Comment Math.
Univ. St. Pauli
XXVIII—1, 1979

p-Pure enveloppes of pairs in torsion free abelian groups

by

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(Received October 30, 1979)

With the work of O. Mutzbauer [2] there seems to be a renewed interest in rank two torsion free abelian groups. In this article, we give the structure of the smallest p-pure subgroup containing a given pair of independant elements of a group. Such a p-pure subgroup is called the p-pure enveloppe of the given pair. We define the p-indicator of an ordered pair or elements. This turns out to be a p-adic number together with a pair of non negative integers or ∞. Although p-adic numbers seem to crop up in various ways in the study of rank two groups (see [2] and [3]) our method of obtaining them involves only the rather natural concept of variations of p-heights of certain sets of elements. Aside from the structure of p-pure enveloppes we give also some results on their endomorphism rings. Finally we establish a formula for computing p-heights of linear combinations of a pair of elements in terms of their coefficients and the p-indicator of the pair. The notation and symbols used here without explanation follow closely [1]. $Z^+=\{n\in Z\mid n>0\}$ and $Q^{(p)}=\{a/b\mid b=p^n,\ n\in Z^+\}$.

1. The p-indicator of an ordered pair of elements

Throughout this section G is a fixed torsion free group.

Let $a, b\in G$ and let $h_p^G(a)=\alpha$ and $h_p^G(b)=\beta$, where p is a prime number. We will use the preceeding notation all through this section. If α and β are finite then there exists an element $y\in G$ such that $p^\beta y=p^\alpha b$. Such a y is unique and is denoted by $p^{\alpha-\beta}b$. We want to study the behavior of the p-heights of elements in G of the form $a+ny$ where $n\in Z^+$. In order to avoid constant consideration of special cases, we agree that, if α or β is infinite, we set $n=0$. We chose to study the elements $a+ny$ because they reflect faithfully the variations of the p-height of all other combinations of a and b in the sense of the following lemma.

LEMMA 1.1. *Let $a, b\in G$ and let $a'\in\langle a\rangle_*$ and $b'\in\langle b\rangle_*$. If for some $r\in Z^+$, $h_p^G(a'+b')\geqq h_p^G(a')+r$ then there exists $n\in Z^+$ such that $0\leqq n<p^r$ and $h_p^G(a+ny)\geqq\alpha+r$.*

* Work done under C.N.R.S.C. Grant no A5991.

Proof. If $r=0$, we take $n=0$. Let then $r>0$. If $\alpha=\infty$ there is nothing to prove, so we may suppose $\alpha<\infty$. In this case, $h_p^G(a')$ is also finite and since $h_p^G(a'+b')>h_p^G(a')$ we must have $h_p(a')=h_p(b')<\infty$. It follows then that $h_p(b)=\beta<\infty$. We have thus reduced the problem to the case where $r>0$ and $\alpha, \beta<\infty$. Let $a'=up^s a$ and $b'=vp^t b$ where $s, t \in \boldsymbol{Z}$ and $u, v \in Q_p$ and $v_p(u)=0=v_p(v)$. Upon multiplying by the common denominator m of u and v, we may assume that $u, v \in \boldsymbol{Z}$. (Note that $(m, p)=1$ so $h_p(ma')=h_p(a')$.) Now, $h_p^G(a')=s+\alpha<h_p^G(a'+b')$ and $p^{-s}(a'+b')=ua+vp^{t-s}b$ is an element of G whose p-height in G is $\geqq r+\alpha$. Let $\gamma, \delta \in \boldsymbol{Z}$ such that $\gamma u+\delta p^{r+\alpha}=1$ then $\gamma ua+\delta p^{r+\alpha}a=a$ and $\gamma(ua+vp^{t-s}b)=a+\gamma vp^{t-s}b-\delta p^{r+\alpha}a$ is also of p-height in G greater or equal to $r+\alpha$. Therefore

$$h_p(a+vp^{t-s}b)\geqq r+\alpha \tag{1}$$

however $s+\alpha=t+\beta$ and thus $t-s=\alpha-\beta$.

