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A progression of consistency proofs

by

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We have formulated in [8] a theory of self-iterating schemes of ordinal diagrams $\boldsymbol{Od}(\Omega)$, whose formal accessibility proof has been attempted in [9]. It was carried out in the system $\boldsymbol{ASU}$, a π_1^1-arithmetic with a specified uniform inductive definition ($\boldsymbol{UID}$). The so-called "consistency proof", however, is not feasible for this system as it is. Hence, in order to relate the theory $\boldsymbol{Od}(\Omega)$ to second order arithmetic, a more delicate analysis of the accessibility proof in [9] is required. As a result of such endeavor, we here present a "progression" of formal systems formulated uniformly in the elements of $\Omega+1$ which admits a formal accessibility proof of $\boldsymbol{Od}(\Omega)$. We then proceed to a "progression" of consistency proofs of those systems by the help of some well-ordered structures based on the theory of ordinal diagrams.

Such an approach should be a matter of dispute from the constructive standpoint. It can nevertheless be justified, considering the importance of syntactical normalization proofs for formal systems.

We shall employ much of the symbolism and technicalities in [8] and [9], sometimes without quotations.

Chapter I

A progression of π_1^1-systems with certain inductive definitions.

Let Ω be a primitive recursive set with a primitive recursive well-order $<_\Omega$. We define a progression of systems $\mathfrak{S}_r$ uniformly in r to obtain $\mathfrak{S}_\Omega$ as a sort of limit of $\mathfrak{S}_r$'s, and carry out an accessibility proof of $\boldsymbol{Od}(\Omega)$ in a special case of $\mathfrak{S}_\Omega$. For a purely technical reason, we assume Ω starts with 1; we preserve 0 as an additional element.

§1. Systems

Definition 1.1. We assume a formulation of first order arithmetic which is convenient to formulate the structure of Ω and various elementary properties coming into considerations henceforth. Let $\boldsymbol{JNN}$ be the second order system of arithmetic with the isolated comprehension and the isolated induction based on such a first order system. (See §27 of [4] for "isolated" formulas. $\boldsymbol{JNN}$ is essentially $\boldsymbol{SJNN}$ in [2].)

We expand the language by adding H, a new predicate symbol with three first order arguments. Formulas of the language thus expanded are defined as usual.

Definition 1.2. 1) We first quote Definition 1.1 in [9].

Let Ω be a set as described at the outset of this section. Let $\boldsymbol{O}$ and $\boldsymbol{S}$ be primitive recursive sets of pairs whose first entries are elements of Ω and which determine primitive recursive linear orders $<^r$ and $<_r$ respectively (uniformly in r) for the second entries of the pairs belonging to $\boldsymbol{O}$ and $\boldsymbol{S}$ respectively whose first entries are r. The requisites for $\boldsymbol{O}$ and $\boldsymbol{S}$ are as follows.

$$\{a; (s, a) \in \boldsymbol{O}\} \subset \{a; (r, a) \in \boldsymbol{O}\} \quad \text{if} \quad s <_\Omega r .$$

$$\{i; (r, i) \in \boldsymbol{S}\} = \bigcup_{s <_\Omega r} \{a; (s, a) \in \boldsymbol{O}\} .$$

$<^s$ forms an initial segment of $<^r$ if $s <_\Omega r$.

$<_r = \bigcup_{s <_\Omega r} <^s$; $<_1 = <^0$ where $<^0$ is a JNN-provable well-order.

The second entry of any pair in $\boldsymbol{O}$ (hence in $\boldsymbol{S}$) has a successor.

Let us assume that Ω, $\boldsymbol{O}$ and $\boldsymbol{S}$ can be expressed in the first order part of $\boldsymbol{JNN}$, and we shall use the same letters Ω, O and S to denote the formal representations of those sets. Let $<(\Omega; s, r)$ be the formal representations of $s <_\Omega r$, let $<(\Omega; j, i)$ be the formal representation of $j <_r i$ and let $\ll(r; a, b)$ represent $a <^r b$.

2) Le I be a set with a linear order $<^*$. $I^\sim$, $I_* = I \cup I^\sim$ and the linear order $<_*$ are defined as in Definition 28.4 of [4]. For the reader's convenience, we repeat the definition here. Let p and q denote elements of I.

$$I^\sim = \{p^\sim; p \in I\} .$$

$p <_* p^\sim$; $p <^* q$ implies $p <_* q$, $p^\sim <_* q^\sim$, $p^\sim <_* q$ and $p <_* q^\sim$.

$I_\infty = I_* \cup \{\infty\}$ and $<_\infty$ is the order of I_∞ induced from $<_*$ regarding ∞ as maximal.

3) Let S_r denote $\{i; (r, i) \in S\}$, and let T denote the set $\bigcup_{r \in \Omega} S_r$. Notice that $\{S_r\}_{r \in \Omega}$ is an increasing sequence of sets and that $<_s$ forms an initial segment of $<_r$ if $s <_\Omega r$. Thus, a linear order of T can be induced from the orders $<_r$'s, which we write as $<_T$.

4) We can also induce a linear order $<_S^*$ to S by $(s, j) <_S^* (r, i)$ if $s <_\Omega r$, or $s = r$ and $j <_r i$. We can define $S^\sim$, S_*, $<_{S,*}$, S_∞ and $<_{S,\infty}$ as in 2) above. For a notational reason, those will be respectively called I_Ω, $I_\Omega^\sim$, $I_{\Omega,*}$, $<_{\Omega,*}$, $I_{\Omega,\infty}$ and $<_{\Omega,\infty}$.

Let I_r denote the set $\{(s, j); s \leqq_\Omega r \text{ and } (s, j) \in S\}$. $<_r^*$ is the order $<_S^*$ restricted to I_r. $I_r^\sim$, $I_{r,*}$, $<_{r,*}$, $I_{r,\infty}$ and $<_{r,\infty}$ can also be defined as in 2).

Remark. Note that the uniform accessibility of $<_r$, $r \in \Omega$ would imply the same of the orders defined in 3) and 4) above. Note also that every element of each set defined in 3) and 4) except ∞ has a successor.

DEFINITION 1.3. Let A be a formula in our language. stg $(\underline{H}; A)$, the stage of an occurrence of H, denoted by $\underline{H}$, relative to A and stg (A), the stage of A, are defined to be the elements of $\{0\} \cup \Omega \cup \{\Omega\}$, 0 being below Ω and Ω being maximal. We write $<_\Omega$ for this order.

1) $\underline{H}$ occurs in A in the form $H(s; j, x) \wedge (s, j) <^*_s (r, i)$. If $r \in \Omega$, then stg $(\underline{H}; A) = r$. Otherwise stg $(\underline{H}; A) = \Omega$.

2) $\underline{H}$ occurs in A in the form $H(s; j, x)$ but not in the context of 1). If $s \in \Omega$, then stg $(\underline{H}; A) = s$, stg $(\underline{H}; A) = \Omega$ otherwise.

3) stg $(A) = \max\{\text{stg}(\underline{H}; A);\ \underline{H} \text{ occurs in } A\}$.

4) stg $(A) = 0$ if H does not occur in A at all.

COROLLARY. *stg* $(A(n_1, \cdots, n_k)) \leqq_\Omega$ *stg* $(A(a_1, \cdots, a_k))$ *if* $a_1, \cdots, a_k$ *are free variables and* $n_1, \cdots, n_k$ *are constants.*

DEFINITION 1.4. Let us abbreviate

$$\{s, j, x\}(O(s; x) \wedge S(s; j) \wedge H(s; j, x) \wedge (s, j) <^*_s (r, i))$$

to $H[r; i]$.

For each r an element of Ω, $\boldsymbol{JD}_r$, the axiom of the isolated inductive definition of stage r, is formulated in two schemas. Let G be an isolated formula without H, with three first order arguments and one second order argument.

