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Finite element methods for nonlinear variational inequalities

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Abstract

For piecewise linear approximation of the obstacle problem involving nonlinear variational inequalities, we prove that the error estimate for $u-u_h$ in the energy norm is of order h, i.e.,

$$||u-u_h||_{1,D}=0(h) .$$

1. Introduction

Variational inequalities play a very important part in studying the obstacle and unilateral problems arising in fluid dynamics, control theory, continuum mechanics and elasticity. Recently finite element methods are being applied for obtaining numerical solutions of the variational inequalities. For obstacle problems involving linear variational inequalities introduced and studied by Lions and Stampacchia [9], using piecewise linear elements, Falk [7], Mosco and Strang [11], and Brezzi, Hager and Raviart [2] have shown the $0(h)$ convergence.

In this paper, we analyze the error in the finite element approximation to *nonlinear variational inequalities*. After deriving a general error estimate for error in the approximation, we apply our estimate to a model nonlinear problem with homogeneous boundary conditions arising in the study of permanent compressible fluid. For this obstacle problem, using linear trial functions, we prove that the error estimate in the energy norm is $0(h)$.

To be more precise, let H be a real Hilbert space with its dual H'. The norm and inner product in H are denoted by $||\cdot||$ and $((\cdot, \cdot))$ respectively. Let $(\cdot, \cdot)$ be the pairing between H' and H. Let M be a closed convex subset of H and f be a given element of H'.

We consider the following problem:

$$(P) \qquad \begin{cases} \text{Find } u \in M \text{ such that} \\ (Tu, v-u) \geqq (f, v-u) , \quad \forall v \in M , \end{cases}$$

where T is a nonlinear operator.

This problem is due to Browder [4]. Sibony [13] has shown that for a real valued functional F, the minimum of a differentiable nonlinear functional

$$I[v]=F(v)-2(f, v)\ , \quad \forall v \in M$$

on a convex subset M of H can be characterized by the variational inequalitiy considered in (P) with $Tu=F'(u)$, the Fréchet derivative of F at u.

First of all, we define some notions.

DEFINITION The operator $T: H \rightarrow H'$ is said to be

(i) *Strongly monotone*, if there exists a constant $\alpha>0$ such that

$$(Tu-Tv, u-v) \geqslant \alpha \|u-v\|^2 \quad \forall u, v \in H\ .$$

(ii) *Lipschitz continuous*, if there exists a constant $\beta>0$ such that

$$\|Tu-Tv\| \leqslant \beta\|u-v\|\ , \quad \forall u, v \in H\ ,$$

In particular, it follows that $\alpha \leqslant \beta$, see Noor [12]. It is well known that that there exists a unique solution of (P), if T is both strongly monotone and Lipshitz continuous, see [4].

For $M=H$, tne problem (P) is equivalent to finding $u \in H$ such that

$$(Tu, v)=(f, v)\ , \quad \forall v \in H\ ,$$

a case studied by Browder [4].

2. Approximation Scheme and General Error Estimate

Let S_h be a finite dimensional subspace of H with basis $\{\varphi_i\}_{i=1}^{n}$. In addition, we also construct the approximate finite dimensional closed convex subset M_h of M. The construction of the closed convex subset should be subject to the following criteria.

(i) M_h should be a "good approximation" to M, see Mosco [10].

(ii) The approximate problem should be "easy" to solve.

In order to apply the finite element method, we define the discrete form of problem (P) as:

$$(P_h) \qquad \begin{cases} \text{Find } u_h \in M_h \text{ such that} \\ (Tu_h, v_h-u_h) \geqslant (f, v_h-u_h)\ , \quad \forall v_h \in M_h \end{cases}$$

Note that we do not require $M_h \subset M$. It is sufficient to consider that both M and M_h are in H. It is well known [14, 15] that for $Tu=f$, the error estimate depends on the inequality

$$\|u-u_h\| \leqslant \text{Const. dis}(u, S_h)\ .$$

But in general, this is not true for variational inequalities, see Mosco and Strang [11]. Furthermore, suppose that V is a Hilbert space dense in H' and the injection of V into H' is continuous. Then there exists a continuous injection i of H into V' such that $i(H)$ is dense in V' and

$$(i(v), w)_{V,V'} = (v, w)_{H,H'}, \quad \forall v \in H, w \in V'.$$

We identify H with a subspace of V', dense in V' with a continuous injection map. We denote by $(\cdot, \cdot)$, the pairing between V' and V.

Our main aim in this section is to derive the general error estimate for $u-u_h$.

THEOREM 1. *Let $u \in M$ and $u_h \in M_h$ be the solutions of (P) and (P_h), and T, a strongly monotone Lipschitz continuous operator. If f-$Tu \in V$, then*

(a) $$\|u-u_h\|_H^2 \leqslant \frac{\beta^2}{\alpha^2}\|u-v_h\|_H^2 + \frac{2}{\alpha}\|f-Tu\|_V(\|u-v_h\|_{V'} + \|u_h - v\|_{V'}),$$

for all $v \in M$ and $v_h \in M_h$.