Let $y=p^{\alpha-\beta}b$ and since $h_p^G(y)=\alpha$, we let $n\equiv\gamma v(p^r)$ such that $0\leqq n<p^r$ and obtain that $h_p(a+ny)\geqq r+\alpha$.

For an ordered pair $(a, b)\in G\times G$ we consider in the notations used above the following set:

$$I_p(a, b)=\{(n, r)\,|\,h_p^G(a+ny)\geqq r+\alpha \quad \text{and} \quad 0\leqq n<p^r,\ r\in \boldsymbol{Z}^+\}\,.$$

This set has some interesting properties which are listed in the following:

PROPOSITION 1.2. *Let $I=I_p(a, b)$ then:*

(i) $(0, 0)\in I$,

(ii) (n, r) *and* $(m, r)\in I\Rightarrow n=m$,

(iii) $(n, r)\in I$ *and* $r>1\Rightarrow \exists m\in \boldsymbol{Z}^+$ *such that* $(m, r-1)\in I$.

Proof. (i) is obvious.

(ii) If $r=0$ or $\alpha=\infty$ there is nothing to prove in as much as m and n must be zero. However if $r>0$ and $\alpha<\infty$ then $h_p(ny)=\alpha$ and $ny-my=(n-m)y=(a+ny)-(a+my)$ is of p-height greater or equal to $r+\alpha$. Now since $h_p(y)=\alpha$ we must have $n\equiv m(p^r)$ and since $0\leqq n, m<p^r$, we have $n=m$.

(iii) Here again we need only consider the case where $h_p(a)=\alpha<\infty$. Let $h_p(a+ny)\geqq r+\alpha<\infty$ and $r>1$. Dividing n by p^{r-1} we have $n=kp^{r-1}+m$ where $0\leqq m<p^{r-1}$ and $k\in \boldsymbol{Z}^+$. Then, $a+ny=a+my+kp^{r-1}y$ and since $h_p(kp^{r-1}y)\geqq r-1+\alpha$ we must also have $h_p(a+my)\geqq r-1+\alpha$. Clearly $(m, r-1)\in I$.

In view of the preceeding proposition, we see that if we write $n_i=n$ if $(n, i)\in I_p(a, b)$ we obtain a sequence of non-negative integers with the following properties:

LEMMA 1.3. *Let $I=I_p(a, b)$ and let $l=l_p(a, b)=\sup\{r\,|\,(n, r)\in I\}$ and write $n_i=n$ if $(n, i)\in I$ then $n_0=0$ and*

$$n_{i+1}=n_i+s_ip^i \quad where \quad 0\leqq s_i<p \quad for\ all \quad i<l\ .$$

Proof. From proposition 1.2 (ii) the n_i's are well defined and by (i) $n_0=0$. Again by (ii) there is an n_i for all $0<i<l+1$ (as usual if $l=\infty$ we let $\infty+1=\infty$). Clearly $n_{i+1}\equiv n_i(p^i)$ therefore $n_{i+1}=n_i+s_ip^i$ and we need only show that $0\leqq s_i<p$. Let $s_i=kp+r$ where $0\leqq r<p$, then $0\leqq n_i+rp^i<p^{i+1}$ and $h_p(a+(n_i+rp^i)y)\geqq i+1+\alpha$, $(a+n_{i+1}y=(a+(n_i+rp^i)y)+kp^{i+1}y)$ therefore $n_i+rp^i=n_{i+1}$ and $k=0$ thus $0\leqq s_i=r<p$.

DEFINITION 1.4. Let $a, b\in G$ be as in the preceeding development. The sequence $\{n_i\}_{i=0}^l$ described in lemma 1.3 converges to a p-adic number $\sum_{i=0}^l s_ip^i$ in the p-adic completion J_p of $\boldsymbol{Z}$.

We set:

$$\eta_p=\eta_p(a, b)=p^{\alpha-\beta}\sum_{i=0}^{l} s_ip^i=\lim p^{\alpha-\beta}n_i\in K_p$$

$$M_p=M_p(a, b)=\alpha+l_p(a, b)\in \boldsymbol{Z}^+\cup\{\infty\}$$

$$\beta_p=h_p^G(b)\ .$$

Note that if either α or β is infinite we take $\eta_p=0$. K_p is the field of quotients of J_p. The triple (η_p, M_p, β_p) is called the p-indicator in G of the pair (a, b). When p is fixed we will drop the indices in such expressions. The p-indicator contains a good amount of informntion about the way the elements a, b sit in the group G. We describe this more precisely in the next section.

