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On Differential Polynomials

by

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Abstract. Let $f(z)$ be a non-constant meromorphic function in the complex plane, and let $m(r,f)$, $N(r,f)$, $T(r,f)$ have the usual meaning of Nevanlinna theory. Let $S(r,f)$ denote any quantity satisfying $S(r,f)=o(T(r,f))$ as $r\to\infty$ except possibly for a set of r of finite linear measure. Let $a(z)$ be a meromorphic function in the plane satisfying $T(r, a(z))=S(r,f)$ as $r\to\infty$. A finite sum of the form

$$a(z)(f(z))^{l_0}(f'(z))^{l_1}(f''(z))^{l_2}\cdots(f^{(k)}(z))^{l_k}$$

denoted by $P_n(f)$, where $l_0+l_1+l_2+\cdots+l_k\leqq n$ is called a differential polynomial of degree atmost n. And if for all the terms constituting $P_n(f)$, $l_0+l_1+\cdots+l_k=n$, then $P_n(f)$ is called a homogeneous differential polynomial of degree n. In this paper, using Nevanlinna theory some properties of differential polynomials have been deduced and these properties have been used for the study of differential equations involving meromorphic functions, their derivatives and differential polynomials.

1. Let f be a meromorphic function and not constant in the complex plane and let $m(r,f)$, $N(r,f)$ $T(r,f)$ have the usual meaning of Nevanlinna theory. We shall denote by $S(r,f)$ any quantity satisfying $S(r,f)=o(T(r,f)$ as $r\to\infty$ except possibly for a set of r of finite linear measure. Throughout this paper $a=a(z)$, a_0, a_1, etc. will denote meromorphic functions in the complex plane satisfying $T(r, a)=S(r,f)$ etc. as $r\to\infty$. By a differential polynomial $P_n(f)$ we shall mean a finite sum of the form

$$a(z)(f(z))^{l_0}(f'(z))^{l_1}(f''(z))^{l_2}\cdots(f^{(k)}(z))^{l_k}$$

where $l_0+l_1+\cdots+l_k\leqq n$. If $l_0+l_1+\cdots+l_k=n$ for all terms constituting $P_n(f)$, then we call $P_n(f)$ a homogeneous differential polynomial of degree n. In case

$$P_n(f)=a(z)(f(z))^{l_0}(f'(z))^{l_1}\cdots(f^{(k)}(z))^{l_k}$$

with $l_0+l_1+\cdots+l_k=n$, $P_n(f)$ will be called a monomial of degree n. In this paper using Nevanlinna theory we prove a number of theorems on differential polynomials.

2. We prove

THEOREM 1. *No transcendental meromorphic function f with $N(r,f)=S(r,f)$*

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can satisfy an equation

$$a_1(z)(f(z))^n P(f) + a_2(z)P(f) + a_3 = 0 \tag{1}$$

where $a_1(z) \not\equiv 0$, *n is positive integer and* $P(f)$ *is a monomial of degree* $\geqq 1$.

To prove the theorem we need the following.

LEMMA 1. *Suppose that* $f(z)$ *is meromorphic and transcendental in the plane and that*

$$(f(z))^n P(z) = Q(z)$$

where $P(z)$, $Q(z)$ *are differential polynomials in* $f(z)$ *and the degree of* $Q(z)$ *is atmost n. Then*

$$m(r, P(z)) = S(r, f) \quad as \quad r \to \infty .$$

LEMMA 2. *If f is meromorphic and not constant in the plane, if* $g(z) = (f(z))^n + P_{n-1}(f)$, *where* $P_{n-1}(f)$ *is a differential polynomial of degree atmost* $n-1$ *in f and if*

$$N(r, f) + N\left(r, \frac{1}{g}\right) = S(r, f) ,$$

then $g(z) = (h(z))^n$, $h(z) = f(z) + (1/n)a(z)$ *and* $(h(z))^{n-1}a(z)$ *is obtained by substituting* $h(z)$ *for* $f(z)$, $h'(z)$ *for* $f'(z)$ etc. *in the terms of degree* $n-1$ *in* $P_{n-1}(f)$.

For the proves of these lemmas see [1, 68], [1, 69].

Proof of Theorem 1. We first consider the case $n \geqq 2$. Suppose (1) holds. Clearly $a_3 \not\equiv 0$, for otherwise either f is a polynomial or $T(r, f) = S(r, f)$ and both of the above are not possible.

