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On $p^{\omega+2}$-Projective p-Groups*

by

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A characterization of $p^{\omega+2}$-projective abelian p-groups is given, like quotients of $p^{\omega+1}$-projective groups modulo free subsocles with values less then ω. Moreover it is shown that $p^{\omega+2}$-projective groups with a special representation decompose into $A/F_1 \oplus T/F_2$, with A separable $p^{\omega+1}$-projective, T totally projective of length $\leqq \omega+1$ and F_1 and F_2 free subsocles with values less than ω.

Introduction

An abelian p-group G is said p^{σ}-projective, where σ is an ordinal, if p^{σ} Ext$(G, X)=0$ for every p-group X. Recently many authors have studied $p^{\omega+n}$-projective abelian p-groups. Fuchs and Irwin gave in [6] a completely satisfactory structure theorem for $p^{\omega+1}$-projective p-groups, by proving that if A is such a group, then A decomposes into the direct sum of a separable p-group and of a totally projective p-group of length $\leqq \omega+1$, and that two $p^{\omega+1}$-projective p-groups A and B are isomorphic if and only if the socles $A[p]$ and $B[p]$ are isomorphic in the category of valued vector spaces (see [4]); some other properties of $p^{\omega+1}$-projective p-groups are given in [6] and [2]. More recently Fuchs generalised in [5] one of the above results to the $p^{\omega+n}$-projective p-groups, by proving that if A and B are two $p^{\omega+n}$-projective p-groups, then A and B are isomorphic if and only if $A[p^n]$ and $B[p^n]$ are isomorphic in the category V_p^n of valued abelian p-groups annihilated by p^n; some more properties of $p^{\omega+n}$-projective p-groups can be found in [1], where, in particular, it is proved that a p-group G is $p^{\omega+n+1}$-projective if and only if G is isomorphic to H/S, where H is $p^{\omega+n}$-projective and S is a subsocle of $H[p]$. In this paper we make the first approach to the problem of decomposing $p^{\omega+2}$-projective p-groups. We introduce new subclasses of the class of $p^{\omega+2}$-projective p-groups, by means of the representation of a $p^{\omega+2}$-projective p-group G as a quotient B/S, where B is a direct sum of cyclic p-groups (denoted by $\oplus_C$), and $S \leqq B[p^2]$ (see [8]); in particular we call a $p^{\omega+2}$-projective p-group *quotient decomposable* if G has a representation $G \cong B/S$ with S a decomposable sub-p^2-socle of $B[p^2]$, i.e. S is a valued direct sum of a free object in V_p^2 and a free object in V_p^1.

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In this paper we prove that every quotient decomposable $p^{\omega+2}$-projective p-group G decomposes into the direct sum $G=A/F_1 \oplus T/F_2$, where A is separable $p^{\omega+1}$-projective, T is totally projective of length $\leqq \omega+1$, and F_1 and F_2 are free subsocles with values less than ω. Moreover we improve the result of [1] stated above, by proving that a p-group G is $p^{\omega+2}$-projective if and only if $G=X/F$ where X is $p^{\omega+1}$-projective and F is a free subsocle of $X[p]$ with values less than ω.

§1. In the following by "group" we shall always mean "reduced abelian p-group." For unexplained notation and terminology we refer to [3]. Let A be a group, Γ the class of ordinals, ∞ a symbol which is considered larger than any ordinal. We call valuation of A a function v: $A \to \Gamma \cup \{\infty\}$ such that:

1) $v(a)=\infty$ if and only if $a=0$,
2) $v(ma)=v(a)$ if m is prime to p; $v(pa)>v(a)$,
3) $v(a+b) \geqq \min\{v(a), v(b)\}$ for all $a, b \in A$.

It is important to observe that if $v(a) \neq v(b)$ we have $v(a+b)=\min\{v(a), v(b)\}$. A group with a valuation is called valuated group. If A, B are valuated groups, f: $A \to B$ is said to be a morphism of valuated groups if f is a group homomorphism and $v(a) \leqq v(f(a))$ for all $a \in A$. A morphism of valuated groups which is injective, surjective, and preserves the values is called an isometry. Let us notice that every group is valuated by the valuation h=height function, and that if B is a subgroup of the valuated group A, the restriction to B of the valuation of A induces a valuation on B. If $\{A_i\}_{i \in I}$ is a family of valuated groups, we call valuated direct sum of the A_i, and we indicate by

$$\overset{v}{\bigoplus_{i \in I}} A_i ,$$

the group

$$\bigoplus_{i \in I} A_i ,$$

with the valuation

$$v((a_i)_{i \in I})=\min{}_{i \in I} v(a_i) \qquad (a_i \in A_i) .$$

A group A is said to be p^n-bounded if $p^n A=0$.

