Comment Math.
Univ. St. Pauli
XXVIII—2, 1979

When a cyclic and acyclic family of sets can be nonlinear?

by

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(Received December 22, 1978)

Introduction

We consider in the paper a graph theoretical problem connected with consecutive one's property of a matrix, consecutive storage of records in a storage device and interval graphs. Almost all definitions and many results we shall use throughout the paper can be found in [4]. We recall them in section 1.

W. Lipski proved (see [4]) that every linear family of sets is simultaneously cyclic and acyclic. He conjectured the converse, i.e. that every family of sets which is cyclic and acyclic has to be linear. Surprisingly M. Wasowska found a family of sets (see the family $\mathfrak{N}_1$ in the example below) which was cyclic, acyclic and nonlinear. Another example of such a family (the family $\mathfrak{N}_2$) was found by the author.

EXAMPLE. Let $X=\{1, 2, 3, 4, 5\}$ and let $\mathfrak{N}_1$ be a family of subsets of X, $\mathfrak{N}_1=\{\{1, 2, 3\}, \{1, 3, 4, 5\}, \{1, 5\}, \{3, 4\}\}$.

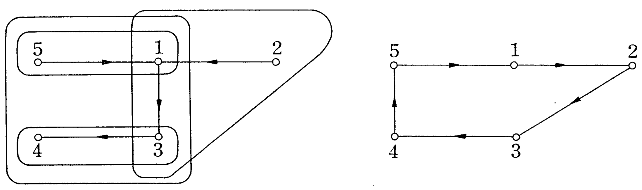

Fig. 1. The family $\mathfrak{N}_1$

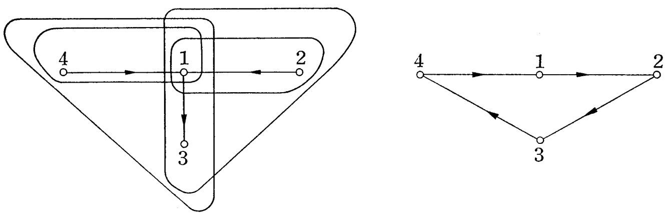

Fig. 2. The family $\mathfrak{N}_2$

Cyclic and acyclic organizations of $\mathfrak{N}_1$ are shown in Fig. 1. One can easily verify that $\mathfrak{N}_1$ is not linear.

For the second example let $X=\{1, 2, 3, 4\}$ and $\mathfrak{N}_2=\{\{1, 2, 3\}, \{1, 2\}, \{1, 3, 4\}, \{1, 4\}\}$. See Fig. 2.

One can easily verify that the deletion of any member of $\mathfrak{N}_1$ or even of $\cup\mathfrak{N}_1$ converts $\mathfrak{N}_1$ into linear family of sets. The same holds for the family $\mathfrak{N}_2$. It means that the families $\mathfrak{N}_1$ and $\mathfrak{N}_2$ are in certain special sense "minimal" between all cyclic acyclic and nonlinear (CAN for short) families of sets. We shall show that $\mathfrak{N}_1$ and $\mathfrak{N}_2$ are (up to isomorphism) the only two minimal CAN families of sets.

Throughout the paper we use the standard mathematical notation. For graph-theoretical notions not defined here see Harary [3].

1. Basic notions

In this section we recall some notions and results, mostly from Lipski [4].

Let X be a set and $S: X\to X$ be a partial function such that $S(x) \neq x$ for every $x \in \mathrm{Dom}\, S$. A pair $\langle X, S\rangle$ will be referrd to as an *f-graph* on X. S is then a *successor function* and $S(x)$ is a successor of x. A set $A\subseteq X$ is said to be a *segment* in $\langle X, S\rangle$ if either $A=\varnothing$ or there exists $x\in A$ such that $A=\{x, S(x), \cdots, S^{|A|-1}(x)\}$. Such an x is a *head* of A in $\langle X, S\rangle$ and $S^{|A|-1}(x)$ is the *end* corresponding to this head. If $S^{|A|}(x)=x$ then B is a *cycle*. If a nonempty segment A is not a cycle then its head and end are unique and denoted $h(A)$ and $e(A)$ respectively.

An *f*-graph on X is *cyclic* if X is a cycle in it, *linear* if X is a segment but not a cycle in it, and *acyclic* if no subset of X is a cycle in it.

