]$: $\alpha[b]\vee<(r;i,a,b)$.

PRE-LEMMA 6.1.1.

(3) $A(r;i,a)\rightarrow M(r;i,a)$.

Proof. It can be easily established straight:

(4) $A(r;i,a),\ \forall y(F(r;i,y)\wedge\forall z(F(r;i,z)\wedge<(r;i,z,y)$
$\supset\tilde{\alpha}[z])\supset\tilde{\alpha}[y])\supset\tilde{\alpha}[a]\rightarrow\forall y(A(r;i,y)\wedge\forall z(A(r;i,z)\wedge$
$<(r;i,z,y)\supset\alpha[z])\supset\alpha[y])\supset\alpha[a]$.

By comprehension on $\{x\}\tilde{\alpha}[x]$ to (4), we obtain (3).

PRE-LEMMA 6.1.2.

(5) $Ac(r),\ \leqq(r;0,\kappa^*,p),\ <(r;0,p,i),\ <(r;p,b,a),$
$F(r;p,b)\rightarrow A(r;i,b)$.

Proof.

(6) $\neg F(r;i,b),\ F(r;p,b),\ l(b)=m,\ \leqq(r;0,\kappa^*,p),\ <(r;0,p,i)$
$\rightarrow\exists k(<(r;0,p,k)\wedge<(r;0,k,i)\wedge\neg F(r;k,b)$
$\wedge\forall q(<(r;0,q,k)\supset F(r;q,b)))$

straight.

(7) $Ac(r),\ \subset(r;q,c,b),\ F(r;q,b)\rightarrow F(r;q,c)$

by Lemma 5.1.

(8) $<(r;i,b,a),\ A(r;i,a),\ F(r;i,b)\rightarrow A(r;i,b)$

by Proposition 3. From (6)~(8) follows

(9) $Ac(r),\ Q(m)\rightarrow\exists n(n<m\wedge Q(n))$,

namely, $\{m\}\neg Q(m)$ is progressive in m, where $Q(m)$ abbreviates:

$\exists x(l(x)=m\wedge\exists j(\leqq(r;0,\kappa^*,j)\wedge<(r;0,j,i)\wedge F(r;j,x)\wedge$
$<(r;j,x,a))\wedge\neg A(r;i,x))$.

By induction on $\{m\}\neg Q(m)$ applied to the contraposition of (9), we obtain (5).

PRE-LEMMA 6.1.3.

(10) $Ac(r)\rightarrow\forall x(A(r;i,x)\wedge\forall y(A(r;i,y)\wedge$
$<(r;i,y,x)\supset N(r;i,y))\supset N(r;i,x))$.

Proof.

(11) $F(r;p,a),\ \neg A(r;p,a)\rightarrow\exists b(<(r;p,b,a)$
$\wedge F(r;p,b)\wedge\neg A(r;p,b))$

by Proposition 1.

(12) $Ac(r),\ \leqq(r; 0, \kappa^*, p),\ \leqq(r; 0, p, i),\ A(r; i, a),\ <(r; p, b, a),$
$F(r; p, b) \rightarrow F(r; i, b)$

by Pre-Lemma 6.1.2 and ***UID***. From (11) and (12) and another application of Pre-Lemma 6.1.2 follows (10).

Proof of Lemma 6.1. Immediate from Pre-Lemmas 6.1.1 and 6.1.3.

PROPOSITION 7.

$Ac(r),\ J(r; i),\ D(r; i),\ A(r; i, a) \rightarrow \forall p(<(r; 0, p, i) \supset A(r; p, a)),$

where D was defined in Proposition 6.

Proof.

Abbreviation.

$U(i)$: $\forall x(BA(r; i, x) \wedge D(r; i) \wedge A(r; i, x) \supset \forall p(<(r; 0, p, i) \supset A(r; p, x)))$.

By Propositions 5 and 6, $\{i\}U(i)$ is shown to be progressive in i. Thus, by an application of the Basic Lemma on $\{i\}U(i)$, we obtain the result.

PROPOSITION 8.

$Ac(r),\ C(r),\ A(r; \infty, a) \rightarrow \forall p(J(r; p) \supset A(r; p, a)),$

where $C(r)$ stands for:

$\forall i \forall j(J(r; i) \wedge J(r; j) \wedge <(r; 0, i, j)$

$\supset \forall x(A(r; j, x) \wedge F(r; \infty, x) \supset A(r; i, x)))$.

