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Imbedded N-High Subgroups of Abelian Groups*

by

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1. Introduction

A subgroup H of an abelian group G is imbedded if its Z-adic topology coincides with the relative one inherited from G. The elementary properties of these subgroups are treated in [7] and [8]. In particular, if $l\colon Z^+ -> Z^+$ (Z^+ denotes the non-negative integers) then H is l-imbedded (written $H <_l G$) if $H \cap l(n)G \subset nH$, for $n \in Z^+$. This notion is clearly a generalization of purity. In this paper, we examine the extent of this generalization in several ways.

In § 2, we determine those subgroups N of an abelian group for which all N-high subgroups are l-imbedded, and specify the smallest such l. This extends results of Pierce [9].

In § 3, we answer the same question, removing the restriction of a particular imbedding function l.

In § 4, we treat the question dually, and establish an interesting property of the neat hulls of a subgroup.

Finally, in § 5, we restrict our attention to primary abelian groups, and show that it is possible for a non-purifiable subsocle to support an imbedded subgroup. In fact, we show that there exist groups which are not pure-complete, but in which every subsocle supports an imbedded subgroup. We hope to study these "imbedded-complete" groups in a future paper.

Throughout, "group" means reduced abelian group, and the notation is that of [2]. In particular, $h_p(x)$ denotes the p-height of x, and we omit the subscript if the context is clear. We use $h_G(x)$ to denote the height of x in the group G.

2. l-Centers and l-cores

Definition 2.1. Let N be a subgroup of G and $l\colon Z^+ -> Z^+$. N is an *l-center* in G if $H <_l G$ whenever H is N-high in G. If l is the pointwise smallest such function, N is called an *l-core*.

We note first that the concept of imbeddedness can be localized to a prime p by

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considering the p-adic topology, and that H is imbedded if and only if it is p-imbedded for every prime p. In this case we write $H<_pG$ and note that there is then a function $l_p: Z^+ -> Z^+$ such that $H \cap p^{l_p(n)}G \subset p^nG$, for $n \in Z^+$. We can now similarly define the l_p-centers and l_p-cores of a group, and proceed to reduce to the local case.

Definition 2.2. A function $l: Z^+ -> Z^+$ is *multiplicative* if $l(mn)=l(m)l(n)$ whenever $(n, m)=1$, and the prime divisors of $l(n)$ are among those of n, for all n.

Note that a multiplicative function l determines functions l_p by

$$l(p_1^{r_1} \cdots p_k^{r_k})=p_1^{l_{p_1}(r_1)} \cdots p_k^{l_{p_k}(r_k)}$$

The proof of the following is similar to that of Prop. 1.2 of [8], but is included for completeness.

PROPOSITION 2.3. *Let G be a group, N a subgroup of G and $l: Z^+ -> Z^+$. If N is an l-core in G, then l is multiplicative.*

Proof. Suppose $(p, n)=1$ and $l(n)=p^rt$, where $(p, t)=1$. If $r>0$, then $t<l(n)$, so there is an N-high subgroup H with $H \cap G \not\subset nH$. But if $x \in H \cap tG$, then

$$p^rx \in H \cap p^rtG=H \cap l(n)G \subset nH .$$

Thus if a, $b \in Z$ satisfy $an+bp^r=1$, we have $x=anx+bp^rx \in nH$, so $H \cap tG \subset nH$. Thus $r=0$, so $(l(n), p)=1$.

Now if $(n, m)=1$, we have $(l(n), l(m))=1$, so if H is N-high in G,

$$\begin{aligned} H \cap l(n)l(m)G &= (H \cap l(n)G) \cap (H \cap l(m)G) \\ &\subset nH \cap mH \\ &= nmH . \end{aligned}$$

Thus by minimality of l, $l(nm) \leqq l(n)l(m)$.

