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Note on Free Subsets

by

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In this note we shall give some remarks on free subsets.

If X is a set, $|X|$ denots its cardinality, and if X is a set and κ is a cardinal, we set:

$$[X]^{<\kappa}=\{Y \mid Y\subset X \text{ and } |Y|<\kappa\}.$$

Let κ, λ be cardinals. We say $\mathrm{Fr}(\kappa, \lambda)$ iff for all $f\colon [\kappa]^{<\omega}\to[\kappa]^{<\omega}$ there is $X\subset\kappa$ such that $|X|=\lambda$ and for all $s\in[X]^{<\omega}$ $f(s)\cap X\subset s$. If $\mathrm{Fr}(\kappa, \lambda)$, we say that κ has a free subset of cardinality λ. Let κ be a cardinal. We say $J(\kappa)$ iff for all $f\colon [\kappa]^{<\omega}\to\kappa$ there is $X\subset\kappa$ such that $|X|=\kappa$ and $f''[X]^{<\omega}\neq\kappa$. If $J(\kappa)$, we call κ a Jonsson cardinal.

PROPOSITION 1. *Let κ, λ be cardinals.* $\mathrm{Fr}(\kappa, \lambda)$ *iff for all* $f\colon [\kappa]^{<\omega}\to[\kappa]^{<\omega}$ *there is* $X\subset\kappa$ *such that*

(1) $|X|=\lambda$,

(2) *for all* $s\in[X]^{<\omega}$ *if* $f(s)\cap X\neq\phi$ *then* $f(s)\cap s\neq\phi$.

Proof. From left to right is obvious. We shall prove right to left. Let $f\colon [\kappa]^{<\omega}\to[\kappa]^{<\omega}$. We define a function $g\colon [\kappa]^{<\omega}\to[\kappa]^{<\omega}$ as follows:

$$g(s)=f(s)-s \quad (s\in[\kappa]^{<\omega}).$$

Then by hypothesis there is $X\subset\kappa$ such that $|X|=\lambda$ and for all $s\in[X]^{<\omega}$ if $g(s)\cap X\neq\phi$ then $g(s)\cap s\neq\phi$. By definition of g for all $s\in[X]^{<\omega}$ $g(s)\cap s=(f(s)-s)\cap s=\phi$. So for all $s\in[X]^{<\omega}$ we must have $g(s)\cap X=\phi$. Hence we have for all $s\in[X]^{<\omega}$,

$$f(s)\cap X\subset(g(s)\cup s)\cap X=(g(s)\cap X)\cup(s\cap X)=\phi\cup s=s.$$

This means $\mathrm{Fr}(\kappa, \lambda)$.

PROPOSITION 2. *If not* $\mathrm{Fr}(\kappa, \kappa)$, *then not* $\mathrm{Fr}(\kappa^+, \kappa^+)$.

Proof. Let not $\mathrm{Fr}(\kappa, \kappa)$. For each γ such that $\kappa\leqq\gamma<\kappa^+$, we have $f_\gamma\colon [\gamma]^{<\omega}\to[\gamma]^{<\omega}$ such that for all $X\subset\gamma$, $|X|=|\gamma|=\kappa$ there is $s\in[X]^{<\omega}$ such that $f_\gamma(s)\cap X\neq\phi$ and $f_\gamma(s)\cap s=\phi$. Define $f\colon [\kappa^+]^{<\omega}\to[\kappa^+]^{<\omega}$ as follows:

$$f(s)=\begin{cases} f_\gamma(s-\{\gamma\}), & \text{if } s\in[\kappa^+]^{<\omega} \text{ and } \gamma=\max(s)\\ \phi, & \text{otherwise.}\end{cases}$$

Assume $\mathrm{Fr}(\kappa^+, \kappa^+)$. Then there is $X\subset\kappa^+$ such that $|X|=\kappa^+$ and for all $s\in[X]^{<\omega}$ if

$f(s)\cap X\neq\phi$ then $f(s)\cap s\neq\phi$. Since cardinality of X is κ^+, there is $\alpha\in X$ such that $\kappa\leqq\alpha<\kappa^+$ and $|X\cap\alpha|=|\alpha|=\kappa$. Let Y be $X\cap\alpha$. Then there is $t\in[Y]^{<\omega}$ such that $f_\alpha(t)\cap Y\neq\phi$ and $f_\alpha(t)\cap t=\phi$. But since $\max(t\cup\{\alpha\})$ is α, $f(t\cup\{\alpha\})=f_\alpha(t)\not\ni\alpha$. Hence $f(t\cup\{\alpha\})\cap X\neq\phi$ and $f(t\cup\{\alpha\})\cap(t\cup\{\alpha\})=\phi$. But this contradicts $t\cup\{\alpha\}\in[X]^{<\omega}$. So not Fr (κ^+, κ^+).

Next is easily proved by model theoretic definitions of Fr (κ, κ) and $J(\kappa)$. (see Devlin [1]). Here we give another proof by Proposition 1.

PROPOSITION 3. *If* Fr (κ, κ), *then* $J(\kappa)$.

Proof. Let $f\colon[\kappa]^{<\omega}\to\kappa$. Define $g\colon[\kappa]^{<\omega}\to[\kappa]^{<\omega}$ as follows.

$$g(s)=\{f(s)\}\ (s\in[\kappa]^{<\omega}).$$

Then there is $X\subset\kappa$ such that $|X|=\kappa$ and for all $s\in[X]^{<\omega}$ if $g(s)\cap X\neq\phi$ then $g(s)\cap s\neq\phi$. Hence for all $s\in[X]^{<\omega}$ if $\{f(s)\}\cap X\neq\phi$ then $\{f(s)\}\cap s\neq\phi$, so $f(s)\in s$. Assume that for each $\alpha\in X$ there is $s\in[X]^{<\omega}$ such that $f(s)=\alpha$. If $\alpha\in X$, $s\in[X]^{<\omega}$ and $f(s)=\alpha$, then $f(s)=\alpha\in s$. Let Y be $X-\{\alpha\}$. Then $|Y|=|X|=\kappa$ and for all $s\in[Y]^{<\omega} f(s)\neq\alpha$. So $f''[Y]^{<\omega}\neq\kappa$. If there is $\alpha\in X$ such that for all $s\in[X]^{<\omega} f(s)\neq\alpha$. Then $f''[X]^{<\omega}\neq\kappa$. Hence $J(\kappa)$.

Reference

[1] DEVLIN, K. J.; Some weak versions of large cardinal axioms, Annals of Mathematical Logic 5 (1973).

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