Proof. Follow the proof of Proposition 6, reading ∞ in the place of i there.

PROPOSITION 9.

$Ac(r),\ J(r; i) \rightarrow D(r; i)$.

PROOF. Put

$V(i)$: $J(r; i) \rightarrow D(r; i)$.

(1) $Ac(r)$, Lim $(i) \rightarrow F(r; i, a) \equiv \forall p(<(r; 0, p, i) \supset F(r; p, a))$

by ***UID***. Using (1) and Proposition 7 for the case $i \equiv p^*$, we can establish that $\{i\}V(i)$ is progressive. Thus, by the Basic Lemma on $\{i\}V(i)$, we obtain the result.

PROPOSITION 10.

$Ac(r) \rightarrow C(r)$,

where $C(r)$ was defined in Proposition 8.

Proof.

(1) $Ac(r),\ \forall p(J(r; p) \supset D(r; p)) \rightarrow C(r)$

is established in a manner similar to Proposition 9. On the other hand,

(2) $Ac(r) \rightarrow \forall p(J(r; p) \supset D(r; p))$

by Proposition 9. (1) and (2) yield the proposition.

PROPOSITION 11.

$Ac(r),\ A(r; \infty, a) \rightarrow \forall p(J(r; p) \supset A(r; p, a))$.

Proof. Immediate from Propositions 8 and 10.

PROPOSITION 12.
$Ac(r),\ F(r;\infty, a)\to A(r;\infty, a)$.

LEMMA 12.1.
(1) $Ac(r),\ \mathrm{Con}(r; b),\ F(r;\infty, b)\to A(r;\infty, b)$.

Proof of Proposition 12.
(2) $Ac(r),\ F(r;\infty, a)\to\forall n(1\leqq n\leqq lh(a)\supset F(r;\infty, \mathrm{comp}(n, a)))$
by Pre-Lemma 4.2.3.
(3) $Ac(r),\ \forall n(1\leqq n\leqq lh(a)\supset A(r;\infty, \mathrm{comp}(n, a)))\to A(r;\infty, a)$
by Proposition 4. It is obvious that (1)~(3) imply the proposition.

Lemma 12.1 states a crucial property the establishment of which requires a special care.

Abbreviations.
$B(r; b)$: $\mathrm{Con}(r; b)\wedge F(r;\infty, b)$.
B: $\{y\}B(r; y)$.
$\boldsymbol{A}(r; b, B)$: $\forall\phi(\forall x(B(r; x)\wedge\forall y(B(r; y)\wedge$
$<(r;\infty, y, x)\supset\phi[y])\supset\phi[x])\supset\phi[b])$;
$\boldsymbol{b}$ is $<_{r,\infty}$-accessible in $\{y\}B(r; y)$.
$\boldsymbol{B}(r; B)$: $\forall z(B(r; z)\supset\boldsymbol{A}(r; z, B))$.

PRE-LEMMA 12.1.1.
(4) $Ac(r),\ \boldsymbol{B}(r; B)\to\forall z(B(r; z)\supset A(r;\infty, z))$.

Proof.
Abbreviations.
$\mathrm{Prg}(r; k, \alpha, \beta)$: $\forall x(\beta[x]\wedge\forall y(\beta[y]\wedge<(r; k, y, x)\supset\alpha[y])\supset\alpha[x])$.
$W(y)$: $B(r; y)\supset A(r;\infty, y)$.
W: $\{y\}W(y)$.
$TI(r;\infty, \alpha, \beta)$: $\forall a(\beta[a]\wedge\mathrm{Prg}(r;\infty, \alpha, \beta)\supset\alpha[a])$.
$TI(r; 0, a, \beta)$: $\forall\phi(\mathrm{Prg}(r; 0, \phi, \beta)\supset\phi[a])$.
By virtue of Pre-Lemma 4.2.3 and Proposition 4, we can deduce
(5) $Ac(r),\ \forall y(<(r;\infty, y, a)\supset W(y))\to W(a)$.
Evidently,
(6) $\mathrm{Prg}(r;\infty, W, B),\ TI(r;\infty, W, B)\to\forall y W(y)$.
(5) and (6) imply
(7) $TI(r;\infty, W, B)\to\forall y W(y)$.
(8) $\boldsymbol{B}(r; B)\to TI(r;\infty, W, B)$
by comprehension on W. (7) and (8) yield (4).