Now from (1) we get

$$(f)^n + \frac{a_2}{a_1} = -\frac{a_3}{a_1 P(f)} = G(z) \quad \text{say} .$$

Then

$$N\left(r, \frac{1}{G}\right) = S(r, f) .$$

Also $N(r, f) = S(r, f)$. Therefore by Lemma 2, $G = (f)^n$, which yields $a_2 \equiv 0$. Thus the equation (1) becomes

$$(f)^n P(f) = -\frac{a_3}{a_1}$$

and hence

$$T(r, (f)^n P(f)) = S(r, f) . \tag{2}$$

Now let

$$\phi=(f)^n P(f)$$
$$=(f)^n(f)^{l_0}(f')^{l_1}\cdots(f^{(k)})^{l_k} \tag{3}$$

then

$$f^{n+l_0+l_1+\cdots+l_k}=\phi\left(\frac{f}{f'}\right)^{l_1}\left(\frac{f}{f''}\right)^{l_2}\cdots\left(\frac{f}{f^{(k)}}\right)^{l_k}.$$

Therefore by Nevanlinna's first fundamental theorem we obtain

$$\begin{aligned}(n+l_0+l_1+\cdots+l_k)T(r,f)&\leqq T(r,\phi)+l_1T\left(r,\frac{f'}{f}\right)+l_2T\left(r,\frac{f''}{f}\right)\\&+\cdots+l_kT\left(r,\frac{f^{(k)}}{f}\right)+S(r,f)\\&=l_1N\left(r,\frac{f'}{f}\right)+l_2N\left(r,\frac{f''}{f}\right)+\cdots+\\&l_kN\left(r,\frac{f^{(k)}}{k}\right)+S(r,f)\end{aligned} \tag{4}$$

using (2), (3) and a result of Milloux [1, 55]. But $N(r,f)=S(r,f)$, and so

$$\begin{aligned}N\left(r,\frac{f^{(k)}}{f}\right)&\leqq N(r,f^{(k)})+N\left(r,\frac{1}{f}\right)\\&\leqq T(r,f)+S(r,f).\end{aligned}$$

Therefore from (4)

$$(n+l_0+l_1+\cdots+l_k)T(r,f)\leqq(l_1+l_2+\cdots+l_k)T(r,f)+S(r,f).$$

Hence $(n+l_0)T(r,f)=S(r,f)$ which is not possible. We now consider the case $n=1$. Let

$$F=f+\frac{a_2}{a_1},$$

then $P(f)=Q(F)$ where $Q(F)$ is a differential polynomial in F. Then (1) can be written as

$$FQ(F)=-\frac{a_3}{a_1}$$

and hence by Lemma 1

$$\begin{aligned}m(r,Q(F))&=S(r,F)\\&=S(r,f)\end{aligned}$$

Also $N(r,Q(F))=S(r,f)$ and so $T(r,Q(F))=S(r,f)$, from which it follows that

$T(r,f)=S(r,f)$ a contradiction. This proves the theorem.

C. C. Yang [2] has stated the following:

THEOREM 2. *Let $f(z)$ be a transcendental meromorphic function with $N(r,f)=S(r,f)$. Then*

$$T(r, (f)^k+a_1\pi_{k-1}(f)+a_2\pi_{k-2}(f)+\cdots+a_k) \\ =kT(r,f)+S(r,f) \tag{5}$$

where $\pi_i(f)$ are homogeneous differential polynomials of degree i.

In his proof of the above theorem he uses the assertion that $N(r,f)=S(r,f)$ implies $T(r, \phi/f)=S(r,f)$ where

$$\phi(z)=\sum_{i=0}^{k} a_i(z)f^{(i)}(z), \qquad f^{(0)}=f,$$

which is obviously wrong. A very simple counter example is the following. Let $f(z)=\sin z$, $\phi=f+f'$, then $N(r,f)=S(r,f)$ *and* $T(r, \phi/f)\sim T(r,f)$. However Theorem 2 is correct. We give below a correct proof of the above theorem on the lines of C. C. Yang and deduce an interesting result from it.

