COMMENTARII MATHEMATICI
UNIVERSITATIS SANCTI PAULI
Vol. 29, No. 1, 1980

On a Subclass of Close-to-Convex Functions

by

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(Received February 20, 1980)

Let S, S^*, K and C be the classes of univalent, starlike, close-to-convex and convex functions. The class Q of quasi-convex univalent functions has been defined and studied in [1]. A function f is analytic in the unit disc E with $f(0)=0$ and $f'(0)=1$ belongs to Q if and only if there exists a convex univalent function g such that for $z\in E$ and $g(0)=0$, $g'(0)=1$,

$$\mathrm{Re}\frac{(zf'(z))'}{g'(z)}>0. \tag{1}$$

It is also established in [1] that $f\in Q$ if and only if $zf'\in K$. The geometric interpretation of $f\in Q$ is that zf' maps each circle $|z|=r<1$ onto a simple closed curve whose tangent rotates, as θ increases, either in the counterclockwise direction or clockwise direction, in such a way that it never turns back on itself so much as to completely reverse its direction. It has been shown in [1] that the class C of convex functions forms a proper subclass of Q, and Q itself is included in K.

Waadeland [6] proved that every starlike m-fold symmetric function g, with

$$g(z)=z+\sum_{k=1}^{\infty} b_{mk+1}z^{mk+1}$$

satisfies

$$|b_{mk+1}|\leqq\binom{2/m+k-1}{k}, \qquad k=1, 2, \cdots.$$

Since $g=zf'$ is starlike if f:

$$f(z)=z+\sum_{k=1}^{\infty} a_{mk+1}z^{mk+1}$$

is convex, it follows that

$$\begin{aligned}|a_{mk+1}|&\leqq\frac{1}{mk+1}\binom{2/m+k-1}{k}\\ &\simeq\frac{1}{m\Gamma(2/m)}k^{2/m-2}. \end{aligned}\tag{2}$$

In order to extend this result to Q, we need only to extend Waadeland's result to K and then use the relationship between Q and K. However this was done by

Pommerenke [3] and so (2) is true for $f \in Q$.

In [2], Livingstone has shown that if $F \in C$, S^* or K, then f defined for $z \in E$ by $f(z) = \frac{1}{2}[zF(z)]'$ is also in the same class for $|z| < 1/2$. The constant 1/2 cannot be improved. We prove this result for the class Q.

THEOREM 1. *Let $F \in Q$ and for $z \in E$, define $f(z) = \frac{1}{2}[zF(z)]'$. Then $f \in Q$ for $|z| < 1/2$. The constant 1/2 is best possible.*

Proof. Since $F \in Q$, there exists a function $G \in C$ such that for $z \in E$,

$$\operatorname{Re} \frac{(zF'(z))'}{G'(z)} > 0 .$$

Let $g(z) = \frac{1}{2}[zG(z)]$, and consider

$$\frac{(zf'(z))'}{g'(z)} = \frac{[z\{(zF(z))'\}']'}{[(zG(z))']'} . \tag{3}$$

Now

$$\begin{aligned} [z\{(zF(z))'\}']' &= zF''(z) + 2F'(z) + z[zF'''(z) + 3F''(z)] \\ &= (zF'(z))' + F'(z) + z[(zF'(z))']' + zF''(z) \\ &= z((zF'(z))')' + (zF'(z))' + zF''(z) + F'(z) \\ &= z((zF'(z))')' + 2(zF'(z))' \end{aligned}$$

Let $zF'(z) = H(z)$, then from (3), we have

$$\frac{(zf'(z))'}{g'(z)} = \frac{((zH(z))')'}{((zG(z))')'} .$$

Since $H \in K$, Livingstone's results show that $\frac{1}{2}(zH(z))' \in K$ for $|z| < 1/2$. Thus for $|z| < 1/2$,

$$\operatorname{Re} \frac{(zf'(z))'}{g'(z)} > 0 ,$$

i.e., $f \in Q$ for $|z| < 1/2$. The function $f(z) = z/(1-z)$ shows that 1/2 is best possible.