PRE-LEMMA 12.1.2.
(9) $Ac(r),\ J(r; b)\to\boldsymbol{A}(r; b, B)$.

Proof. From the definition of $Ac(r)$, we obtain
(10) $Ac(r)\to\forall p(J(r; p)\supset Ac(r;\infty, p))$.

Also,

(11) $\forall x(Od(r; x) \wedge \forall y(Od(r; y) \wedge <(r; \infty, y, x) \supset (B(r; y) \supset \beta[y])) \supset (B(r; x) \supset \beta[x])) \supset (B(r; b) \supset \beta[b]),\ B(r; b) \rightarrow \forall x(B(r; x) \wedge (\forall y(B(r; y) \wedge <(r; \infty, y, x) \supset \beta[y]) \supset \beta[x]) \supset \beta[b]$.

By comprehension on $\{x\}(B(r; x) \supset \beta[x])$ applied to (11),

(12) $B(r; b),\ Ac(r; \infty, b) \rightarrow A(r; b, B)$.

(10) and (12) imply (9).

PRE-LEMMA 12.1.3.

(13) $Ac(r),\ B(r; b),\ \neg J(r; b) \rightarrow A(r; b, B)$.

Proof. In order to render the reader some concrete idea of how to reach (13), we shall present somewhat informal account of the proof.

$B(r; b) \wedge \neg J(r; b)$ implies that b is of the form $(r; i, c, a)$.

Case 1. $i \equiv 0 \wedge c \equiv 0$. Put $1 \equiv 0 \# 0$.

(14) $Ac(r) \rightarrow F(r; 1, (r; 0, 0, a)) \equiv Ac(r; 0, a)$

by ***UID***, from which follows

(15) $Ac(r),\ F(r; 1, (r; 0, 0, a)) \rightarrow \forall \phi(\forall x(\forall y(<(r; 0, y, x) \supset \phi[y]) \supset \phi[x]) \supset \phi[a])$.

(16) $\forall \phi(\forall x(\forall y(<(r; 0, y, x) \supset \phi[y]) \supset \phi[x]) \supset \phi[a]),\ F(a) \rightarrow \forall \phi(\forall x(F(x) \wedge \forall y(F(y) \wedge <(r; 0, y, x) \supset \phi[y]) \supset \phi[x]) \supset \phi[a])$

by comprehension on $\{y\}(F(y) \supset \alpha[y])$, where $F(y)$ abbreviates $F(r; 1, (r; 0, 0, y))$. (15) and (16) imply

(17) $Ac(r),\ F(r; 1, (r; 0, 0, a)) \rightarrow TI(r; 0, a, \{y\}F(y))$.

It can be established straight

(18) $F(r; \infty, (r; 0, 0, a)),\ \exists \phi(\forall x(B(r; x) \supset \forall y(<(r; \infty, y, x) \wedge B(r; y) \supset \phi[y]) \supset \phi[x]) \wedge \neg \phi[(r; 0, 0, a)]) \rightarrow \exists y(<(r; 0, y, a) \wedge F(r; \infty, (r; 0, 0, y)) \wedge \neg A(r; (r; 0, 0, , y), B))$.

Let $T(a)$ abbreviate $F(r; \infty, (r; 0, 0, a)) \supset A(r; (r; 0, 0, a), B)$.

(18) implies

(19) $\forall y(<(r; 0, y, a) \supset T(y)) \rightarrow T(a)$.

(20) $Ac(r; 0, a),\ \forall y(<(r; 0, y, a) \supset Ac(r; 0, y)) \supset T(a) \rightarrow \forall y(<(r; 0, y, a) \wedge Ac(r; 0, y) \supset T(y)) \supset T(a)$

is a consequence of Proposition 2. (19) and (20) imply

(21) $Ac(r; 0, a) \rightarrow \forall y(<(r; 0, y, a) \wedge Ac(r; 0, y) \supset T(y)) \supset T(a)$,

or $\{y\}(Ac(r; 0, y) \supset T(y))$ is progressive in y with regards to $\prec_{r,0}$. (17) and (21) with an application of comprehension on $\{y\}T(y)$ imply

(22) $Ac(r),\ F(r; \infty, (r; 0, 0, a)) \rightarrow A(r; (r; 0, 0, a), B)$.