On the other hand, since $(n, m)=1$, and the prime divisors of $l(nm)$ are among those of nm, we can find integers r and s with $l(nm)=rs$ and $(n, s)=1$. Choose a, $b \in Z$ such that $as+bn=1$, and let H be N-high in G. Then if $x \in H \cap rG$, we have

$$sx \in H \cap rsG=H \cap l(nm)G \subset nmH \subset nH .$$

Then $x=asx+bnx \in nH$, so $l(n) \leqq r$. Similarly, $l(m) \leqq s$, so $l(n)l(m) \leqq rs=l(nm)$.

PROPOSITION 2.4. *Let N be a subgroup of G, and l a multiplicative function on Z^+ determining functions l_p. Then N is an l-core if and only if it is an l_p-core for every prime p.*

Proof. If N is an l-core, it is clearly an l_p-center for every p. If N is an $\bar{l}_p$-core and $\bar{l}_p(n)<l_p(n)$, then for all N-high H, $H \cap p^{\bar{l}_p(n)}G \subset p^nH$, where $p^{\bar{l}_p(n)}<p^{l_p(n)}=l(p^n)$, contradicting minimality of l.

Conversely, if $n=p_1^{r_1} \cdots p_k^{r_k}$, and H is N-high,

$$\begin{aligned} H \cap l(n)G &= H \cap (p_1^{l_{p_1}(r_1)} \cdots p_k^{l_{p_k}(r_k)})G \\ &= \bigcap_{i=1}^{k} (H \cap p^{l_{p_i}(r_i)}G) \end{aligned}$$

$$\subset \bigcap_{i=1}^{k} p_i^{r_i}H = nH .$$

Thus H is an l-center, so an $\bar{l}$-core, for $\bar{l} \leqq l$. Then $\bar{l}$ is multiplicative, determining $\bar{l}_p$. Of course, $\bar{l}_p \leqq l_p$ and since N is an $\bar{l}_p$-center, $l_p \leqq \bar{l}_p$. Thus $l_p = \bar{l}_p$, for all p, so $l = \bar{l}$.

We thus need only determine the l_p-cores of a group, and in this case, l_p is trivially nondecreasing. Further, Moore [7] has noted that the minimal p-imbedding function l_p of a subgroup H is stationary at n [i.e. $l_p(n) = l_p(n+1)$] only if p^nH is p-divisible. We will therefore adopt Moore's convention and take all functions strictly increasing. We treat the primary case first.

LEMMA 2.5. *Let N be a subgroup of the p-group G, and l_p a strictly increasing function on Z^+. Suppose there exist $x \in N$, $g \in G$, $m \in Z$, $m \geqq 1$, such that*
i) $O(g) > O(x)$
ii) $\langle g \rangle \cap N = 0$
iii) $h(g) = h(x) < m-1 \leqq l_p(m-1) \leqq l_p(m) - 1 \leqq h(g+x)$.
Then N is not an l_p-center.

Proof. Similar to the Lemma in [9].

Since H is N-high if and only if H is $N[p]$-high, we need only consider subsocles of G. Further, it is well known that if $N \subset P_\infty = \{x \in G[p] \| h(x) = \infty\}$, then all N-high subgroups are pure, so we may disregard this case.

THEOREM 2.6. *Let G be a p-group and N a subsocle of G with $N \not\subset P_\infty$. Then N is an l_p-center for some $l_p : Z^+ -> Z^+$ if and only if $P_m = \{x \in G[p] \| h(x) \geqq m\} \subset N$, for some $m \in Z^+$.*

Proof. If $P_m \subset N$, define $l_p(i) = m + i$, for $i \in Z^+$. Then if H is N-high in G,

$$(H \cap p^{l_p(i)}G)[p] \subset H \cap P_m \subset H \cap N = 0 ,$$

so $H <_{l_p} G$. Conversely, if $P_m \not\subset N$, for all m, and l_p is strictly increasing, there are elements $g_k \in G$, $k > 0$, with
i) $O(g_k) = p^2$
ii) $\langle g_k \rangle \cap N = 0$
iii) $h(g_k) \geqq k$.
Then taking $x \in N$ with $h(x) = t < \infty$, we have

$$h(x) = h(x - g_{l_p(t+2)}) < t + 1 \leqq l_p(t+1) \leqq h(g_{l_p(t+2)}) ,$$

so N is not an l_p-center.

