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The trace class of an arbitrary Hilbert algebra

by

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Abstract

The trace class $\tau(A)$ of an arbitrary Hilbert algebra A is defined to be the span in A of products ab, where $a, b \in A$, and is equipped with the trace class norm $\|\cdot\|_\tau$ it inherits from the dual $\mathfrak{A}^*$ of the C^*-algebra $\mathfrak{A}$ of A. The completion of $(\tau(A), \|\cdot\|_\tau)$ is identified with a subspace Φ_H of $\mathfrak{A}^*$; if G is a unimodular locally compact group and $A=L^1(G)\cap L^2(G)$, then $\Phi_H \cong A(G)$, the Fourier algebra of G. Several of the known results for $\tau(A)$ when A is a full Hibert algebra are extended to the general case, while others are extended to replete Hilbert algebras. For example, it is shown for replete A that $\tau(A)=\{ab: a, b \in A\}$, and that a positive element of A is in $\tau(A)$ if and only if it has a positive square root in A.

Intoduction

If G is a unimodular locally compact group with Hilbert algebra $A=L^1(G)\cap L^2(G)$ whose C^*-algebra completion is $\mathfrak{A}$, then the Fourier algebra $A(G)$ of G is isometrically isomorphic as a Banach space to a subspace of $\mathfrak{A}^*$, and is the pre-dual of the von Neumann algebra $\mathfrak{A}''$ generated by $\mathfrak{A}$ [4]. In fact, $A(G)$ is isometrically isomorphic to the completion of the trace class $\tau(A)$ of the Hilbert algebra A. In this paper, the trace class $\tau(A)$ of an arbitrary Hilbert algebra A is defined and studied. It is shown that its completion can be identified with a closed subspace Φ_H of the dual $\mathfrak{A}^*$ of the C^*-algebra $\mathfrak{A}$ of A; Φ_H plays the role of $A(G)$ for an arbitrary Hilbert algebra A (here H is the Hilbert space completion of A). For example, it is proved that the Banach space Φ_H is isometrically isomorphic to a quotient space of the Banach space of trace class operators $H\hat{\otimes}H^*$ on H. Moreover, many of Schatten's basic results [14] concerning the Banach algebra of trace class operators on a Hilbert space are shown to extend to arbitrary Hilbert algebras without assuming that A is either an H^*-algebra [11, 12] or a full Hilbert algebra [6]. For instance, when A is a replete Hilbert algebra, a positive element $a \in A$ is integrable if and only if

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$a \in \tau(A)$ if and only if $a=b^2$ for some positive b in A; that is, the positive square root of positive element of the repletion A_r of A lies in A_r, which is generally smaller than the fulfillment A_b of A. Our methods of analysis are necessarily different from those used in [11, 12, 6], where an abundance of projections permits a concept of summability (arbitrary Hilbert algebras may have no nonzero projections). Instead, our analysis depends on properties of Φ_H and $\mathfrak{A}$.

The organization of the paper is as follows. Section 1 contains, in addition to some notation and preliminary results, the key (see Theorem 1) to the development of the trace class without summability and projection bases. In Section 2, the fundamental properties of the trace class $(\tau(A), \|\cdot\|_\tau)$ of an arbitrary Hilbert algebra A are determined. The trace class is identified with a linear subspace Φ_A of the dual $\mathfrak{A}^*$ of the C^*-algebra $\mathfrak{A}$ of A, and the completion Φ_H of Φ_A in $\mathfrak{A}^*$ is the main subject of Section 3. Finally, in Section 4, replete Hilbert algebras are considered, and the trace class of such an algebra is related to its integrable elements.

