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Some properties of Tor and δ-cyclic p-groups

by

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In this article, we investigate various properties of pure resolutions of totally projective primary abelian groups. We construct for every natural number n a group P_n very similar to the generalized Prüfer group $H_{\omega+n}$ and use it to obtain some structural results on $\mathrm{Tor}(G, P_n)$, notably that $\mathrm{Tor}(G, P_n)$ is a direct sum of countable (d.s.c.) groups when G is totally projective. In the last part, we study groups all of whose subgroups of cardinality less than a given cardinal δ are direct sums of cyclic groups. We call such groups *δ-cyclic*. We show G is δ-cyclic if and only if $\mathrm{Tor}(G, P_n)$ is δ-cyclic. All concepts not explicitly defined here can be found in [2]. All groups are primary groups. The notation is essentially that of [2]. The symbol $\bigoplus_c$ denotes direct sums of cyclic groups. A pure exact sequence $0\to K\to F\to G\to 0$ is said to be a pure resolution of G if $F=\bigoplus_c$.

§1. The groups P_n and $\mathrm{Tor}(G, P_n)$

LEMMA 1.1. *Let $B=\bigoplus Z(p^n)$ and let S be a basic subgroup of B of corank* 1. *For each natural number n let $P_n=B/S[p^n]$ then the basic subgroups of P_n are isomorphic to B, $P_n^1\simeq Z(p^n)$ and $P_n/P_n^1\simeq B$.*

Proof. $S/S[p^n]$ is a basic subgroups of P_n and is isomorphic to p^nS. But $p^nS\simeq p^nB\simeq B$.

Now $P_n^1=B[p^n]/S[p^n]=B[p^n]/(S\cap(B[p^n]))\simeq(B[p^n]+S)/S$ and since S is pure in B, this is $(B/S)\,[p^n]$. But $B/S\simeq Z(p^\infty)$, therefore $P_n^1\simeq Z(p^n)$ and $P_n/P_n^1\simeq B/B[p^n]\simeq p^nB\simeq B$.

Note that P_n is quite similar to $H_{\omega+n}$ the generalized Prüfer group defined in [2]. However, they are not isomorphic, except for $n=1$. Note also that P_n is $p^{\omega+n}$-projective.

Before we proceed with $\mathrm{Tor}\,(G, P_n)$, we need to recall the following facts:

LEMMA 1.2. [1] *Let H be a pure subgroup of G. Then, $H/H[p^n]$ is $p^{\omega+n}$ pure in $G/H[p^n]$.*

LEMMA 1.3. [6] *If $0\to A\to B\to C\to 0$ is a p^α-pure exact sequence with $\alpha\geqq\omega$ and E is any group, then,* $0\to\mathrm{Tor}\,(A, E)\to\mathrm{Tor}\,(B, E)\to$

$\mathrm{Tor}(C, E)\to 0$ *is a* p^{α}*-pure exact sequence.*

THEOREM 1.4. *Let* $0\to K\to F\to G\to 0$ *be a pure resolution of* G, *then there exist for each natural number* n, *groups* $C_{n,i}=\oplus_c$, $i=1, 2$ *such that* $(F/K[p^n]\oplus C_{n,1})\simeq(\mathrm{Tor}(G, P_n)\oplus C_{n,2})$.

Proof. Consider the $p^{\omega+n}$-pure (by Lemma 1.2) exact sequence $0\to K/A\to F/A\to G\to 0$ where $A=K[p^n]$, then by Lemma 1.3 $0\to \mathrm{Tor}(K/A, P_n)\to \mathrm{Tor}(F/A, P_n)\to \mathrm{Tor}(G, P_n)\to 0$ is $p^{\omega+n}$-pure exact. Now, $\mathrm{Tor}(G, P_n)$ is $p^{\omega+n}$-projective and therefore:

(i) $\mathrm{Tor}(F/A, P_n)\simeq \mathrm{Tor}(G, P_n)\oplus \mathrm{Tor}(K/A, P_n)$. Let $S[p^n]=R$ and consider the $p^{\omega+n}$-pure exact sequence $0\to S/R\to B/R\to Z(p^{\infty})\to 0$, where B and S are as in Lemma 1.1. Then $0\to \mathrm{Tor}(F/A, S/R)\to \mathrm{Tor}(F/A, P_n)\to F/A\to 0$ is $p^{\omega+n}$-pure exact and since F/A is $p^{\omega+n}$-projective it is split exact. Therefore,