$\boldsymbol{JD}_r$

1) $O(r; a),\ S(r; i),\ H(r; i, a) \rightarrow G(r; i, a, H[r; i])$.

2) $O(r; a),\ S(r; i),\ G(r; i, a, H[r; i]) \rightarrow H(r; i, a)$.

Here a and i are arbitrary terms.

Notice that $\boldsymbol{JD}_r$ has a form similar to $\boldsymbol{UID}$ in [9], but with a different flavor. Note also that $\boldsymbol{JD}_r$ is formulated uniformly in r.

COROLLARY. *In* $\boldsymbol{JD}_r$, stg $(H(r; i, a)) = r$ *and*

$$\text{stg}(G(r; i, a, H[r; i])) \leqq_\Omega r\ .$$

DEFINITION 1.5. The systems $\mathfrak{S}_r$ and $\mathfrak{S}_\Omega$. Define $\mathfrak{S}_0$ to be $\boldsymbol{JNN}$ (cf. Definition 1.1). Notice that stg $(A) = 0$ for every formula occurring in any proof in $\mathfrak{S}_0$.

Let r be an element of Ω and let us assume that the systems $\mathfrak{S}_s$, $s <_\Omega r$, have been defined so as to satisfy stg $(A) \leqq_\Omega s$ for every sequent-formula A in any proof in $\mathfrak{S}_s$. Let $\mathfrak{S}(r)$ be $\bigcup_{s <_\Omega r} \mathfrak{S}_s$.

$\forall^r$ will denote a specified infinite rule of stage r:

$\forall^r$

$$\frac{\Gamma \rightarrow \Delta, \quad A(s), \quad s <_\Omega r}{\Gamma \rightarrow \Delta,\ \forall x(<(\Omega; x, r) \supset A(x))}\ ,$$

where $A(e)$ is an isolated formula without H and without second order parameters, the variable e occurs in $A(e)$ only at the indicated places and there is a recursive method $M(e)$ such that $M(s)$ provides a proof in $\mathfrak{S}_s$ of the sequent $\Gamma \to \Delta,\ A(s)$ for every $s <_\Omega r$.

Now let $\mathfrak{S}_r$ be the system $\mathfrak{S}(r)$ augmented by JD_r and $\forall^r$, where every sequent-formula A in any proof satisfies $\mathrm{stg}(A) \leqq_\Omega r$, and closed up with regards to the inferences of *JNN*.

Let $\mathfrak{S}(\Omega)$ be $\bigcup_{s \in \Omega} \mathfrak{S}_s$. $\forall^\Omega$ can be defined in a manner similar to $\forall^r$; assume a recursive method $M(e)$ such that $M(s)$ provides an appropriate proof in $\mathfrak{S}_s$, $s \in \Omega$. Thus $\forall^\Omega$ looks:

$$\frac{\Gamma \to \Delta, \quad A(s), \quad s \in \Omega}{\Gamma \to \Delta, \quad \forall x(\Omega(x) \supset A(x))},$$

where $A(e)$ is isolated without H and without second order parameters. $\mathfrak{S}_\Omega$ is $\mathfrak{S}(\Omega)$ augmented by $\forall^\Omega$ and closed up with regards to the inferences of *JNN*; $\mathrm{stg}(A) \in \{0\} \cup \Omega$ be satisfied for every formula A in $\mathfrak{S}_\Omega$.

We shall say that a sequent with a free variable e, say

$$S(e)\colon A_1(e), \cdots, A_m(e) \to B_1(e), \cdots, B_n(e)$$

is provable in $\mathfrak{S}_\Omega$ Ω-uniformly (in e) if there is a recursive method $M(e)$ such that $M(s)$ provides a proof of $S(s)$ for every $s \in \Omega$. Ω-uniform provability implies that

$$\to \forall x(\Omega(x) \supset (A_1(x) \wedge \cdots \wedge A_m(x) \supset B_1(x) \vee \cdots \vee B_n(x)))$$

is probable in $\mathfrak{S}_\Omega$.

(See [3] and [6] also for the definition of the systems with infinite rules.)

We can arithmetize those systems by assigning Gödel numbers to the proofs; it is done by transfinite induction along $\{0\} \cup \Omega \cup \{\Omega\}$. For example, $\forall^r$ can be formulated as follows. Let f be Gödel number of a recursive function such that $\{f\}(s, r)$ is a proof of $\Gamma \to \Delta,\ A(s)$ in $\mathfrak{S}_s$ for $s <_\Omega r$. Then $2 \cdot 5^{\Lambda s \{f\}(s,r)}$ is Gödel number of the conclusion.

Let $\ulcorner P \urcorner$ denote Gödel number of P.

COROLLARY. *Let $r \in \Omega \cup \{\Omega\}$. Along any branch in $\mathfrak{S}_r$, there can be at most one application of $\forall^s$, $s \leqq_\Omega r$, and there is no application of $\forall^t$ under $\forall^s$ if $t <_\Omega s \leqq_\Omega r$.*

DEFINITION 1.6. Requisite. Let $\mathfrak{S}_\Omega$ be the system defined above. We place a requisite on $\mathfrak{S}_\Omega$, (R), stated below.

(R) $\to Ac(O; e)$ is provable $\mathfrak{S}_\Omega$ Ω-uniformly in e, where $Ac(O; e)$ is an abbreviation of

$$\forall z(O(e; z) \supset \forall \varphi(\forall x(O(e; x) \wedge \forall y(O(e; y) \wedge \ll(e; y, x) \supset \varphi[y]) \supset \varphi[x]) \supset \varphi[z])) .$$

$Ac(O; r)$ expresses the accessibility of $<^r$ for r an element of Ω.

For r an element of Ω, $\rightarrow Ac(O; r)$ being provable in $\mathfrak{S}_r$ is denoted by (R_r).

When $\mathfrak{S}_\Omega$ is known to satisfy (R), it will be called $\mathfrak{S}_\Omega(R)$.

DEFINITION 1.7. Specific systems $\mathfrak{S}_r^\circ$ and $\mathfrak{S}_\Omega^\circ$.

We here consider the case where $\boldsymbol{O}$ and $\boldsymbol{S}$ are $\boldsymbol{Od}(\Omega)$ and $\bigcup_r \boldsymbol{J}_r$ respectively, which were defined in §1 of [8]. We refer to §2 and §3 of [9] for the notations concerning $\boldsymbol{Od}(\Omega)$ and Proposition 2.1 there. Thus, $O(r; a)$ and $S(r; i)$ respectively are $Od(r; a)$ and $J(r; i)$; $<^s$ is the order of $<_{s,0}$ of $\boldsymbol{Od}_s$ ond $<_r$ is the order $<_{r,0}$ of $\boldsymbol{J}_r$. $\boldsymbol{JD}_r$ specified to arithmetization of $\boldsymbol{Od}(\Omega)$ will be denoted by $\boldsymbol{JD}_r^\circ$. Let G be the matrix defined in Definition 3.2 of [9]. The intended meaning of G is the definition of $(\boldsymbol{r}, \boldsymbol{i})$-fans. Let F be a new predicate symbol and let $BA^\circ(r; i, a)$ abbreviate $Od(r; a) \wedge (J(r; i) \vee i \equiv \infty)$. $\boldsymbol{JD}_r^\circ$ looks like this:

1) $BA^\circ(r; i, a),\ F(r; i, a) \rightarrow G(r; i, a, \{j, x\}(F(r; j, x) \wedge <(r; 0, j, i)))$.

2) $BA^\circ(r; i, a),\ G(r; i, a, \{j, x\}(F(r; j, x) \wedge <(r; 0, j, i))) \rightarrow F(r; i, a)$.