1. Notation and preliminaries

All vector spaces and algebras are over the complex field $\boldsymbol{C}$. Let $A=(A, \|\cdot\|)$ be a Hilbert algebra with inner product $(\cdot|\cdot)$ and involution $a \mapsto a^*$. Let $H=(H, \|\cdot\|)$ be the Hilbert space completion of A, and let $J: H \to H$ be the involution on H extending that on A; then, $J^2=I$ and $(Jk|Jh)=(h|k)$, for all h, k in H. For each a in A, let L_a, R_a be the operators in $\boldsymbol{B}(H)$, the algebra of all bounded linear operators on H with operator norm $|\cdot|$, defined on A by $L_a b=ab$, $R_a b=ba$, respectively. Let $\mathfrak{A}=(\mathfrak{A}, |\cdot|)$ denote the closure in $\boldsymbol{B}(H)$ of the set $L_A=\{L_a: a \in A\}$; $\mathfrak{A}$ is called the C^*-algebra of A. Finally, let $\mathfrak{A}''$ be the double commutant of $\mathfrak{A}$ in $\boldsymbol{B}(H)$ (i.e., $\mathfrak{A}''$ is the von Neumann subalgebra of $\boldsymbol{B}(H)$ generated by $\mathfrak{A}$ and by L_A). For definitions and basic facts about Hilbert algebras, see [2, 8, 16].

For h, k in H, define $\varphi_{hk}: \mathfrak{A}'' \to \boldsymbol{C}$ by $\varphi_{hk}(S)=(Sh|Jk)$, where $S \in \mathfrak{A}''$. Then $|\varphi_{hk}(S)| \leqq |S|\,\|h\|\,\|k\|$; hence $\varphi_{hk} \in (\mathfrak{A}'')^*$ with $\|\varphi_{hk}\| \leqq \|h\|\,\|k\|$. Let $\Phi_H=\{\varphi_{hk}: h, k \in H\} \subseteqq (\mathfrak{A}'')^*$ be equipped with the norm it inherits from $(\mathfrak{A}'')^*$. Because $\mathfrak{A}''$ is a standard [2, Def. 7, p. 79] von Neumann algebra, Φ_H is the Banach space pre-dual $(\mathfrak{A}'')_*$ of $\mathfrak{A}''$ [2,Rem. 1, p. 266, Thm. 1, p. 38].

Next, let $\mathfrak{A}^{**}$ be the bidual of $\mathfrak{A}$, realized as its enveloping von Neumann algebra (i.e., as the double commutant of $\pi(\mathfrak{A})$ in $\boldsymbol{B}(H_\pi)$, where $\pi: \mathfrak{A} \to \boldsymbol{B}(H_\pi)$ is the universal representation of $\mathfrak{A}$) [3, Sec. 12.1]. Since $\mathfrak{A} \subseteqq \mathfrak{A}''$ and $\mathfrak{A}$ is ultraweakly dense in $\mathfrak{A}''$, it follows from the Kaplansky Density Theorem [10, Thm. 1.9.1, p. 22] that Φ_H imbeds isometrically in $\mathfrak{A}^*$. Further, using [3, Prop. 12.1.5, p. 237] (with ρ the identity representation of $\mathfrak{A}$ in $\boldsymbol{B}(H)$ and $\tilde{\rho}$ the corresponding normal representation of $\mathfrak{A}^{**}$ onto $\mathfrak{A}''$ in $\boldsymbol{B}(H)$), it follows that $\mathfrak{A}^{**}/\Phi_H^{\perp} \cong \Phi_H^* \cong \mathfrak{A}'' \overset{*}{\simeq} \mathfrak{A}^{**}/\ker(\tilde{\rho})$. Thus, $\Phi_H^{\perp}$ is a closed *-ideal of $\mathfrak{A}^{**}$. These facts are collected in the following:

THEOREM 1. *The Banach space $\Phi_H = \{\varphi_{hk}: h, k \in H\}$, equipped with the norm it inherits from $\mathfrak{A}^*$, is the Banach space pre-dual $(\mathfrak{A}'')_*$ of $\mathfrak{A}''$. Moreover, $\Phi_H^{\perp}$ is a closed *-ideal of $\mathfrak{A}^{**}$, and the von Neumann algebras $\mathfrak{A}^{**}/\Phi_H^{\perp}$ and $\mathfrak{A}''$ are isometrically *-isomorphic.*

The Banach space Φ_H is the analogue, for arbitrary Hilbert algebras, of Eymard's Fourier algebra $A(G)$ of a locally compact group G [4]. Since $A(G)$ is defined even when G is nonunimodular, many of the results in this paper may also obtain for left Hilbert algebras.