2. The structure of the p-pure enveloppe of $\{a, b\}$

Let $a, b\in G$, we denote by $\langle a, b\rangle_p$ the p-pure subgroup of G generated by $\langle a, b\rangle$. Note that $\langle a, b\rangle_p/\langle a, b\rangle$ is simply the p-primary part of $G/\langle a, b\rangle$, thus if rank $(G)=2$, $G/\langle a, b\rangle=\bigoplus_{p\in P}\langle a, b\rangle_p/\langle a, b\rangle$. This last equation implies that the knowledge of $\langle a, b\rangle_p$ is useful in the study of rank two torsion free groups. We proceed to the description of the generators of $\langle a, b\rangle_p$ in the following:

LEMMA 2.1. *Let $a, b\in G$, n_i, s_i, l be as in lemma* 1.3. *Let*

$$x_i=p^{-i-\alpha}(a+n_ip^{\alpha-\beta}b)\ , \qquad 0\leqq i<l+1 \quad \text{then :}$$

$$\langle a, b\rangle_p=\langle\{\{x_i\}_{i=0}^{i=l}, p^{-\beta}b\}\rangle\ . \tag{1}$$

Moreover $x_i=px_{i+1}-s_ip^{-\beta}b$ *for* $0\leqq i<l$.

Proof. We recall the following notation: $\langle p^{-\infty}x\rangle=\langle\{p^{-i}x\}_{i=0}^{\infty}\rangle$. This way of writing allows us to use the same formula even when either α or β is infinite and a quick check shows that in that case 1 is true. We assume then α, β finite and let $p^{\alpha-\beta}b=y$ then:

$$\begin{aligned} px_{i+1}-s_ip^{-\beta}b &= p^{-i-\alpha}(a+n_{i+1}y)-p^{-i-\alpha}(s_ip^iy) \\ &= p^{-i-\alpha}(a+(n_{i+1}-sp^i)y) \\ &= p^{-i-\alpha}(a+n_iy)=x_i \end{aligned}$$

for all $0\leqq i<l$.

Now let $H_i=\langle x_i, p^{-\beta}b\rangle$ then $H_i\subset H_{i+1}$ and the right hand side of (1) can be written as $H=\bigcup_{i=0}^{l} H_i$. Clearly $a, b\in H$, and in fact $H\subset \langle a, b\rangle_p$. We need only show that H is p-pure in G. Let $g\in G$ be such that $pg\in H$ then there exists i such that $pg\in H_i$. Say $pg=nx_i+mp^{-\beta}b$.

After multiplying this equality by $p^{\alpha+i}$ and replacing x_i by its expression in terms of a and b we obtain:

$$p^{\alpha+i+1}g=na+nn_iy+mp^iy=na+(nn_i+mp^i)y\ .$$

Now, using lemma 1.1, there exists n_{i+1} such that

$$h_p(a+n_{i+1}y)\geqq\alpha+i+1$$

therefore $i<l$, that is to say x_{i+1} exists and we can write

$$pg=npx_{i+1}+(m-ns_i)p^{-\beta}b\ .$$

But $h_p(p^{-\beta}b)=0$ therefore p divides $m-ns_i$ say $pk=m-ns_i$ then, $g=nx_{i+1}+kp^{-\beta}b\in H_{i+1}\subset H$ and thus H is p-pure.

We are now ready to study the structure of $\langle a, b\rangle_p$. We do this by looking at the p-indicator (η, M, β) of (a, b) in G. There are two main cases according to wether η is rational or not. We begin by the case $n\notin Q$.