$\mathfrak{S}_r$ and $\mathfrak{S}_\Omega$ thus specified will be denoted by $\mathfrak{S}_r^\circ$ and $\mathfrak{S}_\Omega^\circ$ respectively. $<(r; 0, j, i)$ can be regarded as $(r, j) <_S^* (r; i)$, hence the stage of $F(r; j, x) \wedge <(r; 0, j, i)$ is $\boldsymbol{r}$.

In this specified case, the requisite (R) means the provability of the accessibility of $(\boldsymbol{Od}_r, <_0)$ in $\mathfrak{S}_r$.

§2. Accessibility proof

THEOREM 1. 1) *The following sequent is provable in $\mathfrak{S}_\Omega^\circ$ Ω-uniformly (in e).*

$$(1) \qquad BA^\circ(e; i, a) \rightarrow Ac(e; i, a) ,$$

where $Ac(e; i, a)$ is defined in 3° of Definition 3.1 in [9], the intended meaning of which is the accessibility of the order $<_{e,i}$.

2) *$\mathfrak{S}_\Omega^\circ$ satisfies the requisite (R) in Definition 3.1, namely $\mathfrak{S}_\Omega^\circ(R)$.*

CONCLUSION. The following sequent is provable in $\mathfrak{S}_\Omega^\circ(R)$.

$$(2) \qquad \boldsymbol{Od}(\Omega; a), \boldsymbol{Od}(\Omega; i) \rightarrow Ac(i, a) ,$$

where $Ac(i, a)$ stands for the accessibility of $\boldsymbol{a}$ with regards to $<_i$ in $\boldsymbol{Od}(\Omega)$.

The proof of the Conclusion from 1) of Theorem 1 can be established in a manner similar to the corresponding proof in §3 of [9]. (3)~(5) there are provable in $\boldsymbol{JNN}$.

$$(3) \qquad Od(\Omega; a), Od(\Omega; i) \rightarrow \exists e(\Omega(e) \wedge e = \max_\Omega(\mathrm{stg}\,(a), \mathrm{stg}\,(i)+1)) .$$

(4) $$Od(\Omega; a), Od(\Omega; i), \Omega(e), e = \max_\Omega(\mathrm{stg}(a), \mathrm{stg}(i)+1) \rightarrow BA^\circ(e; i, a).$$

(5) $$BA^\circ(e; i, a), Ac(e; i, a) \rightarrow Ac(i, a).$$

Thus (1) and (5) yield

$$BA^\circ(e; i, a) \rightarrow Ac(i, a)$$

in $\mathfrak{S}_\Omega^\circ$ Ω-uniformly in e. Apply a cut to this sequent and (4) to obtain Ω-uniformly

$$Od(\Omega; a), Od(\Omega; i), \Omega(e) \wedge e = \max_\Omega(\mathrm{stg}(a), \mathrm{stg}(i)+1) \rightarrow Ac(i, a).$$

$\Omega(e) \wedge e = \max_\Omega(\mathrm{stg}(a), \mathrm{stg}(i)+1)$ is isolated without F and without second order parameters. Thus by an application of $\forall^\Omega$ (in its symmetric form), we obtain

$$Od(\Omega; a), Od(\Omega; i), \exists e(\Omega(e) \wedge e = \max_\Omega(\mathrm{stg}(a), \mathrm{stg}(i)+1)) \rightarrow Ac(i, a).$$

Now apply a cut to this sequent and (3) to obtain (2), q.e.d.

1) and 2) of Theorem 1 can be established simultaneously from the following

MAIN PROPOSITION.

(6) $$BA^\circ(e; i, a), Ac(e) \rightarrow Ac(e; i, a)$$

is provable in $\mathfrak{S}_\Omega^\circ$ *Ω-uniformly, where $Ac(e)$ stands for the $<_{e,0}$-accessiblity of J_e.*

From the Main Proposition follows that

(7) $$\rightarrow A(e) \equiv Ac(e)$$

is provable in $\mathfrak{S}_\Omega^\circ$ Ω-uniformly (in e), where $A(e)$ stands for

$$\forall x \forall i(BA^\circ(e; i, x) \supset Ac(e; i, x)).$$

Thus it suffices to establish that

(*) $$\rightarrow Ac(e)$$

is provable in $\mathfrak{S}_\Omega^\circ$ Ω-uniformly (in e), since (*) and (7) imply $\rightarrow A(e)$, and $A(e)$ is the 1) in Theorem 1; 2) is a special case of 1) for our particular theory. (*) can be established as follows.

The *JNN*-provability of

(8) $$\forall w(<(\Omega; w, e) \supset A(w)) \rightarrow Ac(e)$$

can be easily seen (cf. Lemma 2, §3 of [9]). Assume a recursive method $M(e)$ such that $M(s)$ provides an $\mathfrak{S}_s^\circ$-proof of $\rightarrow Ac(s)$ for every $\boldsymbol{s} <_\Omega \boldsymbol{r}$. (7) then implies that there is a recursive method to provide a proof in $\mathfrak{S}_s^\circ$ of $\rightarrow A(s)$ for $\boldsymbol{s} <_\Omega \boldsymbol{r}$.

Notice that $A(e)$ is isolated without F and without second order parameters.

Thus, by $\forall^r$,

$$(9) \qquad \rightarrow \forall w(<(\Omega; w, r) \supset A(w))$$

in $\mathfrak{S}_r^\circ$. Since (8) is provable in *JNN*,

$$\forall w(<(\Omega; w, r) \supset A(w)) \rightarrow Ac(r)$$

is provable in *JNN* with the same type of proof for all r. Combining this with (9), we obtain

$$\rightarrow Ac(r) \ .$$

The entire scheme described above provides a recursive method of constructing a proof in $\mathfrak{S}_r^\circ$ uniformly in r (See the end of this section), hence

$$\rightarrow Ac(e)$$

is provable in $\mathfrak{S}_\Omega^\circ$ Ω-uniformly, or (*), q.e.d.

We now examine the process of establishing the Main Proposition. Proposition 0 in §4 of [9] is valid here too. Propositions 1 through 16 there can be stated as they are, only the letter r be replaced by e a free variable for a notational reason, and those sequents be claimed to be provable in $\mathfrak{S}_\Omega^\circ$ Ω-uniformly in e. Let us state, as exemplary cases, Propositions 15 and 16.

PROPOSITION 15.

$$Ac(e),\ BA^\circ(e; i, a),\ F(e; 0, a) \rightarrow A(e; i, a) \ .$$

PROPOSITION 16.

$$Ac(e),\ BA^\circ(e; i, a),\ A(e; i, a) \rightarrow Ac(e; i, a) \ .$$

For each instance of those pairs of sequents with regards to $e=r$, we obtain, by applications of the cut, an $\mathfrak{S}_r^\circ$-proof of the sequent

$$Ac(r),\ BA^\circ(r; i, a) \rightarrow Ac(r; i, a)$$

uniformly in r. Thus

$$Ac(e),\ BA^\circ(e; i, a) \rightarrow Ac(e; i, a)$$

is provable in $\mathfrak{S}_\Omega^\circ$ Ω-uniformly in e, q.e.d.

BASIC LEMMA at the end of §4 of [9],

$$\Omega(e),\ Ac(e),\ \forall i(J(e; i) \wedge \forall j(<(e; 0, j, i) \supset V(j)) \supset V(i)) \rightarrow \forall i(J(e; i) \supset V(i))$$

is provable in JNN by comprehension on $\{i\}V(i)$ *(with or without the letter* F*).*

It is only a matter of routine (though lengthy) to verify that

Propositions 1 through 16 are provable in $\mathfrak{S}_\Omega^\circ$ Ω-uniformly in e (cf. §5 of [9]). Note that $\boldsymbol{JD}_r^\circ$ is formally (apparently) stronger than the specified $\boldsymbol{UID}$ (for each fixed r) in Definition 3.3 of [9], hence an application of $\boldsymbol{JD}_r^\circ$ suffices where $\boldsymbol{UID}$ (for r) was applied before. Note also that in those proofs there are no applications of $\forall^r$, and the proofs are identical for all r up to the numeral r.