2. The trace class

Let $\tau(A)$ be the linear span in the Hilbert algebra A of elements of the form ab, where $a, b \in A$. It is immediate that $\tau(A)$ is a dense *-ideal of A and is itself a Hilbert algebra; however, we wish to study $\tau(A)$ under another norm. Let $\Phi: \tau(A) \to \mathfrak{A}^*$ denote the linear map defined by $\Phi(\sum_{j=1}^n a_j b_j) = \sum_{j=1}^n \varphi_{a_j b_j}$, and let $\Phi_A = \Phi(\tau(A))$; clearly, $\Phi_A \subseteq \Phi_H$. Note that, for λ in $\boldsymbol{C}$ and a, b in A, $\lambda\varphi_{ab} = \varphi_{(\lambda a)b} = \varphi_{a(\lambda b)}$. Also, from the fact that $\varphi_{ab}(L_c) = (L_c a \mid b^*) = (c \mid (ab)^*)$, for all a, b, c in A, it follows routinely that $\sum_{j=1}^n \varphi_{a_j b_j} = 0$ if and only if $\sum_{j=1}^n a_j b_j = 0$. Thus, Φ is an injection, and so Φ_A is an isomorphic copy of $\tau(A)$. The norm $\|\cdot\|_\tau$, defined on $\tau(A)$ by the formula $\|\sum_{j=1}^n a_j b_j\|_\tau = \|\sum_{j=1}^n \varphi_{a_j b_j}\|$ (in $\mathfrak{A}^*$), is called the trace class norm and, equipped with this norm, $\tau(A)$ is called the trace class of A.

It is not difficult to show (using [6, Thm. 2.2 and 3.2]) that this construction and norming of $\tau(A)$ is consistent with that given in [6] when A is full. However, it is worth noting that, when A is full, every element of $\tau(A)$ is of the form ab, for some a, b in A. In Section 4, we show that this also occurs when A is a replete Hilbert algebra.

Returning now to the case in which A is an arbitrary Hilbert algebra, it is an easy consequence of Theorem 1 and the fact that A is dense in H that the closure of Φ_A in $\mathfrak{A}^*$ is Φ_H. Consequently, the trace class norm of an element $\sum_{j=1}^n a_j b_j$ in $\tau(A)$ is the norm of $\sum_{j=1}^n \varphi_{a_j b_j}$ in $\mathfrak{A}^*$, in $(\mathfrak{A}'')^*$, or in $(\mathfrak{A}_b)^*$ (since $\mathfrak{A}_b'' = \mathfrak{A}''$), where $\mathfrak{A}_b$ is the C^*-algebra of the fulfillment A_b of A. In general, $\mathfrak{A} \neq \mathfrak{A}_b$; nonetheless, by the above, facts about the trace class norm established in [6] may be freely used.

In the following discussion of how the results for the trace class of a full Hilbert algebra translate to the setting of an arbitrary

Hilbert algebra A, we will continue to let A_b denote the fulfillment of A, $\mathfrak{A}_b$ the C^*-algebra of A_b, and $\tau(A_b)$ its trace class. It is shown in [5, Prop. 3.1] that the algebra $M_L(A)$ (resp., $M_R(A)$) of continuous left (resp., right) multipliers of A may be regarded as a linear subspace of $\mathfrak{A}''$ (resp., $\mathfrak{A}'$, the commutant of $\mathfrak{A}$ in $\boldsymbol{B}(H)$). (If A is full, then $M_L(A)=\mathfrak{A}''$ and $M_R(A)=\mathfrak{A}'$.) Define a continuous linear functional tr on $\tau(A)$ by the formula: $\operatorname{tr}(a)=\varphi_a(I)$, where I is the identity operator on H, and $a\in\tau(A)$. Since, for $a=\sum_{j=1}^{n} b_j c_j$ in $\tau(A)$,

$$\operatorname{tr}(a)=\varphi_a(I)=\sum_{j=1}^{n}\varphi_{b_j c_j}(I)=\sum_{j=1}^{n}(b_j\,|\,c_j^*)\ ,$$

this definition is consistent with that in the full case (see [6, Thm. 2.2]). The following theorem is a routine extension of Lemma 2.3 and Theorem 2.4 in [6].