THEOREM 2.2. *Let (η, M, β) be the p-indicator of the paire (a, b) where $a, b\in G$ are independant elements. If $\eta\notin Q$ then all non-zero endomorphisms of $H=\langle a, b\rangle_p$ are monomorphisms and*

(1) *If η is not quadratic over Q then H is rigid and $E(H)\simeq Z$.*

(2) *If η is quadratic over Q then $r(E(H))=2$ and $E(H)$ is a commutative domain.*

Proof. Since $\eta\notin Q$, α and β are necessarily both finite and $M=\infty$. Let $f\in E(H)$, $f\neq0$, and D a divisible enveloppe of H. Then f extends naturally to D and since $\{a, b\}$ is a vector basis of D as a space over Q, f is completely determined by its values $f(a)$ and $f(b)$. Let

(1) $\quad f(a)=A_1a+A_2b$ and $f(b)=B_1a+B_2b,\ A_i,\ B_i\in Q,\quad i=1, 2\ .$

Recall that $\ker f\neq0$ if and only if $A_1B_2-A_2B_1=0$. Without loss of generality we may assume that A_i and B_i are in $p^{2(\alpha+\beta)}Z$. Recall also that

$$x_i=p^{-i-\alpha}(a+n_ip^{\alpha-\beta}b)\ ,\quad \text{(see Lemma 2.1)}\ .$$

Then we have:

$$f(x_i)-(A_1+n_ip^{\alpha-\beta}B_1)x_i=p^{-i-\alpha}[A_2+p^{\alpha-\beta}n_i(B_2-A_1)-(p^{\alpha-\beta}n_i)^2B_1]b \ .$$

The number between square brackets in the right hand side of this equation is an integer m_i such that $h_p^G(m_ib)\geqq i+\alpha$ and since $h_p^G=\beta$, m_i must be a multiple of $p^{i+\alpha-\beta}$ for all i. Thus, taking limits in K_P we find that

$$(2) \qquad A_2+\eta(B_2-A_1)-\eta^2B_1=0 \ .$$

Note that every endomorphisme of H gives rise to an equation such as (2) even when $\eta\in Q$ provided α, β are finite and $M=\infty$. Now if η is not quadratic over Q, we must necessarily have $B_1=0$ and since $\eta\notin Q$, $B_2-A_1=0$ and $A_2=0$, this means that f is simply a multiplication by A_1. Therefore H is rigid and every non-zero endomorphism is a monomorphism. Further more the only possible multiplications are by integers since H contains elements of p-height 0.

Therefore $E(H)\simeq \boldsymbol{Z}$.

Now, if $B_1\neq 0$ since $\eta\notin Q$, $\ker f=0$ for otherwise we have $A_1B_2-A_2B_1=0$ and (2) gives $\eta=((B_2-A_1)\pm(B_2+A_1))/2B_1\in Q$, which is a contradiction. Therefore f is a monomorphism in all cases.

It remains to show that if η satisfies an equation of degree two with integral coefficients and $\eta\notin Q$ that $E(H)$ is of rank two. Suppose η satisfies

$$(3) \qquad C_0+C_1\eta+C_2\eta^2=0 \quad \text{with} \quad C_2\neq 0 \ .$$

Without loss of generality we may assume that the coefficients are in $p^{2(\alpha+\beta)}Z$. Let $C_2=-B_1$, $C_1=B_2$, $C_0=A_2$ and $A_1=0$ then (3) assumes the same form as (2). We define an endomorphism f of D by the formula

$$f(a)=A_2b \text{ and } f(b)=B_1a+B_2b \ .$$

Then, a straight forward computation shows that $f(x_i)\in H$ and thus f applies H into itself. This f is not a multiplication since $B_1\neq 0$ and we infer that rank $(E(H))\geqq 2$. Now if we set up the correspondence $\theta(f)=(A_1, A_2)$ where A_1, A_2 are as in equation (1), we obtain a homomorphism θ between $E(H)$ and $Q\oplus Q$ and $\theta(f)=0$ implies $\ker f\neq 0$ and $f=0$. Therefore θ is a monomorphism and rank $(E(H))=2$. It follows then that $E(H)$ is a commutative domain.

An immediate consequence of this theorem is that if $\eta\notin Q$, $\langle a, b\rangle_p$ is indecomposable. The converse is also true, however we need first to study the situation where $\eta\in Q$ before we can prove this.