THEOREM 2.7. *Let G be a p-group, and suppose $P_{k+r} \subset N \subset P_k$, where k is the largest and r the smallest such integer. Then N is an l_p-core, where*

$$l_p(i) = \begin{cases} i \quad , & i \leqq k+1 \\ \max(i, k+r), & i=k+2 \\ l_p(i-1)+1 \quad , & i \geqq k+3 \end{cases}$$

Proof. N is an l_p-core, for some l_p, and if H is N-high in G, H is p^{k+1}-pure. If $r \leqq 2$, then N is a center of purity by [9], Thm. 1. If $r > 2$, and $l_p(k+2) < k+r$, then there is a $g \in G$ with

i) $O(g) = p^2$

ii) $\langle g \rangle \cap N = 0$

iii) $h(g) \geqq k+r-2 > k$.

Then taking $x \in N$ of height $k < \infty$, we have

$$h(x) = h(x-g) < k+1 \leqq l_p(k+1) \leqq l_p(k+2) - 1 \leqq k+r < h(g),$$

so N is not an l_p-center. Hence $l_p(k+2) \geqq k+r$, but

$$(H \cap p^{l_p(k+2)}G)[p] \subset H \cap P_{k+r} = 0,$$

so $l_p(k+2) \leqq k+r$. The rest follows since l_p is strictly increasing.

For completeness, we note the extension to arbitrary groups.

THEOREM 2.8. *Let $l_p: Z^+ \to Z^+$ be strictly increasing, and N a subgroup of G. Then N is an l_p-center in G if and only if*

1) *$T(N)$ is an l_p-center in $T(G)$*

2) *Either G/N is torsion or $N[p] \subset p^\omega G$.*

Proof. Similar to Thm. 2 of [9].

Finally, since condition 2 of 2.8 is independent of l_p, the theorem holds for l_p-cores as well.

3. *p*-Adic centers

In this section, we remove the restriction of a particular imbedding function l_p.

Definition 3.1. Let N be a subgroup of a group G. N is a *Z-adic* (*p-adic*) *center* in G if all N-high subgroups of G are imbedded (p-imbedded).

Again, to characterize the Z-adic centers of a group, we need only consider the local case. We begin with primary groups.

LEMMA 3.2 *Let G be a p-group and N a subsocle of G with $N \not\subset P_\infty$. If there is a subsocle H of G disjoint from N containing nonzero elements of arbitrarily large height, then N is not a p-adic center.*

Proof. Let $x \in N$ have finite height m. If $H \cap P_\infty = 0$, we can inductively choose $g_i \in H$ as follows:

i) $h(g_1) > m+1$

ii) $h(g_i) > h(g_{i-1})$ and $g_1 \cdots g_i$ are independent.
Let $H' = \langle g_i | i > 0 \rangle$. We may decompose $G[p] = K \oplus N$, where K contains H' and a complementary summand to $P_{m+1} \cap N$ in P_{m+1}. For each i, choose x_i with $px_i = g_i$ and $h(x_i) = h(g_i) - 1$. Let $L = \langle x_i - x | i > 0 \rangle$ and $M' = < K, L >$. Then $M' \cap N = 0$, and $L[p] = H' \subset K$, so we may take M N-high containing M'. Say $M <_{l_p} G$ and choose i so that $h(g_i) > l_p(m+2)$. If $h_M(g_i) \geqq m+2$, take $y_i \in M$ with $p^{m+2} y_i = g_i = p(x_i - x)$. Then

$$g = p^{m+1} y_i - (x_i - x) \in M \cap G[p] = K .$$

Now $h(x_i) \geqq m+1$, so $x_i = p^{m+1} z_i$, $z_i \in G$. Then

$$p^{m+1}(z_i - y_i) = x_i - (g + x_i - x) = x - g \in P_{m+1} ,$$

and hence $x \in P_{m+1}$, a contradiction. The case $0 \neq g \in H \cap P_\infty$ is similar.