THEOREM 2. *If $(\tau(A), \|\cdot\|_\tau)$ is the trace class of an arbitrary Hilbert algebra A, then*

(1) $|\operatorname{tr}(Ta)|\leqq|T|\,\|a\|_\tau$, *for all in $\tau(A)$, T in $M_L(A)\cup M_R(A)$.*

(2) $|T|_\tau=|T|$, *for all T in $M_L(A)\cup M_R(A)$, where $|\cdot|_\tau$ is the operator norm on $\boldsymbol{B}(\tau(A))$.*

(3) *multiplication in $\tau(A)$ is separately continuous; indeed, $\|ab\|_\tau\leqq\min\{|L_a|\,\|b\|_\tau, |R_b|\,\|a\|_\tau\}$, for all a, b in $\tau(A)$.*

(4) $\|a^*\|_\tau=\|a\|_\tau$, *for all a in $\tau(A)$.*

(5) $\|ab\|_\tau\leqq\|a\|\,\|b\|$, *and* $\|aa^*\|_\tau=\|a\|^2=\|a^*a\|_\tau$, *for all a, b in A.*

In the language of [5], parts (3) and (4) of Theorem 2 establish that $\tau(A)$ is an involutive quasi-normed algebra. Theorem 3 below shows that $\|\cdot\|_\tau$ is an algebra norm on $\tau(A)$ if and only if $\|\cdot\|$ is an algebra norm on A (compare with [6, Thm. 2.5]).

THEOREM 3. *For A an arbitrary Hilbert algebra, the following statements are equivalent:*

(1) *multiplication in $\tau(A)$ is jointly continuous.*

(2) *there is a constant $M>0$ such that $\|ab\|_\tau\leqq M\|a\|_\tau\|b\|_\tau$, for all a, b in $\tau(A)$.*

(3) *there is a constant $N>0$ such that $\|a\|\leqq N\|a\|_\tau$, for each a in $\tau(A)$.*

(4) *the trace class $\tau(A_b)$ of A_b is complete.*

(5) *the Hilbert algebra A_b is complete (i.e., $A_b=H$).*

(6) *multiplication in A is jointly continuous.*

(7) *the Hilbert algebra A_b is trivially renormable to be an H^*-algebra.*

(8) *there is a constant $P>0$ such that $|L_a|\leqq P\|a\|$, for all a in A.*

(9) *there is a constant $Q>0$ such that $|L_a|\leqq Q\|a\|_\tau$, for all a in*

$\tau(A)$.

Proof. That statements (1) through (7) are equivalent can be shown by modifying, in a straightforward manner, the proof of Theorem 2.5 [6] (note: it seems to be necessary to use Theorem 2.5 to establish the equivalence (4)$\Rightarrow$(1)). As to statements (8) and (9), it is clear that (8) is equivalent to (6); thus, (3)$\Leftrightarrow$(8) together imply (9), while, in view of (3) of Theorem 2, (9) implies (2). □

The import of Theorem 3 is that there is, in general, no constant $N>0$ such that $||a||\leqq N||a||_\tau$, for all a in $\tau(A)$. The next theorem concerns the existence of a reverse inequality (see [6, Thm. 4.1] for the situation when A is full).

THEOREM 4. *For A an arbitrary Hilbert algebra, the following statements are equivalent*:

(1) *there is a constant $N>0$ such that $||a||_\tau\leqq N||a||$, for all a in* $\tau(A)$.

(2) *there is a constant $N>0$ such that $||a||\leqq N|L_a|$, for all a in* A.

(3) *there is a constant $M>0$ such that $||a||_\tau\leqq M|L_a|$, for all a in* $\tau(A)$.