We denote by V_p^n the category whose objects are the p^n-bounded valuated groups, and the morphisms between objects are the morphisms of valuated groups. The objects in V_p^1 are the valued vector spaces over the prime field with p elements, which have been studied by Fuchs in [4].

If $A \in V_p^n$ and $B \leqq A$, we define, following Richman and Walker [9], the quotient valuation on A/B as the minimal valuation which makes the canonical surjection π: $A \to A/B$ a morphism of valuated groups. Let us notice that, since A and B are p^n-bounded, the proofs of Lemma 2 and Theorem 3 of [9] become simpler. It is easy to verify that the quotient valuation coincides with the one given by Fuchs in [5]:

$$v(a+B)=\max\{\sup_{b\in B}v(a+b),\ h_{A/B}(a+B)\}\,.$$

A cyclic group $\langle a\rangle\in V_p^n$ is called fundamental if $v(x)=v(a)+h_{\langle a\rangle}(x)$ for all $x\in\langle a\rangle$ (see [5]).

Definition. $F\in V_p^n$ is said to be *free* if

$$F=\overset{v}{\bigoplus_{i\in I}}\langle a_i\rangle$$

where for all $i\in I$ $\langle a_i\rangle\cong Z(p^n)$ and $\langle a_i\rangle$ is fundamental.

F is free in the sense that every function f from the basis $\{a_i\}_{i\in I}$ to $G\in V_p^n$ such that $v(a_i)\leqq v(f(a_i))$ extends to a unique morphism $\bar f\in\mathrm{Hom}_{V_p^n}(F,G)$.

Definition. $A\in V_p^n$ is said to be *decomposable* if

$$A=\overset{v}{\bigoplus_{i\in I}}\langle a_i\rangle$$

with the $\langle a_i\rangle$ fundamental. We call $A\in V_p^n$ *completely decomposable* if

$$A=\overset{v}{\bigoplus_{1\leqq i\leqq n}}F_i$$

with F_i free in V_p^i for all i, and, if $i\leqq n-1$ and $x\in F_i\setminus\{0\}$, then $v(x)<i$.

In the investigation of the structure of $p^{\omega+2}$-projective groups, it is useful to study the properties of purification of the sub-p^2-socles of a $\bigoplus_C$. In the sequel we shall always consider, on the sub-p^2-socles of a group X, the valuation $v=h_X=$height function in X.

PROPOSITION 1. *Given $B=\bigoplus_C$ and $S\leqq B[p^2]$, S is purifiable,* i.e. *there exists a pure subgroup H of B such that $H[p^2]=S$, if and only if S is completely decomposable.*

Proof. Let

$$S=U\overset{v}{\oplus}C=\overset{v}{\bigoplus_{i\in I}}\langle u_i\rangle\overset{v}{\oplus}C$$

be completely decomposable, where U is free in V_p^2 and C is free in V_p^1 with $v(c)=0$ if $c\in C$, $c\neq 0$. For all $i\in I$, if $v(u_i)=n$, let $h_i\in B$ be such that $p^nh_i=u_i$ and let $H=\langle h_i\colon i\in I, C\rangle=\bigoplus_i\langle h_i\rangle\oplus C$; celarly $H[p^2]=S$ and the elements of $H[p^2]$ have the same height in H and in B, hence H is pure (see [3], § 26).

Conversely let H be pure in $B=\bigoplus_C$. Then also $H=\bigoplus_C$; so we can write

$$H=\bigoplus_{i\in I}\langle h_i\rangle\oplus C$$

where $ph_i\neq 0$ for all $i\in I$, and $pc=0$ for all $c\in C$. If h_i has order p^{n+2} ($n\geqq 0$) and $u_i=p^nh_i$, from H pure it follows easily that the $\langle u_i\rangle$ are fundamental for all $i\in I$, and

$$H[p^2]=\overset{v}{\bigoplus_{i\in I}}\langle u_i\rangle\overset{v}{\oplus}C\,.$$

A weaker property than the purification for a sub-p^2-socle is given by the following result.