THEOREM 3.3 *Let G be a p-group, and N a subsocle of G with $N \not\subset P_\infty$. The following are equivalent:*

(1) *N is a p-adic center in G*

(2) *If $H \cap N = 0$ then $\bar{H}[p] \subset H \oplus N$, where $\bar{H}$ is the closure of H in the p-adic topology*

(3) *$P_\infty \subset N$ and $(P_k + N)/N$ is finite for some k*

(4) *All N-high subgroups are of bounded height.*

Proof. $(1 => 2)$: Let $H \cap N = 0$ and suppose $0 \neq x \in \bar{H}[p] \sim (H \oplus N)$. Then $\langle H, x \rangle \cap N = 0$, since if $ax + h = y \neq 0$, $a \in Z$, $h \in H$, $y \in N$, then $(a, p) = 1$, so $x \in H \oplus N$. However, we can find $h_i \in H$, for all i, so that $x + h_i \in p^i G$, so $\langle x, H \rangle [p]$ contains elements of arbitrarily large height, and N is not a p-adic center.

$(2 => 3)$: If $x \in P_x \sim N$, $x \in \bar{0}[p]$, but $x \notin 0 + N$. Also, if $(P_k + N)/N$ is infinite for all k, we can find elements $g_i \in P_i$ which are independent mod N. Let $H = \langle g_1 - g_i | i > 1 \rangle$. Then $g_1 \in \bar{H}[p]$ and since we can decompose $G[p] = H \oplus \langle g_1 \rangle \oplus N \oplus K$, $g_1 \notin H \oplus N$.

$(3 => 4)$: Let H be N-high of unbounded height. Since $H[p] \cap P_\infty = 0$, we can inductively choose $g_i \in H[p]$ such that $h(g_i) > i$ *and* $g_1 \cdots g_i$ are independent. Since $H \cap N = 0$, they are independent mod N, so in fact $(P_k + N)/N$ is infinite for all k.

$(4 => 1)$: If H is of bounded height, $H \cap p^k G = 0$, for some k, so $H <_{l_p} G$, where $l_p(i) = i + k$.

COROLLARY 3.4. *Let G be a p-group, and $N \subset G[p]$ a p-adic center in G with $N \not\subset P_\infty$. Then N is an l_p-center for some l_p if and only if N is closed in $G[p]$.*

Proof. It is well known that

$$(G[p]/N) \cap p^\omega(G/N) = \bigcap_{k>0} (P_k + N)/N .$$

If N is an l_p-center, $P_m \subset N$, for some m, so

$$\bigcap_{k>0} (P_k + N)/N \subset (P_m + N)/N = 0 ,$$

and N is closed. Conversely, if $(P_m+N)/N$ is finite, so is $(P_j+N)/N$ for $j>m$. Thus if

$$\bigcap_{k>0} (P_k+N)/N=0, \quad \text{then} \quad (P_r+N)/N=0,$$

for some r, and $P_r \subset N$.

As before, we now extend our characterization to arbitrary groups.

THEOREM 3.5. *Let N be a subgroup of G. Then N is a p-adic center in G if and only if*

1) *$T(N)$ is a p-adic center in $T(G)$*
2) *Either $N[p] \subset p^{\omega}G$ or*
   i) *$\langle \bar{H}, N\rangle / \langle H, N\rangle$ is torsion whenever $H \cap N=0$*
   ii) *G/N has finite torsion-free rank.*