This is (13) for Case 1.

Case 2. $i \equiv 0 \wedge (\neg c \equiv 0)$.

Abbreviations.

$F'(c, a)$: $\exists u(\leqq(r; 0, u, c) \wedge F(r; 1, (r; 0, u, a)))$.

$C(r; i, a, \beta)$: $\forall \phi(\forall x(\beta[x] \wedge \forall y(\beta[y] \wedge <(r; i, y, x) \supset \phi[y]) \supset \phi[x]) \supset \phi[a])$.

$K(d, e)$: $F(r; \infty, (r; 0, d, e)) \supset A(r; (r; 0, d, e), B)$.
$<(r; i, (d, b), (c, a))$: $<(r; 0, d, c) \vee (d \equiv c \wedge <(r; i, b, a))$.
$G(c, a)$: $J(r; c) \wedge Od(r; a) \wedge F(r; \infty, (r; 0, c, a))$.
$TI(<(r; 0), \gamma)$: $\forall c \forall a \forall \phi(\gamma[(c, a)] \wedge \forall u \forall v(\gamma[(u, v)]$
$\wedge \forall w \forall x(<(r; 0, (w, x), (u, v))$
$\wedge \gamma[(w, x)]) \supset \phi[(u, x)]) \supset \phi[(u, v)]) \supset \phi[(c, a)])$.
$TI(<(r; 0), \gamma, c)$: $\forall a \forall \phi(\gamma[(c, a)] \wedge \forall v(\gamma[(c, v)] \wedge \forall x(<(r; 0, x, v)$
$\wedge \gamma[(c, x)] \supset \phi[(c, x)]) \supset \phi[(c, v)]) \supset \phi[(c, a)])$.

(23) $Ac(r) \to F(r; 1, (r; 0, d, a)) \equiv Ac(r; 0, a)$

by ***UID***, hence

(24) $Ac(r) \to F'(c, a) \equiv Ac(r; 0, a)$.

(25) $F'(c, a),\ F'(c, a) \supset \forall x(F'(c, x) \wedge \forall y(F'(c, y) \wedge <(r; 0, y, x)$
$\supset \alpha[y]) \supset \alpha[x]) \supset \alpha[a] \to \forall x(F'(c, x) \wedge \forall y(F'(c, y) \wedge <(r; 0, y, x)$
$\supset \alpha[y]) \supset \alpha[x]) \supset \alpha[a]$.

Applying comprehension on $\{y\}(F'(c, y) \supset \alpha[y])$ to (25), we obtain

(26) $Ac(r),\ F'(c, a),\ Ac(r; 0, a) \to C(r; 0, a, \{y\}F'(c, y))$.

Using (24), (26) can be rewritten as

(27) $Ac(r) \to \forall z(F'(c, z) \supset C(r; 0, z, \{y\}F'(c, y)))$.

This means $F'(c, z)$ is accessible in z with respect to $<_{r,0}$. It can be established straight

(28) $F(r; \infty, (r; 0, c, a)),\ \neg A(r; (r; 0, c, a), B)$
$\to \exists u \exists v(F(r; \infty, (r; 0, u, v)) \wedge \neg A(r; (r; 0, u, v), B)$
$\wedge <(r; 0, (u, v), (c, a)))$,

from which follows

(29) $F(r; \infty, (r; 0, c, a)),\ \forall u \forall v(<(r; 0, (u, v), (c, a))$
$\wedge F(r; \infty, (r; 0, u, v)) \supset A(r; (r; 0, u, v), B)) \to A(r; (r; 0, c, a), B)$,

or

$\forall u \forall v(<(r; 0, (u, v), (c, a)) \supset K(u, v)) \to K(c, a)$.

By comprehension on $\{(u, v)\}K(u, v)$ in (29), we obtain

(30) $Ac(r),\ TI(<(r; 0), \{u, v\}G(u, v))$
$\to \forall c \forall a(G(c, a) \supset A(r; (r; 0, c, a), B))$.