(4) *the Hibert algebra A_b is projection bounded from above.*

(5) *the Hilbert algebra A_b has an identity.*

Proof. The implications (1)$\Rightarrow$(2), (3)$\Rightarrow$(2) are computational (see [6]). The equivalence (2)$\Leftrightarrow$(4) is [16, Thm. 4.2]. Finally, the equivalence (4)$\Leftrightarrow$(5), and the implications (4)$\Rightarrow$(3), (4)$\Rightarrow$(1) follow directly from [6, Thm. 4.1]. □

3. The Banach space Φ_H

In this section, the Banach space completion Φ_H of the trace class $\tau(A)$ of an arbitrary Hilbert algebra A is studied. It is shown that the C^*-algebra $\mathfrak{A}$ of A being a dual C^*-algebra is closely related to Φ_H being all of $\mathfrak{A}^*$. In addition, the Banach space Φ_H is realized as a quotient of the Banach algebra of all trace class operators on H.

By [1, Prop. 4.2], the algebra $M_L(\mathfrak{A})$ of (automatically continuous) left multipliers of $\mathfrak{A}$ is imbeddable in $\mathfrak{A}''$. The fact that $M_L(\mathfrak{A})$ also imbeds isometrically as a closed subalgebra of $\mathfrak{A}^{**}$ [7, Thm. 1.1], thus yields a characterization of $\mathfrak{A}$ as dual C^*-algebra (see [15] for the definition).

PROPOSITION 5. *If A is an arbitrary Hilbert algebra with C^*-algebra $\mathfrak{A}$, then $\mathfrak{A}$ is dual if and only if $M_L(\mathfrak{A})=\mathfrak{A}''$ and Φ_A is dense in $\mathfrak{A}^*$ (i.e., $\mathfrak{A}^*=\Phi_H$).*

Proof. From [15, Thm. 5.1], $\mathfrak{A}$ is a dual C^*-algebra if and only if $M_L(\mathfrak{A})\cong\mathfrak{A}^{**}$. However, as noted above, $M_L(\mathfrak{A})\subseteq\mathfrak{A}''$ and by Theorem 1, $\mathfrak{A}''\overset{*}{\simeq}\mathfrak{A}^{**}/\Phi_H^{\perp}$; whence, the result follows. □

When A is a full Hilbert algebra, A (hence, $\mathfrak{A}$) is a twosided ideal in $\mathfrak{A}''$ [8, Cor. 1.8]; thus, in this case, $\mathfrak{A}$ is dual if and only if Φ_A is dense in $\mathfrak{A}^*$. This fact sheds new light on Example 3.3 in [6].

Now, let $\bar{\mathfrak{A}}$ be the C^*-algebra conjugate to $\mathfrak{A}$ [3, 1.9.1]. View H as a left $\bar{\mathfrak{A}}$-module by setting $W\cdot h=JWJh$, where $W\in\bar{\mathfrak{A}}$, $h\in H$. Then the dual H^* of H becomes a right $\bar{\mathfrak{A}}$-module via the adjoint action; more precisely, $k\cdot W=JW^*Jk$, for all W in $\bar{\mathfrak{A}}$, k in H^*. Following [9, p. 72], let $H\otimes_{\bar{\mathfrak{A}}}H^*=(H\hat{\otimes}H^*)/K$, where $H\hat{\otimes}H^*$ is the projective tensor product of H, H^* (and, as such,is identifiable withe Banach spaces of trace class operators on H [13, Thm. 5.12, p. 119]), and K is the closed linear span in $H\hat{\otimes}H^*$ of elements of the form $W\cdot h\otimes k-h\otimes k\cdot W$, where $h, k\in H$, $W\in\mathfrak{A}$.