Proof. Since u and u_n are solutions of (P) and (P_h), then

$$(Tu, v-u) \geqslant (f, v-u), \qquad \text{for all } v \in M$$

and

$$(Tu_h, v_h-u_h) \geqslant (f, v_h-u_h) \qquad \text{for all } v_h \in M_h.$$

Adding these inequalities and rearranging the terms, we get

$$(Tu, u) + (Tu_h, u_h) \leqslant (f, u-v_h) + (f, u_h - v) + (Tu, v) + (Tu_h, v_h).$$

Subtracting from both sides $(Tu, u_h) + (Tu_h, u)$, we obtain

$$\begin{aligned}(Tu-Tu_h, u-u_h) &\leqslant (f, u-v_h) + (f, u_h-v) + (Tu, v-u_h) + (Tu_h, v_h-u) \\ &= (f-Tu, u-v_h) + (f-Tu, u_h-v) + (Tu-Tu_h, u-v_h)\end{aligned}$$

Since T is a strongly monotone and Lipschitz continuous operator, it follows by using the Cauchy-Schwarz inequality that

$$\alpha\|u-u_h\|^2 \leqslant \|f-Tu\|_V(\|u-v_h\|_{V'} + \|u_h-v\|_{V'}) + \beta\|u-u_h\|_H\|u-v_h\|_H.$$

Using the inequality

$$ab \leqslant \frac{1}{2\varepsilon}a^2 + \frac{1}{2\varepsilon}b^2$$

for positive a, b and $\varepsilon > 0$, we get

$$\beta\|u-u_h\|_H\|u-v_h\|_H \leqslant \frac{\alpha}{2}\|u-u_h\|_H^2 + \frac{\beta^2}{\alpha^2}\frac{1}{2}\|u-v_h\|_H^2$$

Thus by using this inequality, we obtain (a), the required result.

Remark. Note that for $M=H$, the last two terms in (a) drop out, and we have

$$||u-u_h|| \leqslant \frac{\beta}{\alpha}||u-v_h|| , \quad \text{for all } v_h \in S_h ,$$

a well known result Zlamal [15], and Strang and Fix [14]. The inequality (a) for $u-u_h$ is a generalization of a result of Falk [7]. Theorem 1 enables us to consider a class of nonlinear variational inequalities associated with nonlinear elliptic boundary value problems satisfying some extra constrained conditions.

3. Applications

We consider the following problem:

$$(1) \qquad \begin{cases} \left\{f - \sum_{i=1}^{2} \frac{\partial}{\partial x^i}\left[\frac{1}{q}\,\frac{\partial(u+\varphi)}{\partial x^i}\right]\right\}(u-\psi)=0 , & \text{on } D \\ u \geqslant \psi \text{ on } D \text{ and } u=0 \text{ on } E , \end{cases}$$

where $D \subset R^2$ is a bounded convex domain with baundary E, $f \in L_2(D)$ and ψ is a given function in $H_0^1(D) \cap H^2(D)$.

This problem occurs in the study of permanent compressible irrotational flow. Here $X=u+\varphi$ is the stream function of the motion. It is assumed that the flow is subsonic and $1/q$ is given by (see Fig 1)

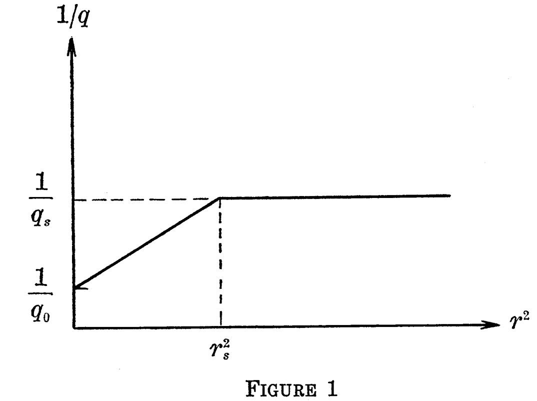

FIGURE 1

$$\frac{1}{q} = \begin{cases} 1/q_0 + \frac{r^2}{r_s^2}(1/q_s - 1/q_0) , & \text{if } 0 < r^2 < r_s^2 \\ 1/q_s , & \text{if } r_s^2 \leqslant r^2 \end{cases}$$

where $r^2 = |\text{grad}\, X|^2$, r/q is the scalar speed and r_s/q_s is the speed of the sound, for more details, see [8].

The space $W_2^k(D) = H^k$ is taken to be the usual Sobolev space, where for $u \in H^k$, k a positive integer, we have the norm

$$||u||_{k,D}^2 \equiv \sum_{|\alpha| \leq k} ||D^\alpha u||_{L_2(D)}^2$$

The space of functions from H^k, which in a generalized sense satisfies the homogeneaus boundary conditions on E is denoted by H_0^k.