By ***UID*** and (27),

(31) $Ac(r),\ J(r; c) \to TI(<(r; 0), \{u, v\}G(u, v), c)$,

from which follows

(32) $Ac(r) \to TI(<(r; 0), \{u, v\}G(u, v))$.

(30) and (32) imply (13) for Case 2.

Case 3. $\neg i \equiv 0$.

Abbreviations.

$F''(i, a)$: $\exists c F(r; i^*, (r; i, c, a))$.
$S(i, a)$: $\forall c(F(r; \infty, (r; i, c, a)) \supset A(r; (r; i, c, a), B))$.
$K(j, d, e)$: $F(r; \infty, (r; j, d, e)) \supset A(r; (r; j, d, e), B)$.
$<(r; 0, (j, d, e), (i, c, a))$: $(r; 0, j, i) \vee (j \equiv i \wedge <(r; 0, (d, e), (c, a)))$.
$H(i, c, a)$: $J(r; i) \wedge J(r; c) \wedge Od(r; a) \wedge F(r; \infty, (r; i, c, a))$.

$TI^*(<(r; 0), \gamma)$: $\forall i \forall c \forall a \forall \phi(\gamma[(i, c, a)] \wedge \forall p \forall u \forall v(\gamma[(p, u, v)] \wedge \forall q \forall w \forall x(<(r; 0, (q, w, x), (p, u, v)) \wedge \gamma[(q, w, x)] \supset \phi[(q, w, x)]) \supset \phi[(p, u, v)]) \supset \phi[(i, c, a)])$.

Similarly to (28), we can derive

(33) $F(r; \infty, (r; i, c, a)),\ \neg \boldsymbol{A}(r; (r; i, c, a), B) \rightarrow \exists j \exists d \exists e(<(r; 0, (j, d, e), (i, c, a)) \wedge F(r; \infty, (r; j, d, e)) \wedge \neg \boldsymbol{A}(r; (r; j, d, e), B))$.

Similarly to (27), by an application of comprehension on $\{y\}(F'''(i, y) \supset \alpha[y])$ we obtain

(34) $Ac(r) \rightarrow \forall a(F'''(i, a) \supset \boldsymbol{A}(r; i, a, \{y\}F'''(i, y)))$.

(35) $F(r; \infty, (r; i, c, a)),\ \neg \boldsymbol{A}(r; (r; i, c, a), B) \rightarrow \exists j \exists d \exists e(<(r; 0, (j, d, e), (i, c, a)) \wedge F(r; \infty, (r; j, d, e)) \wedge \neg \boldsymbol{A}(r; (r; j, d, e), B))$,

from which follows

(36) $\forall j \forall d \forall e(<(r; 0, (j, d, e), (i, c, a)) \supset K(j, d, e)) \rightarrow K(i, c, a)$.

(37) $Ac(r),\ \forall i(\forall j(<(r; 0, j, i) \supset \forall d \forall e K(j, d, e)) \supset \forall d \forall e K(i, d, e)) \rightarrow \forall i \forall d \forall e K(i, d, e)$

by comprehension on $\{i\}\forall d \forall e K(i, d, e)$.

(38) $Ac(r),\ \forall c(\forall d(<(r; 0, d, c) \supset \forall e K(i, d, e)) \supset \forall e K(i, c, e)) \rightarrow \forall c \forall e K(i, c, e)$

by comprehension on $\{c\}\forall e K(i, c, e)$.

(39) $\forall a(F'''(i, a) \supset \boldsymbol{C}(r; i, a, \{y\}F'''(i, y))),\ \forall a \forall b(F'''(i, a) \wedge (F'''(i, b) \wedge <(r; i, a, b) \supset K(i, c, a)) \supset K(i, c, b)) \rightarrow \forall a K(i, c, a)$

by comprehension on $\{a\}K(i, c, a)$. By the definition of F''' and (37)~(39),

(40) $Ac(r) \rightarrow TI^*(<(r; 0), \{i, c, a\}H(i, c, a))$.

(36) and (40) complete the proof.

PRE-LEMMA 12.1.4.

(41) $Ac(r) \rightarrow \boldsymbol{B}(r; B)$.

Proof. Immediate from Pre-Lemmas 12.1.2 and 12.1.3.

Proof of Lemma 12.1. Immediate from Pre-Lemmas 12.1.1 and 12.1.4.