A family of sets is *segmental* over an *f*-graph if every member of the family is a segment in the *f*-graph.

A family $\mathfrak{M}$ of subsets of a set X is said to be *linear* (cyclic, acyclic) if $\mathfrak{M}$ is segmental over certain linear (cyclic, acyclic) *f*-graph $\langle X, S\rangle$. We say that $\langle X, S\rangle$ realizes linearity (cyclicity, acyclicity) of $\mathfrak{M}$, or that $\langle X, S\rangle$ is a linear (cyclic, acyclic) organization of $\mathfrak{M}$.

We have the following obvious theorem:

THEOREM 1.1. *A family $\mathfrak{M}\subseteq\mathfrak{P}(X)$ is liner (cyclic, acyclic) iff every subfamily of $\mathfrak{M}$ is linear (cyclic, acyclic).*

Let's consider a family $\mathfrak{M}$ consisting of three sets, say M_1, M_2, M_3. Such a family is said to be a *triangle* if $\bar{M}_1\cap M_2\cap M_3\neq\varnothing$, $M_1\cap \bar{M}_2\cap M_3\neq\varnothing$ and $M_1\cap M_2\cap \bar{M}_3\neq\varnothing$, i.e., none of the sets M_1, M_2, M_3 contains the intersection of the other two (recall that $\bar{M}_i$ denotes $X\backslash M_i$). The following two theorems can be found in Lipski [4]:

THEOREM 1.2. *The family $\mathfrak{M}$ consisting of three sets is acyclic iff it is not a triangle.*

A nondirected graph is a *rigid circuit graph* if for each of its elementary cycles of length greater then 3 there is an edge joining two vertices of this cycle, joined by no edge of the cycle.

The *intersection graph* of a family $\mathfrak{M}$ is a nondirected graph with $\mathfrak{M}$ as a set of vertices, two different vertices M, N joined by an edge iff $M \cap N \neq \varnothing$.

THEOREM 1.3. *The intersection graph of an acyclic family of sets is rigid circuit graph.*

Now we are ready to introduce our results.

2. Results

First we shall prove a fundamental lemma:

LEMMA 2.1. *Let $\mathfrak{M}$ be a CAN family of subsets of a set X. There exist then two members of $\mathfrak{M}$, say M and N, such that the set $M \cap N$ is not a segment in any realization of cyclicity of $\mathfrak{M}$ and for every $K \in \mathfrak{M}$ we have $K \subseteq M$ or $K \subseteq N$.*

Proof. Let $\langle X, S\rangle$ be a realization of cyclicity of $\mathfrak{M}$. If for certain $x \in X$ there exists no set in $\mathfrak{M}$ containing both x and $S(x)$ then the family $\mathfrak{M}$ is linear. Hence, there exists a family $\mathfrak{N} \subseteq \mathfrak{M}$ such that:

(1) For every $x \in X$ there is a set in $\mathfrak{N}$ containing both x and $S(x)$.

(2) Every member of $\mathfrak{N}$ is maximal in $\mathfrak{M}$ (i.e. is not a proper subset of any member of $\mathfrak{M}$).

(3) For every family $\mathfrak{N}'$ satisfying conditions (1) and (2), $|\mathfrak{N}'| \geqslant |\mathfrak{N}|$.

Denote $|\mathfrak{N}| = k$. Since $\mathfrak{M}$ is acyclic and nonlinear, the set X cannot be a member of $\mathfrak{N}$, and therefore $k \geqslant 2$.

Let M_0 be a set from the family $\mathfrak{N}$. Since the family $\mathfrak{N}$ satisfies (1), there exists in $\mathfrak{N}$ a set M_1, different from M_0 and containing $e(M_0)$. If there exists another such a set, say M_1', then, since M_0, M_1, M_1' are segments in $\langle X, S\rangle$ and $\mathfrak{N}$ satisfies (2), we have $h(M_1), h(M_1') \in M_0$ and either $e(M_1) \in M_1'$ or $e(M_1') \in M_1$. If $e(M_1) \in M_1'$ then for every x such that $x, S(x) \in M_1$ we have $x, S(x) \in M_0$ or $x, S(x) \in M_1'$. Thus the family $\mathfrak{N} \setminus \{M_1\}$ satisfies (1) and (2). This implies that the family $\mathfrak{N}$ does not satisfy (3). The same contradiction arises in the case $e(M_1') \in M_1$. So there exists exactly one such a set M_1. Let M_2 be the unique set in $\mathfrak{N}$ different from M_1 and containing $e(M_1)$, etc. Clearly, M_i is the unique element of $\mathfrak{N}$ containing $h(M_{(i+1) \bmod k})$ and different from $M_{(i+1) \bmod k}$.