The condition on the stages of the formulas (cf. Definitions 1.3 and 1.5) occurring in the proof (for r) is easily verified, since every occurrence of the symbol F is of the form $F(r; i, x)$ where r is a constant and $r \in \Omega$, or of the form $F(r; j, x) \wedge <(r; 0, j, i)$ where $r \in \Omega$.

This complete the demonstration of Theorem 1.

Let us now confirm the legitimacy of the proof process.

Let $p(r)$ be the arithmetization of the proof of the Main Proposition (p is a recursive function of r). Let f be a number variable and let c be Gödel number of the function g, where

$$g(f, r, s) = \begin{cases} \{f\}(s) & \text{if } \ s <_\Omega r \\ 0 & \text{otherwise} . \end{cases}$$

The intended meaning of $\{f\}(s)$ is a proof of $A(s)$ and the intended meaning of $g(f, r, s)$ is a proof of $A(s)$ when $s <_\Omega r$. Let $M(f, r)$ be the partial recursive function which is defined in terms of $S_1^2(c, f, r)$ and $p(r)$ to form (Gödel number of) a proof of $A(r)$. By the recursion theorem, there is a number f_0 such that

$$M(f_0, r) \simeq \{f_0\}(r) .$$

Define

$$\mu(r) \simeq \{f_0\}(r) .$$

μ is partial recursive. By transfinite induction along $<_\Omega$, we can show μ is totally defined. It can also be seen that μ is $<_\Omega$-recursive. In particular for each r, $\lambda s g(f_0, r, s)$ (=the μ restricted to $s <_\Omega r$) is recursive in s.

Chapter II

A progression of consistency proofs

Let $\mathfrak{S}_\Omega(R)$ be the system considered in §1 for given Ω, $\boldsymbol{O}$ and $\boldsymbol{S}$; thus we assume the requisite (R) for $\mathfrak{S}_\Omega$. Let $\mathfrak{S}_r^*$ and $\mathfrak{S}_\Omega^*$ be the enlargements of $\mathfrak{S}_r(R)$ and $\mathfrak{S}_\Omega(R)$ respectively with the full induction (rather than the isolated induction). Thus, $\mathfrak{S}_0^*$ is $\boldsymbol{INN}$ (cf. §27 of [4]). We shall present a consistency proof of $\mathfrak{S}_\Omega^*$ as a progression of consistency proofs of $\mathfrak{S}_r^*$'s each one adopting the reduction method in Chapter 4 of [2] or §28 of [4].

§3. Assignment of diagrams

We first define various notation such as rank and grade, following Chapter 4 of [2], §11 of [5] and §28 of [4]. For $<_S^*$, S_∞ etc., see Definition 1.2.

DEFINITION 3.1. Let $\mathfrak{S}_r^*$ and $\mathfrak{S}_\Omega^*$ be as above. Let A be a formula of $\mathfrak{S}_\Omega^*$. The rank of an occurrence of H in A, $r(\underline{H}; A)$, is an element of $I_{\Omega,\infty}$ (cf. 4) of Definition 1.2), defined as follows.

1) $H(s; j, x) \wedge (s, j) <_S^* (r, i)$ occurs in A.

1.1) $(\boldsymbol{r}, \boldsymbol{i}) \in \boldsymbol{S}$ and one of the following holds: s or j is a variable; $(\boldsymbol{s}, \boldsymbol{j}) \in \boldsymbol{S}$ and $(\boldsymbol{r}, \boldsymbol{i}) \leqq_S^* (\boldsymbol{s}, j)$; s and j are constants but $(\boldsymbol{s}, \boldsymbol{j}) \notin \boldsymbol{S}$. Then $r(\underline{H}; A) = (\boldsymbol{r}, \boldsymbol{i})$.

1.2) $(\boldsymbol{r}, \boldsymbol{i}) \in \boldsymbol{S}$ and $(\boldsymbol{s}, \boldsymbol{j}) <_S^* (\boldsymbol{r}, \boldsymbol{i})$. Then $r(\underline{H}; A) = (\boldsymbol{s}, \boldsymbol{j})^{\frown}$.

1.3) i is a variable, or i is a constant but $(\boldsymbol{r}, \boldsymbol{i}) \notin \boldsymbol{S}$. $r(\underline{H}; A) = \infty$.

2) $H(s; j, x)$ occurs alone in A.

2.1) $(\boldsymbol{s}, \boldsymbol{j}) \in \boldsymbol{S}$. $r(\underline{H}; A) = (\boldsymbol{s}, \boldsymbol{j})^{\frown}$.

2.2) 2.1) is not the case. $r(\underline{H}; A) = \infty$.

Recall that for any A a formula of $\mathfrak{S}_\Omega^*$ $\mathrm{stg}(A) \in \{0\} \cup \Omega$, hence $\boldsymbol{r}$ in 1) and $\boldsymbol{s}$ in 2) are elements of Ω (cf. Definitions 1.3 and 1.5). Thus follows the completeness of the definition.

COROLLARY 1. *Let* $r_0 \in \{0\} \cup \Omega \cup \{\Omega\}$. *If* A *is a formula of* $\mathfrak{S}_{r_0}^*$, *then* $r(\underline{H}; A) \in \boldsymbol{I}_{r_0,\infty}$. (See Definitions 1.3 and 1.5)

COROLLARY 2. *If* a *is a variable and* c *is a constant, then* $r(\underline{H}; A(c)) \leqq_{\Omega,\infty} r(\underline{H}; A(a))$.

DEFINITION 3.2. The γ-degree of a formula, $\gamma(A)$, is a nutural number, expressing the logical complexity of a formula relative to isolated formulas.

Notice that $H(s; j, x)$ is isolated.

COROLLARY. $\gamma(F(\alpha)) = \gamma(F(V))$ *if* V *is isolated.*

DEFINITION 3.3. Let $\boldsymbol{r}$ be an element of $\Omega \cup \{\Omega\}$. Define $\boldsymbol{I}(\boldsymbol{r}) = \{0\} \cup \boldsymbol{I}_{r,\infty}$. 0 is regarded as the least of $\boldsymbol{I}(\boldsymbol{r})$ and the order of $\boldsymbol{I}(\boldsymbol{r})$ induced from $<_{r,\infty}$ will be denoted by $<_{I(r)}$. Consider the set of expressions

$$\boldsymbol{K}(\boldsymbol{r}) = \{X + n;\ X \in \boldsymbol{I}(\boldsymbol{r}),\ n \in \omega\}\ .$$

We define an order $<_{K(r)}$ for $\boldsymbol{K}(\boldsymbol{r})$ as follows.

1) $X + n <_{K(r)} X + m$ if $X \in \boldsymbol{K}(\boldsymbol{r})$ and $n < m$.

2) $X + n <_{K(r)} Y + m$ for any n and m if $X <_{I(r)} Y$.

$<_{K(r)}$ is a linear order and in case $<_r^*$ is a well-order, the order type of $<_{K(r)}$ is $\omega \cdot (2 \cdot |<_r^*| + 1)$.

DEFINITION 3.4. Suppose $\boldsymbol{r} \in \Omega \cup \{\Omega\}$. Let $\boldsymbol{K}^+(\boldsymbol{r})$ denote the set

$K(r)\cup\{\varepsilon\}$ and let $<_r^+$ be the linear order of $K^+(r)$ induced from $<_{K(r)}$, regarding ε as maximal.