LEMMA 2.3. *If $\eta\notin Q$, then $\langle a, b\rangle_p$ does not contain any non-zero element of infinite p-height.*

Proof. It suffices to consider elements of the form $x=ua+vb$ where $u, v\in \boldsymbol{Z}$. Let $\eta=\eta_p(a, b)\notin Q$. We will compute explicitly the p-height of x in G. Now, we have:

$$x=u(a+n_ip^{\alpha-\beta}b)+(v-n_ip^{\alpha-\beta}u)b \quad \text{for all} \quad i<\infty$$

and $h_p((v-n_ip^{\alpha-\beta}u)b)=\beta+v_p(v-n_ip^{\alpha-\beta}u)$. Also, there exists j such that $v_p(v-n_ip^{\alpha-\beta}u)=v_p(v-\eta u)$ for all $i\geqq j$. Since $\eta\notin Q$, $v-\eta u\neq 0$ and thus $v_p(v-\eta u)$ is finite. However, $h_p(u(a+n_ip^{\alpha-\beta}b))$ is an unbounded function of i since $M=\infty$. Therefore, $h_p(x)=\beta+v_p(v-\eta u)$ and since $\eta\notin Q$, $h_p(b)=\beta$ is finite and $h_p(x)$ is finite.

PROPOSITION 2.4. *$\langle a, b\rangle_p$ contains a non-zero element of infinite p-height if and only if $\eta\in Q$ and either M or β is infinite. Furthermore:*

(a) *if $\beta=\infty$ then $\langle a, b\rangle_p=\langle p^{-\alpha}a\rangle\oplus\langle p^{-\beta}b\rangle$,*

(b) *if $\beta<\infty$, and $\eta=\gamma/\delta$ where $\delta\in \boldsymbol{Z}$ and $\gamma\in Q^p$ and $(\gamma, \delta)=1=(\delta, p)$ then $h_p(\delta a+\gamma b)=\infty$ and $\langle a, b\rangle_p=\langle p^{-\infty}(\delta a+\gamma b)\rangle\oplus\langle p^{-m}(\tau a-\sigma b)\rangle$ where $\tau, \sigma\in \boldsymbol{Z}$, $\tau\gamma+\sigma\delta=1$ and $m=h_p(\tau a-\sigma b)<\infty$.*

Proof. If $\beta=\infty$ then $\eta=0$, $M=0$ and $\langle a, b\rangle_p=\langle x_0\rangle\oplus\langle p^{-\beta}b\rangle$ and $x_0=p^{-\alpha}a$. Suppose then that $\beta<\infty$. In this case, using the notation of (b) above, if $\alpha=\infty$ let $\gamma=0$, $\delta=1$, $\sigma=1$ then $\langle a, b\rangle_p=\langle p^{-\infty}a\rangle\oplus\langle p^{-\beta}b\rangle$ and conforms to the formula.

We may therefore consider $\alpha<\infty$ and $M=\infty$. Now $v_p(\gamma)=v_p(\eta)=\alpha-\beta\geqq-\beta$ and therefore $\gamma b\in\langle a, b\rangle_p$. We show that $h_p(\delta a+\gamma b)=\infty$. Indeed:

$$\delta a+\gamma b-\delta(a+n_ip^{\alpha-\beta}b)=(\gamma-\delta n_ip^{\alpha-\beta})b\ .$$

But $\eta-n_ip^{\alpha-\beta}$ is divisible by $i+\alpha-\beta$ and then so is $\gamma-\delta n_ip^{\alpha-\beta}$. Therefore, $h_p(\gamma-\delta n_ip^{\alpha-\beta})b\geqq i+\alpha$ in one hand. In the other hand $h_p(a+n_ip^{\alpha-\beta}b)\geqq i+\alpha$ hence $h_p(\delta a+\gamma b)\geqq i+\alpha$ for all $i<M$ but M is infinite therefore $h_p(\delta a+\gamma b)=\infty$. Now $\langle p^{-\infty}(\delta a+\gamma b)\rangle$ is p-divisible and $\langle p^{-m}(\tau a-\sigma b)\rangle$ is p-pure therefore $\langle p^{-\infty}(\delta a+\gamma b)\rangle\oplus\langle p^{-m}(\tau a-\sigma b)\rangle$ is p-pure and since it contains a and b it is equal to $\langle a, b\rangle_p$.