PROPOSITION 13.

$Ac(r),\ F(r; \infty, a) \rightarrow \forall i(J(r; i) \supset \boldsymbol{A}(r; i, a))$.

Proof. Straight from Propositions 11 and 12.

PROPOSITION 14.

$Ac(r),\ F(r; 0, a) \rightarrow F(r; \infty, a)$.

Proof.

(1) $Ac(r) \rightarrow F(r; \infty, a) \equiv \forall i(J(r; i) \supset F(r; i, a))$

and

(2) $Ac(r) \rightarrow F(r; 0, a) \equiv Od(r; a)$

by ***UID***. (1) and (2) assure us that what must be proved is the following.

LEMMA 14.1.

(3) $Ac(r) \rightarrow F(r; i, a)$.

Proof.

Abbreviations.

$<^*(r; 0, (n, i), (m, j))$: $n < m \vee (n = m \wedge <(r; 0, i, j))$.

$TI(\{i\}J(r; i), \beta)$: $\forall i \forall \phi(\beta[i] \wedge J(r; i) \wedge \forall j(\beta[j] \wedge J(r; j) \wedge \forall k(\beta[k] \wedge J(r; k) \supset \phi[k]) \supset \phi[j]) \supset \phi[i])$.

$TI(<^*(r; 0), \gamma)$: $\forall n \forall i \forall \phi(\gamma[(n, i)] \wedge \forall m \forall j(\gamma[(m, j)] \wedge \forall l \forall k(<^*(r; 0, (l, k), (m, j)) \wedge \gamma[(l, k)] \supset \phi[(l, k)]) \supset \phi[(m, j)]) \supset \phi[(n, i)])$.

$AJ(r; a)$: $\forall \phi(\forall x(J(r; x) \wedge \forall y(<(r; 0, y, x) \supset \phi[y]) \supset \phi[x]) \supset \phi[a])$.

$V(f)$: $\forall x \forall i(Od(r; x) \wedge J(r; i) \wedge f \equiv (l(r; x), i) \supset F(r; i, x))$.

Γ: $Od(r; a) \wedge J(r; i) \wedge \forall f(<^*(r; 0, f, (l(r; a), i)) \supset V(f))$.

(4) $Ac(r) \rightarrow TI(<^*(r, 0), \{f\}V(f))$

by induction on $\{n\}\forall j(J(r; j) \supset V((n, j)))$ and comprehension on $\{j\}(J(r; j) \supset V((n, j)))$. This guarantees that it suffices to establish

(5) $Ac(r),\ \forall f(<^*(r; 0, f, g) \supset V(f)) \rightarrow V(g)$,

or

$$Ac(r),\ \forall f(<^*(r; 0, f(l(r; a), i)) \supset V(f)),\ Od(r; a),\ J(r; i) \rightarrow F(r; i, a).$$

Case 1. $l(r; a) = 0$, or $J(r; a)$.

(6) $J(r; i) \rightarrow F(r; i, a) \equiv (i \equiv 0 \wedge J(r; a)) \vee \exists j(i \equiv j^* \wedge F(r; j, a))$
$\vee(\mathrm{Lim}\,(i) \wedge \forall j(<(r; 0, j, i) \supset F(r; j, a)))$
$\vee(i \equiv \infty \wedge \forall j(J(r; j) \supset F(r; j, a)))$,

since there is no (r, j)-section of a in this case.

(7) $Ac(r) \rightarrow AJ(r; a)$.

(8) $AJ(r; a) \rightarrow TI(\{i\}J(r; i), F^*)$

where F^* stands for $\{i\}((J(r; i) \vee i \equiv \infty) \wedge F(r; i, a))$.

From (6), we obtain

(9) $TI(\{i\}J(r; i), F^*) \rightarrow \forall i((J(r; i) \vee i \equiv \infty) \supset F(r; i, a))$.

(7)~(9) imply (5) for Case 1.

Case 2. $l(r; a) > 0$.

(10) $\Gamma,\ l(r; b) < l(r; a) \rightarrow F(r; \infty, b)$.

Also,

(11) $\Gamma \rightarrow \forall j(<(r; 0, j, i) \supset F(r; j, a))$.

Case 2.1. Suppose $a \equiv (r; k, c, e)$.

Note that $l(r; e) < l(r; a)$.