Proof of Theorem 2. Let

$$\begin{aligned}\phi&=f^k+a_1\pi_{k-1}(f)+a_2\pi_{k-2}(f)+\cdots+a_k\\ &=f^k\left\{1+\frac{a_1\pi_{k-1}(f)}{f^k}+\frac{a_2\pi_{k-2}(f)}{f^k}+\cdots+\frac{a_k}{f^k}\right\}\\ &=f^k\left\{1+\frac{A_1}{f}+\frac{A_2}{f^2}+\cdots+\frac{A_k}{f^k}\right\}\end{aligned} \tag{6}$$

where

$$A_i=\frac{a_i\pi_{k-i}(f)}{f^{k-i}} \qquad i=1, 2, \cdots, k.$$

Since $\pi_n(f)$ is a homogeneous differential polynomial of degree n, it is a finite sum of the form

$$(f)^{l_0}(f')^{l_1}(f'')^{l_2}\cdots(f^{(k)})^{l_k}$$

where $l_0+l_1+l_2+\cdots+l_k=n$.
Hence

$$\begin{aligned}m\left(r, \frac{\pi_n(f)}{f^n}\right)&=m\left(r, \frac{\sum(f)^{l_0}(f')^{l_1}(f'')^{l_2}\cdots(f^{(k)})^{l_k}}{f^n}\right)\\ &=m\left(r, \sum\left(\frac{f'}{f}\right)^{l_1}\left(\frac{f''}{f}\right)^{l_2}\cdots\left(\frac{f^{(k)}}{f}\right)^{l_k}\right)\end{aligned}$$

$$\leqq \sum m\left(r, \left(\frac{f'}{f}\right)^{l_1}\left(\frac{f''}{f}\right)^{l_2}\cdots\left(\frac{f^{(k)}}{f}\right)^{l_k}\right)+O(1)$$

$$=S(r, f) \qquad \text{by Milloux's theorem [1, 55].} \tag{7}$$

Hence for all $i=1, 2, \cdots, k$ it follows that $m(r, A_i)=S(r,f)$. Now on the circle $|z|=r$, let

$$A(re^{i\theta})=\text{Max}\ \{|A_1(re^{i\theta})|, |A_2(re^{i\theta})|^{1/2}, \cdots, |A_k(re^{i\theta})|^{1/k}\} \tag{8}$$

then

$$m(r, A(z))=S(r,f)\ .$$

Let

$$E_1=\{\theta\in[0, 2\pi] \quad : \quad |f(re^{i\theta})|>2A(re^{i\theta})\}$$

and let E_2 be the complementary set. Then on E_1 we have

$$|\phi|=|f|^k\left|1+\frac{A_1}{f}+\frac{A^2}{f^2}+\cdots+\frac{A_k}{f^k}\right|$$

$$\geqq |f|^k\left\{1-\left|\frac{A_1}{f}\right|-\left|\frac{A_2}{f^2}\right|-\cdots-\left|\frac{A_k}{f^k}\right|\right\}$$

$$\geqq |f|^k\left\{1-\frac{1}{2}-\frac{1}{2^2}-\cdots-\frac{1}{2^k}\right\}$$

$$=\frac{1}{2^k}|f|^k\ .$$

Hence on E_1

$$k\log^+|f|\leqq k\log 2+\log^+|\phi|\ . \tag{9}$$

And so

$$km(r, f)=\frac{1}{2\pi}\int_0^{2\pi} k\log^+|f|d\theta$$

$$=\frac{1}{2\pi}\int_{E_1} k\log^+|f|d\theta+\frac{1}{2\pi}\int_{E_2} k\log^+|f|d\theta$$

$$\leqq\frac{1}{2\pi}\int_{E_1}(k\log 2+\log^+|\phi|)d\theta+\frac{k}{2\pi}\int_{E_2}\log^+|2A|d\theta$$

$$\leqq\frac{1}{2\pi}\int_0^{2\pi}\log^+|\phi|d\theta+\frac{k}{2\pi}\int_0^{2\pi}\log^+|A|d\theta+O(1)$$

$$=m(r, \phi)+km(r, A)+O(1)$$
$$=m(r, \phi)+S(r,f) \quad \text{by} \quad (8).$$

Thus

$$km(r,f) \leqq m(r, \phi)+S(r,f).$$

Adding $kN(r,f)$ on both the sides and recalling that $N(r,f)=S(r,f)$ we get

$$kT(r,f) \leqq m(r, \phi)+S(r,f)$$

and hence

$$kT(r,f) \leqq T(r, \phi)+S(r,f). \tag{10}$$

Next from (6)

$$\begin{aligned} m(r, \phi) &= m(r, f^k+A_1f^{k-1}+\cdots+A_{k-1}f+A_k) \\ &\leqq m(r, f^k+A_1f^{k-1}+\cdots+A_{k-1}f)+m(r, A_k)+O(1) \\ &\leqq m(r, f(f^{k-1}+A_1f^{k-2}+\cdots+A_{k-1}))+S(r,f) \\ &\leqq m(r,f)+m(r, f^{k-1}+A_1f^{k-2}+\cdots+A_{k-1})+S(r, f). \end{aligned}$$

Proceeding by induction we obtain

$$m(r, \phi) \leqq km(r,f)+S(r,f).$$

Hence

$$T(r, \phi) \leqq km(r,f)+N(r, \phi)+S(r,f).$$

But $N(r, \phi)=S(r,f)$ since $N(r,f)$ is so, and therefore

$$T(r, \phi) \leqq kT(r,f)+S(r,f). \tag{11}$$

From (10) and (11) we get the desired result.