Proof. For sufficiency, suppose 1) holds. If $N[p] \subset p^{\omega}G$, then N is a center of purity (cf. [9]). Suppose then that 2i) and 2ii) hold, and take H N-high but not p-imbedded. Then we can find an integer k and elements $g_n \in (H \cap p^nG) \sim p^kH$, for $n>0$. We may assume that $O(g_n)=\infty$. We claim that we can choose an infinite independent subset of $\{g_n | n>0\}$. If not, let $\{g_{n_1} \cdots g_{n_m}\}$ be maximally independent and let

$$K=\bigoplus_{i=1}^{m} \langle g_{n_i}\rangle \subset H.$$

If $K \nless_p G$, we can find an integer r and elements

$$h_j \in K'=\bigoplus_{i=2}^{m} \langle g_{n_i}\rangle$$

with $rg_{n_1}+h_j \in p^jG$, so $rg_{n_1} \in \bar{K}'$, violating 2i). Suppose then that $K<_{l_p}G$. If $t>l_p(k)$ and $(m, p)=1$, then $mg_t \notin K$, since then

$$g_t \in K \cap p^{l_p(k)}G \subset p^kH.$$

Now for each j with $l_p(j+1)> \max(l_p(k), n_m)$, we have a smallest $k_{j+1}>0$ such that for some m_{j+1} with $(m_{j+1}, p)=1$,

$$m_{j+1}p^{k_{j+1}}g_{l_p(j+1)} \in K \cap p^{l_p(j+1)}G \subset p^{j+1}K.$$

Then there exist $y_j \in K$ such that

$$0 \neq m_{j+1}p^{k_{j+1}-1}g_{l_p(j+1)}-p^jy_j \in H[p] \cap p^jG,$$

so that $T(N)$ is not a p-adic center in $T(G)$.

Conversely, if N is a p-adic center and H is $T(N)$-high in $T(G)$, we can extend H to an N-high subgroup H' of G, and then $H=T(H')$, so $H<_pH'<_pG$. Now let $x \in N[p]$ have finite p-height. If G/N has infinite torsion-free rank let $\{g_i | i>0\}$ be an infinite independent set with $O(g_i)=\infty$ satisfying

$$\left(\bigoplus_{i=1}^{\infty} \langle g_i\rangle\right) \cap N=0.$$

Then $\{x+p^i g_i\}$ satisfies the same conditions. Let

$$K=\bigoplus_{i=1}^{\infty} \langle x+p^i g_i\rangle$$

and extend K to an N-high subgroup H with $H<_{l_p}G$. Then for each i, $p^{l_p(i)+1}g_{l_p(i)}\in H$, so $p^{l_p(i)+1}g_{l_p(i)}=p^i h_i$, $h_i\in H$.
Then

$$z_i=x+p^{l_p(i)}g_{l_p(i)}-p^{i-1}h_i\in H[p]\,,$$

with

$$y_i=z_{i+1}-z_i\in H[p]\cap p^{i-1}G\,,$$

Note that infinitely many y_i are nonzero, since if $y_i=0$ for $i>i_0$, then

$$0\neq p^{l_p(i_0+1)}g_{l_p(i_0+1)}-p^{i_0}h_{i_0+1}\in G[p]\sim N$$

has infinite p- height, contradicting 3.2. Again, $T(N)$ is not a p-adic center in $T(G)$.

Finally, suppose $\langle\bar{H}, N\rangle/\langle H, N\rangle$ is not torsion for some H disjoint from N, and say $O(g+\langle H, N\rangle)=\infty$. Then we can find $h_j\in H$ for $j>0$ with $g+h_j\in p^jG$. Let $K=\langle g+x\rangle\oplus H$, and take L N-high containing K, so that $L<_{l_p}G$. Now for each j there is $g_j\in L$ with

$$y_j=g+x+h_{l_p(j+1)}-p^jg_j\in H[p]\,.$$

Then $y_j\neq 0$ for $j>h_p(x)$ and $\{y_j|j>h_p(x)\}$ is infinite, since otherwise we could find $y=y_{j_s}$, for arbitrarily large s, and then

$$0\neq y-x\in(G[p]\sim N)\cap p^{\omega}G\,,$$

violating 3.2. Thus we may assume $\{y_j\}$ is independent, and then $\langle y_j-x|j>h_p(x)\rangle\cap N=0$, again contradicting 3.2 and concluding the proof.