It follows from the above that, if $k>2$ then the intersection graph of $\mathfrak{N}$ is a cycle of length k, and no set in $\mathfrak{N}$ contains the intersection of two other sets from $\mathfrak{N}$. Using Theorems 1.2 and 1.1 we obtain $k\neq 3$. Furthermore, it follows from Theorems 1.3 and 1.1 that $k<3$, so we get $k=2$.

Let $\mathfrak{N}=\{M, N\}$. Since $e(M)\in N$ and $e(N)\in M$, the set $M\cap N$ is not a segment in the cyclic f-graph $\langle X, S\rangle$.

Suppose $K\in\mathfrak{M}$ is neither a proper subset of M nor of N. If $K\supseteq M\cap N$ then, since $M\cap N$ contains $e(M)$, $h(M)$, $e(N)$ and $h(N)$, K is a proper superset of M or of N, in contrary to (2). So K does not contain $M\cap N$. If $K\cap M\subseteq N$ or $K\cap N\subseteq M$ then $K\subseteq M$ or $K\subseteq N$, because of $M\cup N=X$. Thus none of K, M, N contains the intersection of the other two, i.e., the family $\{K, M, N\}$ is a triangle. But the this implies that the family $\mathfrak{M}$ is not acyclic.

It remains to prove that $M\cap N$ is not a segment in any realization of cyclicity of $\mathfrak{M}$. But one can easily verify that if $M\cap N$ is a segment in certain realization of cyclicity of $\mathfrak{M}$ then $\mathfrak{M}$ is linear in contrary to our assumption. Q.E.D.

Clearly, M and N are the only two maximal elements of the family $\mathfrak{M}$.

Let $\mathfrak{M}\subseteq\mathfrak{P}(X)$ and $\mathfrak{M}=\{M_1, M_2, \cdots, M_n\}$. For every $i=1, 2, \cdots, n$ denote $M_i^0=X\backslash M_i$ and $M_i^1=M_i$. Every set of the form $S_{\mathfrak{M}}(\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n)=M_1^{\varepsilon_1}\cap M_2^{\varepsilon_2}\cap\cdots\cap M_n^{\varepsilon_n}$, where $\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n\in\{0, 1\}$ is said to be a set-theoretical component of the family $\mathfrak{M}$. W. Lipski proved (see [4]) that the linearity (cyclicity, acyclicity) of a family of sets depends only on which components of the family are nonempty, and does not depend on the cardinalities of the components. We shall need the notion of weak similarity of families of sets, which is nothing but slight modification of that of similarity (see [4]).

DEFINITION 2.2. Let $\mathfrak{M}\subseteq\mathfrak{P}(X)$ and $\mathfrak{N}\subseteq\mathfrak{P}(Y)$. The family $\mathfrak{M}$ is *weakly similar* to $\mathfrak{N}$ if the members of $\mathfrak{M}$ and $\mathfrak{N}$ can be arranged in such a way that $S_{\mathfrak{N}}(\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n)\neq\emptyset$ implies $S_{\mathfrak{M}}(\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n)\neq\emptyset$.

In particular the families $\mathfrak{N}_1$ and $\mathfrak{N}_2$ are not weakly similar one to another.

THEOREM 2.3. *Let $\mathfrak{M}\subseteq\mathfrak{P}(X)$ be a cyclic and acyclic family of sets. Then $\mathfrak{M}$ is nonlinear iff $\mathfrak{M}$ contains a subfamily weakly similar to $\mathfrak{N}_1$ or to $\mathfrak{N}_2$*

Proof. The part "if" is obvious. To prove the part "only if" let's take certain realization $\langle X, R\rangle$ of the acyclicity of $\mathfrak{M}$. Let M and N be the only two maximal elements of $\mathfrak{M}$ (see Lemma 2.1).