COROLLARY *If $\pi_i(f)$ are homogeneous differential polynomials of degree i, then the differential equation*

$$(f'(z))^m=(f(z))^n+a_1(z)\pi_{n-1}(f)+a_2(z)\pi_{n-2}(f)+\cdots+a_n(z) \tag{12}$$

cannot have a transcendental entire solution if $n-1 \geqq m$. And if $n-1 \geqq 2m$ then (12) *cannot have a transcendental meromorphic solution with $N(r,f)=S(r,f)$. Further if $n-1=m$ and if the solution is transcendental meromorphic then it must have an infinity of poles.*

Proof. Let $n-1 \geqq m$. Suppose f is transcendental entire function satisfying (12). Then by Theorem 2,

$$T(r, f^n+a_1\pi_{n-1}(f)+\cdots+a_n)=nT(r,f)+S(r,f) \tag{13}$$

Also

$$\begin{aligned} T(r,(f')^m) &= mT(r,f') \\ &= m\,m(r,f') \qquad \text{since } f \text{ is entire} \\ &\leqq m\,m\left(r,\frac{f'}{f}\right)+m(r,f) \\ &\leqq m\,T(r,f)+S(r,f) \qquad \text{by Milloux's theorem}. \end{aligned}$$

And therefore

$$T(r,(f')^m) \leqq mT(r,f)+S(r,f). \tag{14}$$

From (13) and (14) we obtain

$$nT(r,f)+S(r,f) \leqq mT(r,f)+S(r,f), \quad \text{a contradiction}$$

since $n-m \geqq 1$.

Next let $n-1 \geqq 2m$ and suppose f is transcendental meromorphic function with $N(r,f)=S(r,f)$ satisfying (12), then

$$\begin{aligned} T(r,(f')^m) &= mT(r,f') \\ &\leqq m\{2T(r,f)+S(r,f)\} \\ &= 2mT(r,f)+S(r,f). \end{aligned} \tag{15}$$

Hence from Theorem 2 and (15) we get

$$(n-2m)T(r,f)=S(r,f), \qquad \text{a contradiction}.$$

Finally let $n-1=m$, and let equation (12) have a transcendental meromorphic solution f. If possible let f have a finite number of poles, then $N(r,f)=S(r,f)$. Let

$$E_1=\{\theta\in[0,2\pi] \;:\; |f(re^{i\theta})|\leqq 1\}$$

and let E_2 be the complementary set. Then

$$\begin{aligned} m(r,f) &= \frac{1}{2\pi}\int_{E_1}\log^+|f|\,d\theta+\frac{1}{2\pi}\int_{E_2}\log^+|f|\,d\theta \\ &= \frac{1}{2\pi}\int_{E_2}\log^+|f|\,d\theta \end{aligned} \tag{16}$$

since on E_1, $|f|\leqq 1$.

Now from (12) we have

$$f^n=(f')^m-a_1\pi_{n-1}(f)-a_2\pi_{n-2}(f)-\cdots-a_n.$$

Hence

$$f=\frac{(f')^m}{f^{n-1}}-\frac{a_1\pi_{n-1}(f)}{f^{n-1}}-\frac{a_2\pi_{n-2}(f)}{f^{n-1}}-\cdots-\frac{a_n}{f^{n-1}}.$$

Hence

$$|f|\leqq\frac{|f'|^m}{|f|^{n-1}}+\frac{|a_1||\pi_{n-1}(f)|}{|f|^{n-1}}+\frac{|a_2||\pi_{n-2}(f)|}{|f|^{n-2}}\frac{1}{|f|}+\cdots+|a_n|\frac{1}{|f|^{n-1}}$$

$$=\left|\frac{f'}{f}\right|+|a_1|\left|\frac{\pi_{n-1}(f)}{f^{n-1}}\right|+|a_2|\left|\frac{\pi_{n-2}(f)}{f^{n-2}}\right|\frac{1}{|f|}+\cdots+|a_n|\frac{1}{|f|^{n-1}}.$$