PROPOSITION 2. *Given $B=\bigoplus_C$ and $S\leqq B[p^2]$, there exists a pure subgroup H in B such that $H[p]=S[p]$ and $H[p^2]\supseteq S$ if and only if S is decomposable.*

Proof. The proof of the sufficiency is like that of Proposition 1. Conversely, let

$$S[p]=\overset{v}{\bigoplus_{i\in I}}\langle s_i\rangle\overset{v}{\oplus}\overset{v}{\bigoplus_{j\in J}}\langle s'_j\rangle$$

where $h_S(s_i)=1$ for all $i\in I$, while $h_S(s'_j)=0$ for all $j\in J$. If $v(s_i)=n$, let $h_i\in H$ be such that $p^n h_i=s_i$; let $t_i=p^{n-1}h_i$ and

$$T=\overset{v}{\bigoplus_{i\in I}}\langle t_i\rangle .$$

Let $t\in T$; then

$$pt\in\overset{v}{\bigoplus_{i\in I}}\langle s_i\rangle ,$$

from which $pt=ps$, $s\in S\subseteq H[p^2]$; consequently $t-s\in H[p]=S[p]$, so that $T\subseteq S$. Let now $s\in S\backslash S[p]$. Then $ps=\sum m_i pt_i$ and $s-\sum m_i t_i\in S[p]$, so that

$$s\in T\overset{v}{\oplus}\overset{v}{\bigoplus_{j\in J}}\langle s'_j\rangle .$$

Now we have

$$S=T\overset{v}{\oplus}V$$

with T free in V_p^2 and

$$V=\overset{v}{\bigoplus_{j\in J}}\langle s'_j\rangle$$

free in V_p^1, i.e. S is decomposable.

From Proposition 2 one immediately deduces the following

COROLLARY. *Let F be a free object in V_p^2 and S a subobject of F. Then S is decomposable if $S[p]=F[p]$.*

Some important results on $p^{\omega+1}$-projective groups cannot be extended to $p^{\omega+2}$-projectives, because of the difficulties which arise in the passage from the valued vector spaces to p^2-bounded valuated groups.

For instance, Fuchs and Irwin in ([6], Th. 1) proved that if G is a $p^{\omega+1}$-projective and $P\leqq G[p]$ is such that $G/P=\bigoplus_C$, then

$$G[p]=P\overset{v}{\oplus}S$$

where S is a free valued vector space with values less than ω.

Let now

$$P=\langle x\rangle \overset{v}{\oplus} P' \in V_p^2$$

with values less than ω and $px=0$, $v(x)=m>0$. Theorem 1 of [5] ensures that there exists a $p^{\omega+2}$-projective group G such that P is a sub-p^2-socle of G and $G/P=\bigoplus_C$. It is easy to check that P cannot be a direct summand of $G[p^2]$, hence the result cannot be extended.

Definition. A $p^{\omega+2}$-projective group G is said *quotient-free* (resp. *quotient-completely decomposable, quotient-decomposable*) if G has a representation $G\cong B/S$ with $B=\bigoplus_C$ and $S\leqq B[p^2]$ free (resp. completely decomposable, decomposable) in V_p^2.

PROPOSITION 3. *A group G is quotient-free if and only if it is quotient-completely decomposable.*

Proof. Let $G=B/S$ with $B=\bigoplus_C$ and $S\leqq B[p^2]$ completely decomposable. Then

$$S=U\overset{v}{\oplus} C$$

with U free in V_p^2 and C free in V_p^1 with $v(c)=0$ for all $c\in C$, $c\neq 0$. From Proposition 27.1 of [3] it follows that C is a direct summand of B; consequently

$$G\cong B/S=(B'\oplus C)/(U\overset{v}{\oplus} C)=B'/U$$

is quotient-free.

We notice that from Theorem 2.3 of [1] it follows that every $p^{\omega+2}$-projective group G is a direct summand of a quotient-free group G', and its complement in G' is a direct sum of cyclic groups. We give now a characterization of the p^2-socle of quotient-free $p^{\omega+2}$-projective groups.

PROPOSITION 4. *Let G be a $p^{\omega+2}$*-projective group, *$G\cong B/S$ with $B=\bigoplus_C$ and S a free subobject of $B[p^2]$. Let H be a pure subgroup of B such that $H[p^2]=S$. Then*

$$G[p^2]\cong B[p^2]/S\overset{v}{\oplus} H[p^4]/S\,.$$

Proof. It is easy to verify that $G[p^2]\cong B[p^2]/S\oplus H[p^4]/S$. We show that the direct sum is valuated. Let us observe that $H[p^4]/S$ has values less than ω. Let $v(h+x+S)\geqq n$, $h\in H[p^4]$, $x\in B[p^2]$. It is enough to check that $v(h+S)\geqq n$. We have $h+x+s=p^n y$ with $s\in S$, $y\in B$. Then $p^2h=p^{n+2}y=p^{n+2}h'$ with $h'\in H$ for H is pure; consequently $h-p^n h'\in H[p^2]=S$ from which $h+S=p^n h'+S$.