Conversely if $\langle a, b\rangle_p$ contains a non-zero element of infinite p-height then by lemma 2.3 $\eta\in Q$ and if $\beta<\infty$, $M=\infty$, for, if not then $\langle a, b\rangle_p=\langle x_l\rangle\oplus\langle p^{-\beta}b\rangle$, is free and hence contains no non-zero element of infinite p-height.

We gather in the next proposition some remarks about the endomorphism ring of $\langle a, b\rangle_p$.

PROPOSITION 2.6. *Let $r=\operatorname{rank}(E(\langle a, b\rangle_p))$. Then if $\langle a, b\rangle_p$ is not p-divisible,*

$r=1$ *if and only if $n\notin Q$ and is not quadratic over Q ,*

$r=2$ *if and only if $\eta\notin Q$ and is quadratic over Q ,*

$r=3$ *if and only if $\eta\in Q$ and either M* or *β is infinite ,*

$r=4$ *if and only if both* M, β *are finite.*

The chart below summarizes the results that we have obtained.

We conclude with an explicit formula for computing p-heights of linear combinations of a and b.

THEOREM 2.7. *Let* (η, M, β) *be the p-indicator of* (a, b) *in* G *and let* $x=ua+vb \in G$, u, $v \in Q$, *then*

$$h_p^G(ua+vb)=\min\{v_p(u)+M, \beta+v_p(v-\eta u)\} .$$

Proof. If $n \notin Q$ then $M=\infty$ and the result follows from the proof of Lemma 2.3. If $\eta \in Q$, it can easily be seen that $\langle a, b\rangle_p=\langle p^{-M}(\delta a+\gamma b)\rangle \oplus \langle p^{-\beta}b\rangle$ in all cases and writing $\delta x=\delta ua+\delta vb=u(\delta a+\gamma b)+(\delta v-u\gamma)b$. These terms are respectively in the factors of the direct decomposition of $\langle a, b\rangle_p$ given above, thus $h_p(\delta x)=\min\{h_p(u(\delta a+\gamma b)), h_p((\delta v-u\gamma)b)\}$ and upon dividing by δ we obtain the desired formula.

The indicators of a pair have been shown here to be of some usefullness. They can be used to provide invariants for rank two torsion free groups in a similar way to the so called 2-characteristics introduced in [2]. We reseve for a subsequent article further applications of these concepts. In particular we have used them to obtain classes of indecomposable groups of all ranks $\leqq \aleph_0$ with special properties to be published subsequently.

	η	M	β	α	$\langle a, b\rangle_p$	structure	$E(\langle a, b\rangle_p)$
$\eta \notin Q$	quadratic	(∞)	(finite)	(finite)	$\langle\{x_i, p^{-\beta}b\}_{i=1}^{\infty}\rangle$	Strongly inde-composable	rank 2 (commutative)
	non quadratic						$\cong Z$
$\eta \in Q$	(0)	∞	∞	(∞)	$\langle p^{-\infty}a\rangle \oplus \langle p^{-\infty}b\rangle$	$\cong Q^{(p)} \oplus Q^{(p)}$	$\cong Q^{(p)} \oplus Q^{(p)} \oplus Q^{(p)} \oplus Q^{(p)}$
			finite		$\langle p^{-\infty}(\delta a+\gamma b)\rangle + \langle p^{-m}(\tau a-\tau b)\rangle$ where $m<\infty$*	$\cong Q^{(p)} \oplus Z$	rank 3 additive structure $\cong Q^{(p)} \oplus Q^{(p)} \oplus Z$
	(0)	(finite)	∞	(finite)	$\langle p^{-\alpha}a\rangle \oplus \langle p^{-\infty}b\rangle$		
	$(\varepsilon p^{-M}Z)$		finite	(finite)	$\langle p^{-M}(a+\eta b)\rangle \oplus \langle p^{-\beta}b\rangle$	free	$Z \oplus Z \oplus Z \oplus Z$

* See 2.4 (b).

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