Now on E_2, $|f|>1$ and hence on E_2

$$|f|\leqq\left|\frac{f'}{f}\right|^m+|a_1|\left|\frac{\pi_{n-1}(f)}{f^{n-1}}\right|+|a_2|\left|\frac{\pi_{n-2}(f)}{f^{n-2}}\right|+\cdots+|a_n|.$$

Therefore

$$\frac{1}{2\pi}\int_{E_2}\log^+|f|\,d\theta\leqq\frac{1}{2\pi}\int_{E_2}\log^+\left|\frac{f'}{f}\right|^m d\theta+\frac{1}{2\pi}\int_{E_2}\log^+|a_1|\,d\theta$$

$$+\frac{1}{2\pi}\int_{E_2}\log^+\left|\frac{\pi_{n-1}(f)}{f^{n-1}}\right|d\theta+\cdots+\frac{1}{2\pi}\int_{E_2}\log^+|a_n|\,d\theta$$

$$\leqq m\frac{1}{2\pi}\int_0^{2\pi}\log^+\left|\frac{f'}{f}\right|d\theta+\frac{1}{2\pi}\int_0^{2\pi}\log^+|a_1|\,d\theta$$

$$+\frac{1}{2\pi}\int_0^{2\pi}\log^+\left|\frac{\pi_{n-1}(f)}{f^{n-1}}\right|d\theta+\cdots+\frac{1}{2\pi}\int_0^{2\pi}\log^+|a_n|\,d\theta$$

$$=m\,m\left(r,\frac{f'}{f}\right)+m(r,a_1)+m\left(r,\frac{\pi_{n-1}(f)}{f^{n-1}}\right)+\cdots+m(r,a_n)$$

$$=S(r,f) \tag{17}$$

by (7) and Milloux's theorem.

From (16) and (17) we obtain $m(r,f)=S(r,f)$. But $N(r,f)=S(r,f)$ and hence $T(r,f)=S(r,f)$, a contradiction.

Note. The condition $N(r,f)=S(r,f)$ is essential in the second part of the theorem, for there do exist transcendental meromorphic functions satisfying (12) with $n-1\geqq 2m$, for instance, $f(z)=\tan(z)$ is a solution of the differential equation $f'=2f^3+f^2+(2f-f'')+1$ which satisfies the other conditions of the theorem.

In case $n-1=m$, the solution can be transcendental meromorphic, for instance, if we take $m=1$, $n=2$, then the differential equation $f'=f^2+1$ has $f(z)=\tan z$ for its solution. Let us note that $\tan z$ has infinity of poles.

THEOREM 3. *The differential equation*

$$a_1(z)(f(z))^n f'(z)+\pi_{n-1}(f)=0 \tag{18}$$

where $a_1(z)\not\equiv 0$ *has no transcendental meromorphic solution* $f(z)$ *satisfying*

$N(r,f)=S(r,f)$, where $\pi_{n-1}(f)$ is a homogeneous differential polynomial of degree $n-1$.

Proof. Suppose there exists a transcendental meromorphic function f satisfying (18) such that $N(r,f)=S(r,f)$, then

$$(f)^n f' = -\frac{\pi_{n-1}(f)}{a_1}.$$

Hence by Lemma 1, $m(r,f')=S(r,f)$. Also $N(r,f')\leqq 2N(r,f)=S(r,f)$. Therefore

$$T(r,f')=S(r,f).$$

Also from (18) we get

$$(f)^n = -\frac{\pi_{n-1}(f)}{a_1 f'}$$

and hence by Nevanlinna's first fundamental theorem

$$\begin{aligned} nT(r,f) &\leqq T(r,\pi_{n-1}(f))+T(r,f')+T(r,a_1)+O(1) \\ &= T(r,\pi_{n-1}(f))+S(r,f) \qquad \text{by (19)}. \end{aligned}$$

Also since $N(r,f)=S(r,f)$, we have

$$\begin{aligned} T(r,\pi_{n-1}(f)) &= m(r,\pi_{n-1}(f))+S(r,f) \\ &\leqq (n-1)m(r,f)+S(r,f) \qquad \text{as in (7)} \\ &= (n-1)T(r,f)+S(r,f) \end{aligned}$$

and so

$$nT(r,f)\leqq(n-1)T(r,f)+S(r,f), \qquad \text{a contradiction}.$$

References

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