Let us observe that Proposition 4 generalise the decomposition of the socle of the $p^{\omega+1}$-projective groups, which are always quotient-free, because $H[p^4]/S$ is obviously free in V_p^2.

§2. The following theorem will allow us to use the structure of $p^{\omega+1}$-projective groups to study the structure of the $p^{\omega+2}$-projectives. We recall that an exact sequence in V_p^1

$$0\longrightarrow X\longrightarrow Y\overset{\pi}{\longrightarrow} Z\longrightarrow 0\,,$$

is said to be nice if for all $z \in Z$ there is $y \in \pi^{-1}(z)$ such that $v(y) = v(z)$; the free objects in V_p^1 are precisely those valued vector spaces which are projective for every nice exact sequence.

THEOREM 1. *A group G is $p^{\omega+2}$-projective if and only if $G \cong X/F$ with X $p^{\omega+1}$-projective and F a free subsocle of X with values less than ω.*

Proof. Theorem 2.4 of [1] ensures that X/F is $p^{\omega+2}$-projective if X is $p^{\omega+1}$-projective and $F \leqq X[p]$.

Conversely let $G \cong B/S$ with $B = \bigoplus_C$ and $S \leqq B[p^2]$. Let $X = B/S[p]$ and $F = S/S[p]$. Then $G \cong X/F$ and X is $p^{\omega+1}$-projective by Theorem 2.4 of [1]. We show that F is free. As $P = B[p]/S[p]$ is such that $X/P = \bigoplus_C$, by Theorem 1 of [6] we have

$$X[p] = P \overset{v}{\bigoplus} L$$

with L free valued vector space with values less than ω. Let g be the projection of $X[p]$ onto L and let g' be the restriction of g to F. Let us consider the exact sequence of valued vector spaces

$$0 \longrightarrow \ker g' \longrightarrow F \longrightarrow \operatorname{Im} g' \longrightarrow 0.$$

Im g' is free with values less than ω, because it is a subspace of L, therefore it is projective in V_p^1 (see [4]). Furthermore the exact sequence is nice because Im g' has values less than ω.

Consequently

$$F \cong \ker g' \overset{v}{\bigoplus} \operatorname{Im} g';$$

but $\ker g' = F \cap P = S/S[p] \cap B[p]/S[p] = 0$ and so F is isometric to Im g'.

Let $G = X/F$ be like in Theorem 1. It is easy to check that

$$G[p^2] = X[p^2]/F + H[p^3]/F$$

where H is pure in X, $H[p] = F$. Then we have

$$H[p^3]/H[p^2] \cong (H[p^3]/F)/(X[p^2]/F \cap H[p^3]/F) \cong (X[p^2]/F + H[p^3]/F)/(X[p^2]/F).$$

It is easy to verify that these isomorphisms are isometries for the valuations induced by the heights respectively in $X/H[p^2]$, $(X/F)/(H[p^2]/F)$, $(X/F)/(X[p^2]/F)$. Furthermore let $F^n = \{f \in F : v(f) \geqq n\}$ and let $F^{(-2)}$ be the vector space F^2 with the valuation v' defined by: $v'(f) = v(f) - 2$ for all $f \in F^2$. Then map $h + H[p^2] \mapsto p^2 h$ is an isometry between $H[p^3]/H[p^2]$ and $F^{(-2)}$; the above arguments prove the following

PROPOSITION 5. *If $Y \in V_p^2$ is the p^2-socle of a $p^{\omega+2}$-projective group, then there is an exact sequence in V_p^2*

$$0 \longrightarrow X/F \longrightarrow Y \longrightarrow F^{(-2)} \longrightarrow 0,$$

where F is a free valued vector space with values less than ω, and X is the p^2-socle of a $p^{\omega+1}$-projective group.

This is a partial extension to the $p^{\omega+2}$-projective groups of Theorem 4 of [6]. The converse is not true, because the sequence

$$0 \longrightarrow X[p^2]/F \longrightarrow (X/F)[p^2] \longrightarrow F^{(-2)} \longrightarrow 0$$

never splits.

Fuchs and Irwin in [6] proved that a $p^{\omega+1}$-projective group X is the direct sum of a separable $p^{\omega+1}$-projective group A and of a totally projective group T of length less or equal to $\omega+1$. Since G $p^{\omega+2}$-projective implies $G \cong X/F$ with X $p^{\omega+1}$-projective, it is natural to ask if, for a suitable decomposition $X = A \oplus T$, we have

$$X/F = A/(A \cap F) \oplus T/(T \cap F) .$$

This decomposition arises if G is quotient-decomposable.