(ii) $\mathrm{Tor}(F/A, P_n)\simeq F/A\oplus \mathrm{Tor}(F/A, S/R)$ Let $C_{n,1}=\mathrm{Tor}(F/A, S/R)$ and $C_{n,2}=\mathrm{Tor}(K/A, P_n)$. These can be easily seen to be $\oplus_c$ and combining (i) and (ii) we have the desired result. □

An immediate application of this result is the following:

COROLLARY 1.5. *Let* $0\to K_i\to F_i\to G\to 0$ $i=1, 2$ *be pure resolutions of* G, *then, for each natural number* n *there exists* $C_{n,1}$ *and* $C_{n,2}$ *both* $\oplus_c$, *such that* $F_1/K_1[p^n]\oplus C_{n,1}\simeq F_2/K_2[p^n]\oplus C_{n,2}$.

§2. Some properties of totally projective groups

In this section we study the structure of $\mathrm{Tor}(G, P_n)$ and $F/K[p^n]$ for totally projective groups G. We need the following fact:

LEMMA 2.1. *If* G *is totally projective and* H *is* $p^{\omega+n}G$*-high in* G *for some natural number* n, *then* H *is d.s.c..*

Proof. G/G^1 is $\oplus_c$ and thus $H/H^1=\oplus_c$ since it is isomorphic to $(H+G^1)/G^1$. Now, $p^nH^1=0$ therefore $H^1=\oplus_c$ and the result follows from theorem 1 p. 445 in [4].

THEOREM 2.2. *If* G *is totally projective then* $\mathrm{Tor}(G, P_n)$ *is a d.s.c. for all natural numbers* n.

Proof. We consider two cases.

(i) If $\lambda(G)\geqq\omega 2$, then $\mathrm{Tor}(G, P_n)$ is a d.s.c. by a result in [6] (which is quoted below for the reader's convenience as Lemma 2.3), and Lemma 2.1.

(ii) If $\omega<\lambda(G)<\omega 2$, let $\lambda(G)=\omega+n$ where n is a natural number, then, G is $p^{\omega+n}$-projective and a d.s.c.. If $m\geqq n$, then $\mathrm{Tor}(G, P_m)\simeq G\oplus C$ where $C=\oplus_c$ hence $\mathrm{Tor}(G, P_m)$ is a d.s.c.. If $m<n$, all high subgroups of G are $\oplus_c$ and another application of Lemma 2.3 yields the result.

LEMMA 2.3. [6] *Let A and B be groups such that $\lambda(A)>\lambda(B)\geqq\omega$. Then Tor (A, B) is a d.s.c. if and only if* (i) *B is a d.s.c. and* (ii) *every $p^{\beta}A$-high subgroup of A is a d.s.c. for every infinite ordinal β whose corresponding Ulm invariant of B is non-zero.*

THEOREM 2.4. *If $0\to K\to F\to G\to 0$ is a pure resolution of the totally projective group G, then $F/K[p^n]$ is a d.s.c. and hence is totally projective for every natural number n.*

Proof. By Theorem 1.4, $F/K[p^n]$ is a summand of Tor $(G, P_n)\oplus C$, where $C=\oplus_c$. By Theorem 2.2 Tor (G, P_n) is a d.s.c. and by a well known theorem of I. Kaplansky summands of d.s.c. groups are d.s.c..

However, $F/K[p^n]$ may be totally projective for some n although F/K is not as the following counterexample shows:

EXAMPLE. Let $B=\oplus Z(p^n)$ and $\bar{B}$ the torsion completion of B. Define $G_n=\bar{B}/B[p^{n+1}]$ for each natural number n. Since G_n is uncountable whereas its basic subgroups are countable, G_n cannot be totally projective.

Let $0\to K\to F\to G_n\to 0$ be a pure resolution of G_n. $F/K[p^n]$ is a d.s.c. since Tor (G_n, P_n) is a d.s.c. by an application of Lemma 2.3. Indeed, $\lambda(G_n)=\omega+n+1>\lambda(P_n)=\omega+n$, and the only infinite ordinal, whose corresponding Ulm invariant of P_n is non-zero, is $\omega+n-1$. Now $p^{\omega+n-1}G_n$-high subgroups are in fact high in G_n and since one of them is $B/B[p^{n+1}]=\oplus_c$they are all $\oplus_c$ and hence d.s.c. Therefore $F/K[p^n]$ is totally projective although G_n is not.