1°. $i \equiv j^*$.

From (11) follows

(12) $Ac(r),\ \Gamma,\ i \equiv j^* \rightarrow F(r; j, a)$.

(13) $Ac(r),\ i\equiv j^{*}\rightarrow F(r;\ i,\ a)\equiv F(r;\ j,\ a)$
$\wedge\forall x(\subset(r;\ j,\ x,\ a)\supset A(r,\ j,\ x))$

by ***UID***.

1.1°. $j\equiv k$.

(14) $j\equiv k\rightarrow\exists!x(\subset(r;\ j,\ x,\ a)\wedge x\equiv e)$,

where $\exists!x$ means the unique existence upto $\equiv$. By Proposition 13,

(15) $Ac(r),\ F(r;\ \infty,\ e)\rightarrow\forall k(J(r;\ k)\supset A(r;\ k,\ e))$.

So by (1),

(16) $Ac(r),\ \Gamma\rightarrow\forall k(J(r;\ k)\supset A(r;\ k,\ e))$,

in particular,

$Ac(r),\ \Gamma\rightarrow A(r;\ j,\ e)$.

(14), (16), and (11)$\sim$(13) imply

(17) $Ac(r),\ \Gamma\rightarrow A(r;\ i,\ a)$,

hence (5).

1.2°. $<(r;\ 0,\ j,\ k)$.

(10) implies

(18) $\Gamma\rightarrow F(r;\ i,\ e)$,

hence, in particular,

(19) $Ac(r),\ \Gamma,\ \subset(r;\ j,\ c,\ e)\rightarrow A(r;\ j,\ c)$

by ***UID***, from which follows

(20) $Ac(r),\ \Gamma,\ \subset(r;\ j,\ c,\ a)\rightarrow A(r;\ j,\ c)$.

(11) implies

(21) $\Gamma\rightarrow F(r;\ j,\ a)$.

(20), (21), and ***UID*** imply (5).

1.3°. $<(r;\ 0,\ k,\ j)$.

(5) is trivially established for this case, since

$$\rightarrow\neg\exists x(\subset(r;\ j,\ x,\ a))\ .$$

2°. Lim (i).

(22) $Ac(r)\rightarrow F(r;\ i,\ a)\equiv\forall j(<(r;\ 0,\ j,\ i)\supset F(r;\ j,\ a))$.

(23) $\Gamma,\ <(r;\ 0,\ j,\ i)\rightarrow V((l(r;\ a),\ j))$,

hence by (22) we obtain (5).

Case 2.2. $a\equiv b\sharp c$, where $l(r;b)>0$ or $l(r;c)>0$.

Note that $l(r;b),\ l(r;c)<l(r;a)$.

Thus

(24) $\Gamma,\ J(r;\ k)\rightarrow F(r;\ k,\ b)\wedge F(r;\ k,\ c)$

and

(25) $\Gamma,\ <(r;\ 0,\ k,\ i)\rightarrow F(r;\ k,\ a)$.

1°. $i\equiv j^{*}$.

From the fact that

$\rightarrow\subset(r;\ j,\ d,\ a)\equiv\subset(r;\ j,\ d,\ b)\vee\subset(r;\ j,\ d,\ c)$

and (24) and (25), we obtain (5).

2°. Lim (i).

(5) for this case is immediate from (25) and ***UID***.

PROPOSITION 15.

$Ac(r),\ F(r; 0, a) \to A(r; i, a)$.

Proof. This is an immediate consequence of Propositions 12~14.

PROPOSITION 16.

$Ac(r),\ A(r; i, a) \to Ac(r; i, a)$.

Proof. By ***UID*** and Proposition 15 we obtain

(1) $Ac(r) \to A(r; i, b)$,

from which

(2) $Ac(r) \to F(r; i, b)$,

hence by ***UID***,

(3) $J(r; i),\ Ac(r) \to Od(r; b) \equiv F(r; i, b)$.

Therefore we may replace $\{x\}F(r; i, x)$ in the definition of $A(r; i, a)$ by $\{x\}Od(r; x)$, presuming $J(r; i) \wedge Ac(r)$. Therefore

(4) $Ac(r) \to A(r; i, a) \equiv Ac(r; i, a)$,

which implies the proposition straight.

This completes the entire accessibility proof.

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