The following result for the calss Q follows on the similar lines as in [7] for the class C.

THEOREM 2. *Let $f \in Q$ with $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, and $g(z) \prec f(z)$. Then for all n,*

$$S_n(\tfrac{1}{2}z) \prec f(z) ,$$

where

$$S_n(z) = z + \sum_{k=2}^{n} b_k z^k \quad \text{and} \quad g(z) = z + \sum_{k=2}^{\infty} b_k z^k .$$

("$\prec$" means "subordinate to").

Robertson [4] introduced the class C_1 of convex functions in one direction. These are the functions for which the intersection of the image region with each line of a certain fixed direction is either empty or one interval. He has also shown that if f has real coefficients, then $f \in C$ if and only if $zf' \in T$, where T is the class of typically real zf', that is the functions with real coefficients.

THEOREM 3. *If $f \in Q$ in E and has real coefficients, then it is convex in one direction.*

Proof. Let $Q(R)$, $K(R)$ and $C_1(R)$ be the classes of functions which are in Q, K and C_1 respectively having real coefficients.

Let $f \in Q(R)$. This implies that $zf' \in K(R)$. But $K(R) \subset T$. Hence $zf' \in T$ and so $f \in C_1(R)$. Thus it follows that $Q(R) \subset C_1(R)$.

Remark. From the Theorem 3 and the results in [4], the following results follow immediately.

1. Let $f \in Q(R)$. Then $(f(z)/z) > 1/2$ for $z \in E$ and $f(z)/z$ is subordinate to $(1+z)^{-1}$.

2. Let $f \in Q(R)$ and $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$. Then

$$\frac{1+|a_2|r}{1+2|a_2|r+r^2} \leqq \operatorname{Re}\left\{\frac{f(re^{i\theta})}{re^{i\theta}}\right\} \leqq \frac{1+|a_2|r}{1-r^2}$$

3. Let $f \in Q(R)$. If f maps $|z| = r$ onto contour C_r whose length is $L(r)$, then $L(r) \leqq 2\pi r/(1-r^2)$. The equality is obtained for $f(z) = z/(1-z)$.

4. Let $f \in Q(R)$. Then for $z \in E$,
   (i) $|\arg(f(z)/z)| \leqq \arcsin|z|$,
   (ii) $|\arg f'(z)| \leqq 2 \arcsin|z|$.

5. Let $f \in Q(R)$ for $z \in E$. Then so is the function

$$F(z) = \int_0^1 f(tz)d\phi(t) = z + \sum_{n=2}^{\infty} \mu_n a_n z^n ,$$

where $\phi(t)$ is any real function monotone increasing in the interval (0, 1) and the movements sequence $\{\mu_n\}$ is given by

$$\mu_n = \int_0^1 t^n d\phi(t) , \qquad \mu_1 = 1 .$$

It is well known [5] that

$$\operatorname{Re}\frac{(zf'(z))'}{f'(z)} > 0 \implies \operatorname{Re}\frac{zf'(z)}{f(z)} > \frac{1}{2} \quad \text{in } E ,$$

i.e., every convex function is starlike of order 1/2. It is natural to ask if such a relationship exists between Q and K. The following example shows that this is not in fact the case.

Example. Take $f(z) = z$, $g(z) = z/(1-\alpha z)$; $1/2 < \alpha < 1/\sqrt{2}$. Then

$$\operatorname{Re}\frac{(zf'(z))'}{g'(z)}=\operatorname{Re}(1-\alpha)^2>0, \quad z\in E,$$

but

$$\operatorname{Re}\frac{zf'(z)}{g(z)}=\operatorname{Re}(1-\alpha z)$$

and so

$$\operatorname{Inf}\operatorname{Re}\frac{zf'(z)}{g(z)}<\frac{1}{2} \qquad \text{for} \quad \frac{1}{2}<\alpha<\frac{1}{\sqrt{2}}, \quad z\in E.$$

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