THEOREM 6. *Let $\theta\colon H\otimes_{\bar{\mathfrak{A}}}H^*\to\Phi_H$ be defined, for t in $H\otimes_{\bar{\mathfrak{A}}}H^*$, by $\theta(t)=\sum_{j=1}^{\infty}\varphi_{h_j,Jk_j}$, where $\sum_{j=1}^{\infty}h_j\otimes k_j$ is a representative of t such that $\sum_{j=1}^{\infty}\|h_j\|\,\|k_j\|<+\infty$. Then θ is an iometric isomorphism from $H\otimes_{\bar{\mathfrak{A}}}H^*$ onto Φ_H.*

Proof. Define $\Psi\colon H\times H^*\to\Phi_H$ by $\Psi(h,k)=\varphi_{h,Jk}$. Then Ψ is a jointly continuous bilinear map onto Φ_H with $\|\Psi\|=1$ and, as a result, determines a unique continuous linear map ψ from $H\hat{\otimes}H^*$ onto Φ_H with $\|\psi\|=1$ such that $\psi(h,k)=\varphi_{h,Jk}$. Let θ be the unique map from $B=(H\hat{\otimes}H^*)/\ker\psi$ onto Φ_H satisfying $\psi=\theta\circ\eta$, where $\eta\colon H\hat{\otimes}H^*\twoheadrightarrow B$ is the usual quotient map. It is elementary to show that $\|\theta\|=1$; hence, by the open mapping theorem, B and Φ_H have equivalent norms. Thus, set-theoretically, B and Φ_H have the same dual space $\mathfrak{A}''$. However, $B^*\cong(\ker\psi)^{\perp}=\{F\in(H\hat{\otimes}H^*)^*\colon F(t)=0,\ t\in\ker\psi\}$, and $(H\hat{\otimes}H^*)^*\cong\boldsymbol{B}(H)$ [13, Thm. 5.14, p. 119]; hence, $B^*\cong\mathfrak{A}''$ and it follows that $B\cong\Phi_H$ (i.e., that θ is an isometric isomorphism).

To complete the proof, it remains to show that $K=\ker\psi$. The inclusion $K\subseteq\ker\psi$ follows readily by a routine calculation. Next, since $(H^*)^*=H$, the dual $(H\otimes_{\bar{\mathfrak{A}}}H^*)^*$ of $H\otimes_{\bar{\mathfrak{A}}}H^*$ is isometrically isomorphic to $\mathrm{Hom}_{\bar{\mathfrak{A}}}(H,H)\subset\boldsymbol{B}(H)$ [9, p. 72]. If $T\in\mathrm{Hom}_{\bar{\mathfrak{A}}}(H,H)$, then $T(W\cdot h)=W\cdot(Th)$, for all W in $\mathfrak{A}$, h in H. Thus, $T(JWJ)=(JWJ)T$, for all W in $\mathfrak{A}$; hence, by [2, Thm. 1, p. 71], $T\in\mathfrak{A}''$. Meanwhile, every T in $\mathfrak{A}''$ satisfies $T(JWJ)=(JWJ)T$, for all W in $\mathfrak{A}$, and so is in $\mathrm{Hom}_{\bar{\mathfrak{A}}}(H,H)$. Therefore, $K^{\perp}\cong(H\otimes_{\bar{\mathfrak{A}}}H^*)^*\cong\mathrm{Hom}_{\bar{\mathfrak{A}}}(H,H)=\mathfrak{A}''\cong B^*\cong(\ker\psi)^{\perp}$; from which, since $K\subseteq\ker\psi$, the equality $K=\ker\psi$ follows. □

Theorem 6 shows that, if G is a unimodular locally compact group, then the Fourier algebra $A(G)$ of G is isometrically isomorphic as a Banach space to a quotient of the space $L^2(G)\hat{\otimes}L^2(G)^*$ of trace class operators on $L^2(G)$ (cf., [9, p. 81]).

4. The replete case

Throughout this section, A will be a replete Hilbert algebra; that is, a Hilbert algebra which is a Banach subalgebra of A_b in the so-called Rieffel norm $||a||_r=||a||+|L_a|$ (see [16]). It is known [5, Cor. 4.2] that Hilbert algebra A is replete if and only if $A=\mathfrak{A}\cap A_b$ (set-theoretically). Although replete Hilbert algebras need not contain projections as full Hilbert algebras do, there is still a notion of summability associated with the trace class of such an algebra. In fact, a positive element of a replete Hilbert algebra A is in $\tau(A)$ if and only if it is integrable. For the definitions and properties of positive elements and integrable elements of H, the Hilbert space completion of A, consult [8].