The order type of $<_r^+$ when $<^*$ is a well-order is $\omega\cdot(2\cdot|<_r^*|+1)+1$, or $|<_{K(r)}|+1$.

DEFINITION 3.5. Consider $\mathfrak{S}_{r_0}^*$ for each $r_0\in\Omega\cup\{\Omega\}$. We introduce substitutions (of isolated abstracts for second order variables) to the system, retaining the condition that the stages of the formulas do not exceed r_0, the result of which will be denoted again by $\mathfrak{S}_{r_0}^*$. We define a proof with degree to be a proof in $\mathfrak{S}_{r_0}^*$ satisfying the following conditions.

1) There are no logical inferences or inductions under a substitution (hence a substitution occurs only in the end-piece of a proof).

2) We can assign an element of $K^+(r_0)$ to every semiformula or abstract A and every substitution J in the end-piece, which is called the degree of A or of J, denoted by $d(A)$ or $d(J)$, respectively, so as to satisfy the subsequent conditions.

2.1) $d(A)=0$ if A is explicit.

2.2) $d(A)=\varepsilon$ if A is implicit and not isolated.

2.3) Suppose A is implicit and isolated.

2.3.1) $d(A)=0$ if A is atomic and does not contain H.

2.3.2) $d(H(s;j,x))=r(\underline{H};H(s;j,x))$.

2.3.3) $d(A)=r(\underline{H};A)+1$ if A is of the form $H(s;j,x)\wedge(s,j)<_S^*(r,i)$.

2.3.4) $d(\neg B)=d(B)+1$; $d(B\wedge C)=\max(d(B),d(C))+1$ if $B\wedge C$ is not as in 2.3.3). $d(\forall xB(x))=d(B(x))+1$.

2.3.5) $d(\forall\varphi F(\varphi)=\max_{<_{r_0}^+}(d(F(\varphi)),d(J))+1$, where J ranges over all the substitutions which disturb (affect) $\forall\varphi F(\varphi)$.

2.4) $d(\{x_1\cdots x_n\}B)=d(B)$.

2.5) $d(B)<_{r_0}^+d(J)$ for a substitution J and every B a formula in the upper squent of J.

2.6) $d(J)\in K(r_0)$ for any substitution J.

Note. 1) There can be only finitely many substitutions in a proof (by virtue of 1) in the definition above), and so 2.3.5) is well-defined.

2) We can easily show that for each case of the definitions above the assigned diagram belongs to $K^+(r_0)$ (cf. Corollary 1 of Definition 3.1).

PROPOSITION 3.1. *Suppose r and i are constants and $(r,i)\in S$. If 1) and 2) in Definition 1.4 occurs in an $\mathfrak{S}_{r_0}^*$-proof with degree in which $H(r;i,a)$ is implicit, then*

$$d(G(r;i,a,H[r;i]))<_{r_0}^+d(H(r;i,a))\ .$$

Proof. Notice first that no substitution affects $G(r;i,a,H[r;i])$

(since in $G(r; i, a, \gamma)$ γ is the sole second order parameter), and second that only 1.1) and 1.2) in Definition 3.1 apply. Thus, by going over 2.3.1)~2.3.5), we obtain:

$$d(G(r; i, a, H[r; i]))$$
$$=\max\{r(\underline{H}; H[r; i]);\ \underline{H} \text{ occurs in } G\}+n$$
$$=\begin{cases}(s_0, j_0)^{\sim}+n & \text{if } (s, j)<_S^* (r, i) \text{ for every relevant} \\ & (s, j) \text{ and } (s_0, j_0) \text{ is the maximum one ,} \\ (r, i)+n & \text{otherwise .}\end{cases}$$
$$d(H(r; i, a))=(r, i)^{\sim} .$$

In either case the required inequality holds.

From now on we fix an r_0, $r_0 \in \{0\} \cup \Omega \cup \{\Omega\}$ and consider an $\mathfrak{S}_{r_0}^*$-proof with degree, say P. Thus the sequents and the formulas will refer to the ones in such a proof.

DEFINITION 3.6. Let A be a semi-formula or an abstract occuring in P. We define the norm of A, $n(A)$, in $K(r_0)$.

1) $n(A)=0$ if A is atomic without H.

2) $n(H(s; j, x))=r(\underline{H}; H(s, j, x))$.

3) $n(A)=r(\underline{H}; A)+1$ if A is of the form $H(s; j, x)\wedge(s, j)<_S^* (r, i)$.

4) $n(\neg B)=n(B)+1$; $n(B\wedge C)=\max(n(B), n(C))+1$; $n(\forall xB(x))=n(B(a))+1$; $n(\forall\varphi F(\varphi))=n(F(\alpha))+1$.

5) $n(\{x_1\cdots, x_n\}B(x_1, \cdots, x_n))=n(B(a_1, \cdots, a_n))$.

PROPOSITION 3.2. *If* 1) *or* 2) *in Definition* 1.4 *occurs in* P *and* $(r, i) \in S$, *then*

$$n(G(r; i, a, H[r; i]))<_{K(r_0)} n(H(r; i, a)) .$$

DEFINITION 3.7. Let $N(r_0)$ be $\omega\times\omega\times K(r_0)$ and let $\prec_{r_0}$ be the lexicographical order of $N(r_0)$. The grade of (an occurrence of) a formula A, $g(A)$, is $\langle\gamma(A), f, n(A)\rangle$ where f is the number of second order eigenvariables occuring in A. (We may assume they are used as eigen under A.) $g(A)$ is an element of $N(r_0)$.

PROPOSITION 3.3. *If* $(r, i) \in S$ *and* 1) *or* 2) *in Definition* 1.4 *occurs in* P, *then*

$$g(G(r; i, a, H[r; i]))\prec_{r_0} g(H(r; i, a))$$

(*cf. Proposition* 3.2).

DEFINITION 3.8. $Od\{r_0\}$. In [6], we made use of the system $O(\omega+1, \Omega\times\omega^3)$ (Ω is a countable ordinal), $\Omega\times\omega^3$ is ordered lexicographically, and an element of it is expressed as $[u, a]$, $u \in \Omega$ and $a \in \omega^3$. We shall take over a similar idea; consider the diagrams of $O(K^+(r_0), (r_0^*+1)\times N(r_0))$, or $O(K^+(r_0), (r_0^*+1)\times(\omega\times\omega\times K(r_0)))$, for $r_0 \in \Omega$, where

r_0^*+1 denotes $\{0\}\cup(r_0+1)$, 0 being the least element. Let us abbreviate this system of diagrams to $\boldsymbol{Od}\{r_0\}$.

Notice that here we are speaking of the diagrams alone; no orders have been introduced yet. Pending upon the condition that $<^*_{r_0}$ be well-ordered, the order structure can be introduced in the usual manner, yielding a well-ordering $<_i$ for every $i\in\boldsymbol{K}^+(r_0)$.

For an element of $\boldsymbol{Od}\{r_0\}$, say α, we define stg(α) as follows. stg$(0)=0$; stg$((i,[u,a],\alpha))=\max_\Omega(u,\text{stg}(\alpha))$; stg$(\alpha_1\#\cdots\#\alpha_k)=\max_\Omega(\text{stg}(\alpha_1),\cdots,\text{stg}(\alpha_k))$. It is obvious that stg$(\alpha)\leqq_\Omega r_0$ for $\alpha\in\boldsymbol{Od}\{r_0\}$.

DEFINITION 3.9. Let P be a proof in $\mathfrak{S}^*_{r_0}$ and let Q be a sub-proof of P ending with S. If r is the least element of $\{0\}\cup\Omega\cup\{\Omega\}$ such that Q is a proof in $\mathfrak{S}^*_r$, then stg$(S;P)$ is defined to be r ($r\leqq_\Omega r_0$).