We have been, however, unable to decide wether the converse of Theorem 2.4 is true or false.

3. δ-cyclic groups

A group is said to be δ-cyclic for a cardinal number δ if every subgroups of cardinality less than δ is $\oplus_c$. In [7], the existence of groups $\aleph_n$-cyclic not $\oplus_c$ has been established for every natural number n. In [3], it was shown that $\aleph_\omega$-cyclic groups of cadinality $\aleph_\omega$ are $\oplus_c$. Not much else is known about such groups. We show here that Tor (G, P_n) is δ-cyclic if and only if G is δ-cyclic. We show also that for every cardinal for which there exists δ-cyclic groups not $\oplus_c$, there exists a proper $p^{\omega+1}$-projective δ-cyclic group.

THEOREM 3.1. *Let δ be a cardinal and $G=\bigoplus G_\lambda$ then G is δ-cyclic if and only if each G_λ is δ-cyclic.*

Proof. Suppose G_λ is δ-cyclic for every λ and let H be a subgroup of G such that $|H|<\delta$. Let H_λ be the projection of H on G_λ then $|H_\lambda|<\delta$ and thus $H_\lambda=\oplus_c$. Now $H\subset\bigoplus H_\lambda=\oplus_c$ and $H=\oplus_c$. The converse is obvious.

Before we proceed to the main result of this section we need:

LEMMA 3.2. *Let K be a pure subgroup of $F=\bigoplus_c$ such that F/K is not $p^{\omega+(n-1)}$-projective for some natural number n. Then, $F/K[p^n]$ is proper $p^{\omega+n}$-projective.*

Proof. $F/K[p^n]$ is $p^{\omega+n}$-projective and if it is also $p^{\omega+(n-1)}$-projective then, by Theorem 1.4 in [1], F/K is $p^{\omega+n-1}$-projective.

LEMMA 3.3. *Let H be a pure subgroup of G. Then, the natural map $\pi: G/H[p^n]\rightarrow G/H$ is height preserving on $G[p^n]/H[p^n]$. (The heights in $G/H[p^n]$).*

Proof. One needs only to verify that if p^m divides $\pi(x)$ where $x\in G[p^n]$, then p^m divides $x+H[p^n]$.

THEOREM 3.4. *Let $0\rightarrow K\rightarrow F\rightarrow G\rightarrow 0$ be a pure resolution of G. Then G is δ-cyclic if and only if $F/K[p^n]$ is δ-cyclic for some natural number n.*

Proof. Suppose $F/K[p^n]$ is not δ-cyclic for some natural number n. Let $A=K[p^n]$, then there exists a subgroup H of F such that $|H/A|<\delta$ and $H/A\neq\bigoplus_c$. Now then, $(H+K)/A\neq\bigoplus_c$ and therefore $(H+K)/K\neq\bigoplus_c$. Since $A\subset H\cap K$, there exists an epimorphism from H/A onto $H/(H\cap K)\simeq(H+K)/K$ and $|(H+K)/K|<\delta$. Hence F/K is not δ-cyclic.

Suppose that for some natural number n, $F/K[p^n]$ is δ-cyclic and let H/K be a subgroup of F/K of cardinality less than δ. However $(H/K)[p^n]\simeq H[p^n]/K[p^n]$ and therefore $|H[p^n]/K[p^n]|<\delta$ and we can embed it in a pure subgroup, $M/K[p^n]$ of $H/K[p^n]$ of same cardinality. Now $M/K[p^n]=\bigoplus_c$ and every subgroup of it can be written as the union of an ascending chain of subgroups of bounded height in $H/K[p^n]$. Since the natural map π: $H/K[p^n]\rightarrow H/K$ is height preserving on $H[p^n]/K[p^n]$ by Lemma 3.3, $(H[p^n]+K)/K=(H/K)[p^n]$ is the union of an ascending chain of subgroups of bounded height in. Therefore $H/K=\bigoplus_c$ and F/K is δ-cyclic.

COROLLARY 3.5. *If there exists a δ-cyclic group $G\neq\bigoplus_c$ then there exists a proper $p^{\omega+1}$-projective δ-cyclic group.*

COROLLARY 3.6. *A group G is δ-cyclic if and only if* Tor (G, P_n) *is δ-cyclic for some natural number n.*

Proof. It is a straightforward application of Theorems 1.4, 3.1 and 3.4.

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