Recall that if F is a free valued vector space with values less than β, we have

$$F = \overset{v}{\bigoplus_{\alpha<\beta}} F_\alpha$$

where, for all $\alpha < \beta$, F_α is a homogeneous subspace of value α, called the homogeneous component of value α.

THEOREM 2. *Let G be $p^{\omega+2}$-projective, $G \cong B/S$ with $B = \oplus_C$, $S \leqq B[p^2]$, S decomposable. Then $G = A/F_1 \oplus T/F_2$ where A is a separable $p^{\omega+1}$-projective group, F_1 is a free subsocle of A, T is totally projective of length $\leqq \omega+1$ and F_2 is a free subsocle of T with values less than ω.*

Proof. If

$$S = S_1 \overset{v}{\oplus} V$$

with S_1 free in V_p^2 and V free in V_p^1, let $H = H' \oplus H''$ with H' pure in B such that $H'[p^2] = S_1$, H'' pure such that $H''[p] = V$. $S/S[p] = S_1/S_1[p]$ and

$$H[p^2]/S[p] = S_1/S_1[p] \overset{v}{\oplus} H''[p^2]/H''[p]$$

so that $F = S/S[p]$ is a valuated direct summand of $H[p^2]/S[p]$.

Let $X = B/S[p]$, $P = B[p]/S[p]$, $P' \leqq P$ of minimal cardinality such that $X/P' = \oplus_C$. Applying Theorem 2 of [6] we have

$$P = P' \overset{v}{\oplus} P''$$

with P'' free with values less than ω. By Lemma 3 of [7]

$$X[p] = P \overset{v}{\oplus} H[p^2]/S[p] = P' \overset{v}{\oplus} P'' \overset{v}{\oplus} H''[p^2]/H''[p] \overset{v}{\oplus} F = P' \overset{v}{\oplus} E \overset{v}{\oplus} F$$

where

$$E = P'' \overset{v}{\oplus} H''[p^2]/H''[p]$$

is free with values less than ω.

Let

$$E = \overset{v}{\bigoplus}{}_n E_n, \quad F = \overset{v}{\bigoplus}{}_n F_n$$

with E_n, F_n homogeneous components, and let

$$\bar{S} = \overset{v}{\bigoplus}_n S_n$$

with

$$S_n = E_n \overset{v}{\oplus} F_n$$

for all natural n.

From the minimality of $|P'|$, it follows $|\bar{S}^n| \geqq |P'| \cdot \aleph_0$ for all $n \in N$ (see [6]). Consequently it must exist a sequence $n_1 < n_2 < \cdots < n_k < \cdots$ such that if $a_k = |S_{n_k}|$ we have $a_k \geqq \aleph_0$ and $\sum_k a_k \geqq |P'|$.

If $|E_{n_k}| = |F_{n_k}| = a_k$ let $S'_{n_k} = F_{n_k}$, $S''_{n_k} = E_{n_k}$.

If $|E_{n_k}| < a_k$, let $F_{n_k} = F'_{n_k} \overset{v}{\oplus} F''_{n_k}$ with $|F'_{n_k}| = |F''_{n_k}| = a_k$, and let $S'_{n_k} = F'_{n_k}$, $S''_{n_k} = E_{n_k} \overset{v}{\oplus} F''_{n_k}$.

If $|F_{n_k}| < a_k$, let $E_{n_k} = E'_{n_k} \overset{v}{\oplus} E''_{n_k}$ with $|E'_{n_k}| = |E''_{n_k}| = a_k$ and let $S'_{n_k} = E'_{n_k} \overset{v}{\oplus} F_{n_k}$, $S''_{n_k} = E''_{n_k}$.

Let

$$S' = \overset{v}{\bigoplus}_{n \neq n_k} S_n \overset{v}{\oplus} \overset{v}{\bigoplus}_k S'_{n_k}, \quad S'' = \overset{v}{\bigoplus}_k S''_{n_k}.$$

Let

$$P' = P'_\omega \overset{v}{\oplus} P'^\omega$$

where $P'^\omega = \{x \in P' : v(x) = \omega\}$. Then

$$P'_\omega \overset{v}{\oplus} S'$$

is the socle of a separable $p^{\omega+1}$-projective group A, and

$$P'^\omega \overset{v}{\oplus} S''$$

is the socle of a totally projective group T of length $\omega + 1$ (this is proved in [6]). Then from

$$F = (A \cap F) \overset{v}{\oplus} (T \cap F)$$

it follows

$$X/F = A/(A \cap F) \oplus T/(T \cap F).$$

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