We shall assign an element of $\boldsymbol{Od}\{r_0\}$ to every sequent in P. Let S be a sequent in P. The element of $\boldsymbol{Od}\{r_0\}$ assigned to S relative to P will be denoted by $o(S;P)$. $o(S;P)$ will be defined so that if stg$(S;P)=r$, then stg$(o(S;P))\leqq_\Omega r$.

1) $o(S;P)=0$ if S is an initial sequent.

2) A weak, structural inference retains the diagram unchanged.

3) Let S' and S be the upper sequent and the lower sequent of one of the inferences $\neg$, $\wedge$ left, first order $\forall$, second order $\forall$ right and explicit second order $\forall$ left. $o(S;P)=(\varepsilon,[0,\langle 0,0,0\rangle],\sigma)$, where $\sigma=o(S';P)$.

4) S_1 and S_2 are the upper sequents and S is the lower sequent of $\wedge$: right. $o(S;P)=(\varepsilon,[0,\langle 0,0,0\rangle],\sigma_1\#\sigma_2)$ where $\sigma_1=o(S_1;P)$ and $\sigma_2=o(S_2;P)$.

5) S' and S are the upper sequent and the lower sequent respectively of an implicit second order $\forall$ left whose auxiliary formula is B. $o(S;P)=(\varepsilon,[0,\langle m,f,u+2\rangle],\sigma)$ where $\sigma=o(S';P)$ and $\langle m,f,u\rangle=g(B)$.

6) S_1 and S_2 are the upper sequents of a cut and S is its lower sequent. $o(S;P)=(\varepsilon,[0,\langle m,f,u+1\rangle],\sigma_1\#\sigma_2)$ where $\sigma_1=o(S_1;P)$, $\sigma_2=o(S_2;P)$, and $\langle m,f,u\rangle=g$ (the cut formula).

7) Let $r\leqq_\Omega r_0$, let S_s be the s-th upper sequent of a $\forall^r$ for each $s<_\Omega r$, and let S be its lower sequent. Then stg$(S;P)=r$. Notice that stg$(o(S_s;P))\leqq_\Omega s<_\Omega r$. $o(S;P)=(\varepsilon,[r,\langle 0,0,0\rangle],0)$.

8) S' and S are the upper sequent and the lower sequent respectively of a substitution J. $o(S;P)=(d(J),[0,\langle 0,0,0\rangle],\sigma)$ where $\sigma=o(S';P)$.

9) S' and S are the upper sequent and the lower sequent respectively of an induction. $o(S,P)=(\varepsilon,[0,\langle m,f,u+2\rangle],\sigma)$ where $\sigma=o(S;P)$ and $\langle m,f,u\rangle=g(A(a))$, $A(a)$ being the induction formula.

10) $o(P)=o(S;P)$ where S is the end-sequent of P.

REMARK. 1) The soundness of the definition is obvious since all the sequent formulas occurring in P have stages not exceeding r_0.

2) $\mathrm{stg}\,(o(P)) \leqq_{\Omega} r$ if r is the least element of Ω such that P belongs to $\mathfrak{S}^*_r$.

§4. Reduction

Suppose there is a proof of the sequent $\rightarrow$, say P, in $\mathfrak{S}^*_{\Omega}$, and suppose r_0 is the least element of $\Omega \cup \{\Omega\}$ such that P belongs to $\mathfrak{S}^*_{r_0}$. We shall define the so-called reduction procedure for P, following the method in §27 and §28 of [4], to obtain another proof in $\mathfrak{S}^*_{r_0}$ of the same sequent $\rightarrow$. Let us quote from pp. 326–344 of [4] with page numbers and item numbers. Additional cases will be marked with (+). What we can adopt without change will not be mentioned. The distinctions between the cases where $r_0 \in \Omega$ and $r_0 = \Omega$ will be remarked when necessary.

pp. 326–329. (1)∼(4) go through; by virtue of the Corollary of Definition 1.3, substitution of a numeral for a free variable in 3) of (1) and (2) does not increase the stage of a formula, hence the reducts will be legitimate proofs in $\mathfrak{S}^*_{r_0}$.

p. 343. Elimination of $\boldsymbol{JD}_r$ ($r \leqq_{\Omega} r_0$) for the end-piece of P. Suppose, for instance, the end-piece of P contains an initial sequent of the form 1) in Definition 1.4;

$$(*) \qquad O(r;a)\,, \qquad S(r;i)\,, \qquad H(r;i,a) \rightarrow G(r;i,a,H[r;i])\,.$$

Due to the circumstances, r, a and i are all constants. Consider the case where $\rightarrow O(r;a)$ and $\rightarrow S(r;i)$ are provable (in the first order part of JNN). There exists a cut J where a descendant of $H(r;i,a)$ in (*) is cut out. Let P be of the form

$$
\begin{array}{lcc}
 & H(r;i,a) \longrightarrow H(r;i,a) & O(r;a), S(r;i), H(r;i,a) \longrightarrow G(r;i,a,H[r;i]) \\
 & \vee & \vee \\
 & \Gamma \longrightarrow \Delta, H(r;i,a) & H(r;i,a), \Pi \longrightarrow \Lambda \\
\cline{2-3}
J & \multicolumn{2}{c}{\Gamma, \Pi \longrightarrow \Delta, \Lambda} \\
 & \multicolumn{2}{c}{\vee} \\
 & \multicolumn{2}{c}{\longrightarrow}
\end{array}
.
$$

Define the reduct P' as follows.

$$
\begin{array}{cc}
O(r;a), S(r;i), H(r;i,a) & G(r;i,a,H[r;i]) \\
\longrightarrow G(r;i,a,H[r;i]) & \longrightarrow G(r;i,a,H[r;i]) \\
\vee & \vee \\
\Gamma, O(r;a), S(r;i) & G(r;i,a,H[r;i]), \Pi \\
\longrightarrow \Delta, G(r;i,a,H[r;i]) & \longrightarrow \Lambda \\
\hline
\multicolumn{2}{c}{\Gamma, O(r;a), S(r;i), \Pi \longrightarrow \Delta, \Lambda} \\
\multicolumn{2}{c}{\overline{\overline{O(r;a), S(r;i), \Gamma, \Pi \longrightarrow \Delta, \Lambda}}} \\
\multicolumn{2}{c}{\Gamma, \Pi \longrightarrow \Delta, \Lambda} \\
\multicolumn{2}{c}{\vee} \\
\multicolumn{2}{c}{\longrightarrow\,,}
\end{array}
$$

where all the ancestors of the cut formulas $H(r; i, a)$ in P are replaced by $G(r; i, a, H[r; i])$ and the cuts on $O(r; a)$ and $S(r; i)$ are abbreviated from the expression. It is obvious that the stage does not increase. The substitutions are assigned the same degrees as the corresponding ones in P. By virtue of Proposition 3.1, P' can be made into a proof with degree.

pp. 329–335. (5) goes through.

The existence of a suitable cut in (6) is established as usual, since there can be no infinite inferences in the end-piece of P (a proof of $\longrightarrow$), hence the end-piece is finite (cf. 2.7 of [6]).

(7)~(9) go through. As for Case 1 of (7), the existence of the i-resolvent is guaranteed, since $\boldsymbol{K}^{+}(r_0)$ is linearly ordered and there can be only finitely many sequents under a specified one (in the end-piece). Substitution of V for α in Case 2 of (7) does not affect the maximum stage of formulas.