Clearly the set $M\cap N$ and at least one of the sets $M\backslash N$ and $N\backslash M$

are segments in $\langle X, R\rangle$. Without loss of generality we may assume that $N\backslash M$ is the one. The set $M\backslash N$ is then the union of two segments, say K and L. Assume $K\neq\emptyset$. L can be empty or not; if it is then $M\backslash N$ is a segment too.

Denote the head and the end of a segment A in $\langle X, S\rangle$ with $h(A)$ and $e(A)$ respectively. Clearly $S(e(K))=S(e(N\backslash M))=h(M\cap N)$ and $S(e(M\cap N))=h(L)$ (provided $L\neq\emptyset$).

Consider the elements $e(K)$ and $h(M\cap N)$. If M is the only set in $\mathfrak{M}$ containing both of them, then setting $R'=(R\backslash\{(e(K), h(M\cap N))\})\cup\{(e(M), h(K))\}$ we obtain the linear f-graph $\langle X, R'\rangle$. Since $\mathfrak{M}$ is segmental over it and nonlinear, there exists a set A in $\mathfrak{M}$, different from M and containing both $e(K)$ and $h(M\cap N)$.

Since $M\cap N$ is not a segment in any realization of cyclicity of $\mathfrak{M}$, $M\cap N$ is not a subset of A and thus $e(M\cap N)\notin A$ and $L\cap A=\emptyset$. Clearly $A\cap N\backslash M$ is empty, too.

If $M\backslash N$ is a segment in $\langle X, R\rangle$ (i.e. $L=\emptyset$) then by similar consideration, there exists a set $B\in\mathfrak{M}$, different from N and containing both $e(N\backslash M)$ and $h(M\cap N)$. Furthermore $e(M\cap N)\notin B$ and $B\cap M\backslash N=\emptyset$. One can easily verify that the family $\{M, N, A, B\}$ is weakly similar to $\mathfrak{N}_2$.

Now suppose that $M\backslash N$ is not a segment in any realization of acyclicity of $\mathfrak{M}$. In particular we have $L\neq\emptyset$. Consider the f-graph $\langle X, R''\rangle$ where $R''=(R\backslash\{(e(M\cap N), h(L))\})\cup\{(e(L), h(K))\}$. Since $M\backslash N$ is a segment in $\langle X, R''\rangle$, $\mathfrak{M}$ cannot be segmental over it. Thus there exists a set $C\in\mathfrak{M}$ being a segment in $\langle X, R\rangle$ but not in $\langle X, R''\rangle$. Such a set C is different from M and contains both $e(M\cap N)$ and $h(L)$. Clearly the family $\{M, N, A, C\}$ is weakly similar to the family $\mathfrak{N}_1$.

Q.E.D.

The presence or absence of single-element sets in a family of sets has nothing to do with the segmentality of the family over any f-graph. Thus, for studying the segmentality of families of sets, we shall define the notion of isomorphism of families in the followig way:

DEFINITION 2.4. Two families, $\mathfrak{M}\subseteq\mathfrak{P}(X)$ and $\mathfrak{N}\subseteq\mathfrak{P}(Y)$ are *isomorphic* if there exists a bijection $\varphi\colon X\to Y$ such that for every $M\in\mathfrak{M}$, if $|M|>1$ then $\varphi(M)\in\mathfrak{N}$ and for every $N\in\mathfrak{N}$, if $|N|>1$ then $\varphi^{-1}(N)\in\mathfrak{M}$.

Let $\mathfrak{M}\subseteq\mathfrak{P}(X)$ and $C\subseteq X$. The *trace* of the family $\mathfrak{M}$ on C is the family $\mathfrak{M}|_C=\{M\cap C\colon M\in\mathfrak{M}\}$.

The following corollary is a simple cosequence of the Theorem 2.3.

COROLLARY 2.5. *Let $\mathfrak{M}\subseteq\mathfrak{P}(X)$ be a cyclic family of sets. Then $\mathfrak{M}$ is nonlinear iff for certain $C\subseteq X$, $\mathfrak{M}|_C$ is isomorphic to $\mathfrak{N}_1$ or to $\mathfrak{N}_2$.*

An algorithm for verifying whether a given family $\mathfrak{M}$ of subsets of a set X is cyclic, acyclic and nonlinear can be found in [6]. Its worst-case complexity has been shown to be $0(nk)$ where $|\mathfrak{M}|=n$ and $|X|=k$.

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