THEOREM 7. *If a is a positive element of A, then the following statements are equivalent*:

(1) *a is integrable.*

(2) *a is in $\tau(A)$.*

(3) *$a=b^2$, for some positive b in A.*

(4) *$a=c^*c$, for some c in A.*

Proof. The implications (3)$\Rightarrow$(4)$\Rightarrow$(2)$\Rightarrow$(1) are immediate. To see that (1)$\Rightarrow$(3), first note that, by [6, Lemma 1.2], $a=b^2$, for some positive b in A_b. Thus, L_b is a positive operator in $\mathfrak{A}''$ whose square is the positive operator L_a in $\mathfrak{A}$. It follows that $L_b\in\mathfrak{A}$; hence, $b\in\mathfrak{A}\cap A_b=A$. □

In order to apply Theorem 7 to arbitrary elements of A, we show that each element of A admits a polar decomposition.

PROPOSITION 8. *If $a\in A$, then there exists a unique positive element $[a]$ in A such that $||[a]||=||a||$ and $L_{[a]}=[L_a]$, the positive square root of $L_a^*L_a$ in $\mathfrak{A}$. Further, there is a partial isometry W_a in $\mathfrak{A}''$ such that $a=W_a[a]$ and $[a]=W_a^*a$.*

Proof. As to the first statement, since $A\subseteq A_b$, the element a^*a has a unique positive square root $[a]$ in A_b with the properties described [6, p. 261]. Moreover, because $L_{[a]}=[L_a]$ is in $\mathfrak{A}$, $[a]\in\mathfrak{A}\cap A_b=A$. The uniqueness of $[a]$ also follows from the equality $L_{[a]}=[L_a]$ and the fact that the representation L of A is faithful. The second statement is contained in [6, Thm. 1.3]. □

Combining Theorem 7 and Proposition 8 yields the following results.

PROPOSITION 9. *An element a of A is in $\tau(A)$ if and only if $[a]$ is in $\tau(A)$.*

Proof. If $a \in \tau(A)$, then $a \in \tau(A_b)$; hence, by [6, Lemma 2.1], $[a]$ is integrable, and so is in $\tau(A)$ by Theorem 7. Conversely, suppose that $[a] \in \tau(A)$. Then $[a]=b^2$, for some positive b in A (Theorem 7), and $a=W_a[a]=W_a b^2=(W_a b)b$, for some partial isometry W_a in $\mathfrak{A}''$ (Proposition 8). Because $\mathfrak{A}''=M_L(A_b)$, $W_a b \in A_b$. In addition, however, $L_b=\lim_n [L_a]p_n([L_a])$ in $\mathfrak{A}$, where $\{p_n\}$ is a sequence of real polynomials (indeed, if $q_1(t)=t$, $q_{n+1}(t)=q_n(t)+(t-q_n(t)^2)/2$, all $n \geqq 1$, then $q_n(t) \nearrow \sqrt{t}$ for all t in $[0, 1]$; thus setting $p_n(t)=\sqrt{r}\, t^{-1}q_n(t/r)$, for each n, where $r=\|[L_a]\|$, defines such a sequence $\{p_n\}$). Consequently, $W_a L_b=\lim_n W_a[L_a]p_n([L_a])=\lim_n L_a p_n([L_a])$ is contained in $\mathfrak{A}$, and so $W_a b \in \mathfrak{A} \cap A_b=A$. Consequently, $a=(W_a b)b \in \tau(A)$. □

COROLLARY 10. *The trace class $\tau(A)$ of A is the set $\{ab: a, b \in A\}$, and $\tau(A)=\tau(A_b)\cap A$.*

Proof. That $\tau(A)=\{ab: a, b \in A\}$ is shown in the proof of the previous proposition. The inclusion $\tau(A)\subseteqq\tau(A_b)\cap A$ is immediate, and, for the reverse inclusion, if $a \in \tau(A_b)\cap A$, then $[a]$ is integrable [6, Lemma 2.1]. Thus, $[a]\in\tau(A)$ by Theorem 7, and so $a \in \tau(A)$ by Proposition 9. □

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