(+) The case where the boundary inference of the left cut formula is a $\forall^r$, $r \leqq_{\Omega} r_0$. P looks like this:

$$\dfrac{\dfrac{\dfrac{\Gamma_1 \overset{\vee}{\longrightarrow} \Delta_1, F_1(s), s <_{\Omega} r}{\Gamma_1 \longrightarrow \Delta_1, \forall x(<(\Omega; x, r) \supset F_1(x))}}{\Gamma \overset{\vee}{\longrightarrow} \Delta, \forall x(<(\Omega; x, r) \supset F(x))} \qquad \dfrac{\dfrac{<(\Omega; t, r) \supset F_2(t), \Pi_1 \overset{\vee}{\longrightarrow} \Lambda_1}{\forall x(<(\Omega; x, r) \supset F_2(x)), \Pi_1 \longrightarrow \Lambda_1}}{\forall x(<(\Omega; x, r) \supset F(x)), \Pi \overset{\vee}{\longrightarrow} \Lambda}}{\dfrac{\Gamma, \Pi \longrightarrow \Delta, \Lambda}{\overset{\vee}{\longrightarrow}\ .}}$$

Here $F(x)$ is isolated, without H and without second order parameters (cf. Definition 1.5). So F_1 and F_2 differ from F only by term-replacements. In the case where $r = \Omega$, the cut formula is $\forall x(\Omega(x) \supset F(x))$.

Case 1. $t <_{\Omega} r$. Define P' as follows.

P_1:

$$\dfrac{\dfrac{\Gamma_1 \overset{\vee}{\longrightarrow} \Delta_1, F_1(t)}{<(\Omega; t, r), \Gamma_1 \longrightarrow \Delta_1, F_1(t)}}{\Gamma_1 \longrightarrow \Delta_1, <(\Omega; t, r) \supset F_1(t)}\, J$$
$$\approx\approx\approx\approx\approx\approx\approx\approx\approx\approx$$
$$\Gamma_1 \longrightarrow <(\Omega; t, r) \supset F_1(t), \Delta_1, \forall x(<(\Omega; x, r) \supset F_1(x))$$
$$\dfrac{\Gamma \overset{\vee}{\longrightarrow} <(\Omega; t, r) \supset F(t), \Delta, \forall x(<(\Omega; x, r) \supset F(x)) \qquad \forall x(<(\Omega; x, r) \supset F(x)), \Pi \overset{\vee}{\longrightarrow} \Lambda}{\Gamma, \Pi \longrightarrow <(\Omega; t, r) \supset F(t), \Delta, \Lambda\ .}$$

$$P_2: \quad \frac{<(\Omega;t,r)\supset F_2(t),\ \Pi_1 \xrightarrow{\ \Psi\ } \Lambda_1}{\forall x(<(\Omega;x,r)\supset F_2(x)),\ \Pi_1,\ <(\Omega;t,r)\supset F_2(t) \longrightarrow \Lambda_1}$$

$$\frac{\Gamma \xrightarrow{\ \Psi\ } \Delta,\ \forall x(<(\Omega;x,r)\supset F(x)) \qquad \forall x(<(\Omega;x;r)\supset F(x)),\ \Pi,\ <(\Omega;t,r)\supset F(t) \xrightarrow{\ \Psi\ } \Lambda}{\Gamma,\ \Pi,\ <(\Omega;t,r)\supset F(t) \longrightarrow \Delta,\ \Lambda}\ .$$

$$P': \qquad \text{cut}\ \frac{P_1 \qquad P_2}{\Gamma,\ \Pi \longrightarrow \Delta,\ \Lambda} \qquad \xrightarrow{\ \Psi\ }\ .$$

J is a new boundary. Substitutions will be assigned the same degrees as the corresponding ones in P. Due to the restriction on F, the rank is not involved. So P' can be made into a proof with degree.

Case 2. Not $t <_\Omega r$.

P_1':

$$K\ \frac{\dfrac{<(\Omega;t,r)\longrightarrow}{<(\Omega;t,r),\ \Gamma_1 \longrightarrow \Delta_1,\ \forall x(<(\Omega;x,r)\supset F_1(x)),\ F_1(t)}}{\dfrac{\Gamma_1 \longrightarrow \Delta_1,\ \forall x(<(\Omega;x,r)\supset F_1(x)),\ <(\Omega;t,r)\supset F_1(t)}{\Gamma_1 \longrightarrow <(\Omega;t,r)\supset F_1(t),\ \Delta_1,\ \forall x(<(\Omega;x,r)\supset F_1(x))}}$$

$$\frac{\Gamma \xrightarrow{\ \Psi\ } <(\Omega;t,r)\supset F(t),\ \Delta,\ \forall x(<(\Omega;x,r)\supset F(x)) \qquad \forall x(<(\Omega;x,\,,r)\supset F(x)),\ \Pi \xrightarrow{\ \Psi\ } \Lambda}{\Gamma,\ \Pi \longrightarrow <(\Omega;t,r)\supset F(t),\ \Delta,\ \Lambda}$$

$$P': \qquad \text{cut}\ \frac{P_1' \qquad P_2}{\Gamma,\ \Pi \longrightarrow \Delta,\ \Lambda}$$

K is a new boundary.

If $r=\Omega$, then $<(\Omega;t,r)$ should be replaced by $\Omega(t)$ and $t<_\Omega r$ should be replaced by $t\in\Omega$.

This completes the reduction.

It is obvious that there is a recursive ρ such that $\rho(\ulcorner P \urcorner)$ represents the reduct of P when P is an $\mathfrak{S}^*_{r_0}$-proof of $\rightarrow$.

§ 5. Consistency proofs

THEOREM 2. *For every $r_0 \in \Omega \cup \{\Omega\}$, $<^*_{r_0}$ is well-ordered, and the consistency of $\mathfrak{S}^*_{r_0}$ can be established by the well-ordered structure $\boldsymbol{Od}\{r_0\}$. In particular, $\mathfrak{S}^*_\Omega$ is consistent.*

Proof. The theorem can be established by transfinite induction along $\{0\}\cup\Omega\cup\{\Omega\}$. It is known that the consistency of ***INN***, or $\mathfrak{S}^*_0$, can be established with $\boldsymbol{O}(\omega+1, \omega^3)$ (cf. [2] and [4]). Assume that the theorem has been established for every $s <_\Omega r_0$. Thus, $\mathfrak{S}_s^*$ is consistent and this fact is established by $(Od\{s\}, <_0)$, $s <_\Omega r_0$. The consistency of $\mathfrak{S}_s^*$ implies that $<^s$ is a well-order (cf. the requisite (R), Definition 1.6). By the condition on $<^s$ and $<_{r_0}$, it then follows that $<_{r_0}$ is a well-order, hence $<_{K(r_0)}$ and $<_{r_0}^+$ are also well-orders (cf. Definitions 3.3 and 3.4). Thus we can define a linear order $<_i$ for $Od\{r_0\}$ with respect to every $i \in K^+(r_0)$, which is known to be well-ordered. Thus $Od\{r_0\}$ turns out to be the theory of ordinal diagrams based on $K^+(r_0)$ and $(r_0^*+1) \times N(r_0) = (r_0^*+1) \times (\omega \times \omega \times K(r_0))$.

Suppose now there were a proof P of $\rightarrow$ in $\mathfrak{S}_{r_0}^*$. Due to the induction hypotheses, P is not a proof of $\mathfrak{S}_s^*$ if $s <_\Omega r_0$. Thus the reduction in §4 can be defined for P. We shall show that each reduction case of P decreases the ordinal diagrom (with regards to $<_0$). The well-ordering property of $(Od\{r_0\}, <_0)$ ensures that the process must stop, which means the existence of a proof of $\rightarrow$ in $\mathfrak{S}_s^*$ for some $s <_\Omega r_0$ ($s=0$ inclusive), yielding a contradiction with the induction hypotheses.

The proof is more or less parallel to the existing consistency proofs. Thus, we shall concentrate on some new cases. See §4 for some notations and refer to [6] as well as §§26~28 of [4] for evaluations of elements of $Od\{r_0\}$. We shall make quotations from [4] as we did in §4.