The problem (1) can be treated via an variational inequality. To do this, a convex subset M is defined so that

$$M \equiv \{v: v \in H_0^1;\ v \geqslant \psi \quad \text{on} \quad D\}\ .$$

The variational inequality is:

$$(P) \qquad \begin{cases} \text{Find } u \in M \text{ such that} \\ (Tu, v-u) \geqslant (f, v-u)\ , \quad \text{for all} \quad v \in M\ , \end{cases}$$

where

$$(Tu, v) = \sum_{i=1}^{2} \frac{1}{q} \int_D \frac{1}{q} \frac{\partial(u+\varphi)}{\partial x^i} \frac{\partial v}{\partial x^i} dD\ , \quad \text{for all} \quad u, v \in H_0^1\ .$$

The operator T defined by the above relation is nonlinear. In fact, it has been shown by Froideraux [8] that T is strongly monotone and Lipschitz continuous, i.e.,

$$(Tu - Tv, u-v) \geqslant \alpha ||u-v||^2\ , \quad \text{for all} \quad u, v \in H_0^1\ ,$$

and

$$||Tu - Tv|| \leqslant \beta ||u-v||\ , \qquad \text{for all} \quad u, v \in H_0^1\ ,$$

with $\alpha = 1/q_0$ and $\beta = 3/q_s - 2/q_0$. Thus there does exist a unique solution of (P).

We consider the special case, when problem (P) is defined in the following setting. Let $H = H_0^1$, $H' = H^{-1}$, and $V = V' = L_2(D)$.

Let $\{T_h\}_{h>0}$ be a regular family, see [6] of triangulation of D and define

$$S_h \equiv \{v_h: v_h \in C(D),\ v_h|_T \in P_1 \text{ for all } T \in T_h,\ v_h|_T = 0\}\ ,$$

where P_1 is the set of all polynomial on $\boldsymbol{R}^2$ of degree $\leqslant 1$. Clearly S_h is a finite dimensional subspace of H_0^1. Finally, the convex set M_h is defined as:

$$M_h = \{v_h \in S_h: v_h \geqslant \psi \text{ at every vertex of each triangle belonging to } T_h\}\ .$$

It is obvious that M_h is a closed convex subset of H_0^1. In order to apply the finite element method, we define the discrete form of problem (P) as:

$$(P_h) \qquad \begin{cases} \text{Find } u_h \in M_h \text{ such that} \\ (Tu_h, v_h - u_h) \geqslant (f, v_h - u_h), \text{ for all } v_h \in M_h\ . \end{cases}$$

In this setting, the general error estimate (a) becomes:

(b) $$\|u-u_h\|_{1,D}^2 \leq \frac{\beta^2}{\alpha^2}\|u-v_h\|_{1,D}^2 + \frac{2}{\alpha}\|f-Tu\|_{L_2(D)}(\|u-v_h\|_{L_2(D)} + \|u_h - v\|_{L_2(D)}), \quad \text{for all} \quad v \in M, v_h \in M_h .$$

We formulate the following hypothesis about the regularity of $u \in M$.

(A) {For $f \in L_2(D)$, $\psi \in H_0^1 \cap H^2$, $u \in M$ satisfying (P) also lies in H^2 .

We need the following results, which can be found in Falk [7].

LEMMA 1. *Let v_h be the unique element in S_h such that $u=v_h$ at all vertices of each triangle T_h. Then $v_h \in M_h$ and*

$$\|u-v_h\|_{L_2(D)} \leq C_1 h^2 \|u\|_{2,D}$$
$$\|u-v_h\|_{1,D} \leq C_2 h \|u\|_{2,D}$$

where C_1 and C_2 are constants independent of h and u.

LEMMA 2. *Let u_h be the solution of problem (P_h) and $v=\sup\{u_h, \psi\}$. Then $v \in M$ and*

$$\|u_h - v\|_{L_2(D)} \leq C_3 h^2 ,$$

for some constant C_3 independent of u_h and h.

From (b), Lemma 1 and Lemma 2, we obtain the main result of this paper, which gives an error estimate for $u-u_h$ under the hypothesis (A). It is noted that Froideraux's result of the monotony and Lipschitz continuity of T plays a crucial role in this result.

THEOREM 2. *If $u \in M$ and $u_h \in M_h$ are the solutions of problems (P) and (P_h) and assume that the hypothesis (A) holds, then*

$$\|u-u_h\|_{1,D} = 0(h) .$$

Remark. For the linear variational inequalities involving the obstacle problems, using the quadratic elements, Brezzi and Sacchi [3], and Brezzi et al [2] have proved the $0(h^{3/2-\epsilon})$ convergence. We here conjecture that for the nonlinear variational inequalities, the error estimate for $u-u_h$ would be $0(h^{3/2})$ as in the linear case.

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