1)~4) of the Lemma in §2 of [6] can be established for our present case in a similar manner. 4) there corresponds to Lemma 27.1 (the Main Lemma) in [4], which is basic to the comparison of ordinal diagrams.

pp. 326–329. (2) Elimination of the induction, Case 2. The stage of $A(n)$ or $A(n')$ does not exceed that of $A(a)$; the rank does not increase through the substitution of a numeral for a variable (cf. Corollary 2 of Definition 3.1). Thus, the original proof can be adopted.

p. 343. Elimination of JD_r, where $\rightarrow O(r;a)$ and $\rightarrow S(r, i)$ are provable. Let σ denote $o(\Gamma, \Pi \rightarrow \Delta, \Lambda; P)$ and let σ' denote $o(\Gamma, \Pi \rightarrow \Delta, \Lambda; P')$.

$$\sigma = (\varepsilon, [0, \langle m, f, u+1\rangle], \sigma_1 \# \sigma_2)$$
$$\sigma' = (\varepsilon, [0, g_1+1], 0 \# (\varepsilon, [0, g_2+1], 0 \# \sigma'')\ ,$$

where

$$\sigma'' = (\varepsilon, [0, \langle n, g, v+1\rangle], \sigma_1' \# \sigma_2')\ ;$$
$$\langle m, f, u\rangle = g(H(r; i, a))\ ;$$
$$\langle n, g, v\rangle = g(G(r; i, a, H[r; i]))\ ;$$
$$g_1 = g(O(r; a)) = \langle 0, 0, 0\rangle\ ;$$
$$g_1 + 1 = \langle 0, 0, 1\rangle\ ;$$

$$g_2 = g(S(r; i)) = \langle 0, 0, 0\rangle .$$

$$\langle n, g, v\rangle = g(G(r, i, a, H[r; i]))$$
$$\prec_{r_0} g(H(r; i, a)) = \langle m, f, u\rangle$$

by Proposition 3.3 ($(r, i) \in \boldsymbol{S}$ due to the conditions on the proof), and $0 <_{K(r_0)} u$. Thus, by a simple evaluation of diagrams, we obtain $\sigma' <_0 \sigma$ in $\boldsymbol{Od}\{r_0\}$.

pp. 329–335. (7) through (9) in [4] can be followed. (7.1)~(7.5) go through since there are no infinite rules in the end-piece.

(+) Essential reduction of $\forall^r$.

Case 1. $t <_\Omega r$. Notice that $g(<(\Omega; t, r) \supset F(t)) = \langle 0, 0, u\rangle$ and $g(\forall x(<(\Omega; x, r) \supset F(x))) = \langle 0, 0, u+1\rangle$ since $F(x)$ is isolated and without H and without second order parameters (cf. Definitions 1.5, 3.2, 3.6 and 3.7). Let us give names to the concerning diagrams.

$$o(\Gamma_1 \to \Delta_1, F_1(s); P) = \sigma_s , \qquad s <_\Omega r ,$$

where $\mathrm{stg}\,(\sigma_s) \leqq_\Omega s$ (cf. Definition 3.9).

$$o(\Gamma_1 \to \Delta_1, \forall x(<(\Omega; x, r) \supset F_1(x)); P) = (\varepsilon, [r, \langle 0, 0, 0\rangle], 0) = \alpha .$$

$$o(\Gamma \to \Delta, \forall x(<(\Omega; x, r) \supset F(x)); P) = \tau .$$

$$o(<(\Omega; t, r) \supset F_2(t), \Pi_1 \to \Lambda_1; P) = \mu .$$

$$o(\forall x(<(\Omega; x, r) \supset F_2(x)), \Pi_1 \to \Lambda_1; P) = (\varepsilon, [0, \langle 0, 0, 0\rangle], \mu) .$$

$$o(\forall x(<(\Omega; x, r) \supset F(x)), \Pi \to \Lambda; P) = \rho .$$

$$o(\Gamma, \Pi \to \Delta, \Lambda; P) = (\varepsilon, [0, \langle 0, 0, u+1\rangle], \tau \# \rho)$$

where

$$\langle 0, 0, u+1\rangle = g(\forall x(<(\Omega; x, r) \supset F(x))) .$$

$$o(\Gamma_1 \to <(\Omega; t, r) \supset F_1(t), \Delta_1, \forall x(<(\Omega; x, r) \supset F_1(x)); P')$$
$$= (\varepsilon, [0, \langle 0, 0, 0\rangle], \sigma_t) = \beta ,$$

where $\mathrm{stg}\,(\beta) \leqq_\Omega t <_\Omega r$.

$$o(\Gamma \to <(\Omega; t, r), \Delta, \forall x(<(\Omega; x, r) \supset F(x)); P') = \tau' .$$

$$o(\Gamma, \Pi \to <(\Omega; t, r) \supset F(t), \Delta, \Lambda; P') = (\varepsilon, [0, \langle 0, 0, u+2\rangle], \tau' \# \rho) .$$

$$o(\forall x(<(\Omega; x, r) \supset F_2(x)), \Pi_1, <(\Omega; t, r) \supset F_2(t) \to \Lambda_1; P') = \mu .$$

$$o(\forall x(<(\Omega; x, r) \supset F(x)), \Pi, <(\Omega; t, r) \supset F(t) \to \Lambda; P') = \rho'$$

$$o(\Gamma, \Pi, <(\Omega; t, r) \supset F(t) \to \Delta, \Lambda; P')$$
$$= (\varepsilon, [0, \langle 0, 0, u+2\rangle], \tau \# \rho') .$$

$$o(\Gamma, \Pi \to \Delta, \Lambda; P') = (\varepsilon, [0, \langle 0, 0, u+1\rangle], (\varepsilon, [0, \langle 0, 0, u+2\rangle], \tau' \# \rho)$$
$$\# (\varepsilon, [0, \langle 0, 0, u+2\rangle], \tau \# \rho')),$$

where $\langle 0, 0, u\rangle = g(<(\Omega; t, r)\supset F(t))$.

$\beta <_i \alpha$ for every i. Using this fact and following the corresponding case in 2.7 of [6], we can conclude that $o(\rightarrow; P') <_0 o(\rightarrow; P)$.

Case 2 can be dealt with in a manner similar to Case 1.

This completes the consistency proofs.

§6. Note on a remark in [5]

In §13 of [5], we have remarked that the evaluation of the ordinal diagrams of the isolated inductive definitions (given in §11 of [5]) is not optimal, a counter-example of which concerns the inductive definitions along ω (cf. Additional remark in §13, [5]); Theorem 2.1 for $n=1$ gives the ordinal

$$O(\omega^{\omega^{\omega}+1}, \omega^{\omega^{\omega\cdot 2}}) ,$$

while $O(\omega^2+1, \omega^4+1)$ suffices in proving the consistency of the (semi-isolated) inductive definitions along ω (Chapter I of [5]).

A closer observation reveals, however, that this disparity stems from the fact that the reduction of the isolated inductive definitions to the provably-Δ_2^1-systems presented in §9 of [5] deals with the cases where the abstracts V are isolated (cf. 3) in the proof of Theorem 1.5 there). This restriction is essential in applying the provably Δ_2^1-comprehension axiom (cf. 1° in the deduction of $(3.\,i+1)$ and $(4.\,i+1)$, §9 of [5]). On the other hand, the original version of the inductive definitions allows an arbitrary abstract V in $G_n(s, t, V, \{x, y\}(A_n(x, y, V)\wedge x <^* s))$, hence $G_n(s, t, V, \alpha)$ is not necessarily isolated. Thus, the Additional remark in [5] should have read as follows.

The disparity between two evaluations of ordinals, one given in §11 and one in Chapter I for the case $I=\omega$ and $A=\omega$ may be an indication of the essential difference in the powers of the inductive definitions with V arbitrary and ones with V isolated.

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