Finally, we give an example when the somewhat unwieldy condition 2i) is necessary. In the p-adics J_p, for proper choice of p, we have $\sqrt{2}\in\langle\bar{1}\rangle$, so we can choose $g_i\in J_p$ and $a_i\in\langle 1\rangle$ with $\sqrt{2}+a_i=p^ig_i$. Then $K=\langle 1, \sqrt{2}, g_i|i>0\rangle$ has rank 2. If we take $G=N\oplus K$, when $N\simeq Z_p$, then N is not a p-adic center.

4. Imbedded neat hulls

As Megibben [5] has noted in the case of purity, the results of the previous two sections could be obtained by viewing the question dually, in terms of neat hulls. Recall that a subgroup H of G is neat if $H\cap pG=pH$, for every prime p, and that every subgroup is contained in a minimal neat subgroup, called a neat hull.

Definition 4.1. A subgroup H of a group G is an *l_p-kernel* if all the neat hulls of H are l_p-imbedded in G.

In this section we show that in the more general setting of imbeddedness, an interesting phenomenon occurs: there is no need to define the "p-adic kernels" of a group.

THEOREM 4.2. *If all the neat hulls of a subgroup H of G are p-imbedded, then H is an l_p-kernel, for some function l_p.*

The theorem is immediate from the following two lemmas.

LEMMA 4.3. *If $H <_{l_p} G$, then H is an l_p-kernel in G.*

Proof. If $H[p]$ is dense in $G[p]$, then by [5], Thm 3, every neat hull of H is p-pure. We may thus take the largest integer n with $G[p]/H[p] \subset p^n(G/H[p])$. Let K be a neat hull of H, and take $g \in K \cap p^{l_p(m)}G$, with $m > n+1$. Then $ag \in H$, for some $a \in Z$ with $(a, p) = 1$, since otherwise, there is a smallest integer k with $0 \neq p^k bg \in H$, $(b, p) = 1$, since H is essential in K. Then

$$p^k bg \in H \cap p^{l_p(m)}G \subset p^m H,$$

and

$$p^{k-1}bg + z \in H, \qquad \text{for some} \quad z \in G[p].$$

But then $p^{k-1}bg \in K$, so $z \in K \cap G[p] = H[p]$, and $p^{k-1}bg \in H$, contradicting minimality of k. Thus $h_K(g) = h_K(ag) \geqq h_H(ag) \geqq m$, since $H <_{l_p} G$. Therefore $K <_{l_p} G$.

LEMMA 4.4. *Let H be a subgroup of G, and suppose $H[p]$ is not dense in $G[p]$. Then if every neat hull of H is p-imbedded, so is H.*

Proof. Let $x \in G[p]$ have finite p-height n mod $H[p]$. If $H \nless_p G$, we can find an integer i such that for $k > 0$, there is $g_k \in G$ with $p^k g_k \in H \sim p^{i+1}H$.

Case I: Suppose that for all k, $p^{k-1}g_k + x_k \in H$, where $x_k \in G[p]$ and $x_k + H[p] \notin p^t(G/H[p])$, for some t. If K is a neat hull of H and $K <_{l_p} G$, take $k_0 > l_p(t+1)$. Then $p^{t+1}y = p^{k_0}g_{k_0}$, for some $y \in K$. But then

$$p^t y - p^{k_0 - 1}g_{k_0} \in K[p] = H[p],$$

so

$$p^{k_0-1}g_{k_0} + x_{k_0} + h \in p^t G, \qquad \text{for some} \quad h \in H[p].$$

Then $x_{k_0} + h \in p^t G$, since $k_0 > t$, a contradiction.

Case II: Suppose $p^k g_k \in H \sim pH$, for all k. Then H/pH is of unbounded height in G/pH.

Subcase 1: Suppose that $0 \neq g + pH \in p^\omega(G/pH)$, for $g \in H$. If $g + ph \in p^\omega G$, for $h \in H$, let $py = g + ph$, where $h_G(y) \geqq n+1$, and let $K = \langle H, y+x \rangle$. Then H is essential in K, so a neat hull L of K is also one of H. Then $L <_p G$, since if $p^{n+2}z = g + ph$, $z \in K$, then $p^{n+1}z - (y+x) \in H[p]$ and $x + w \in p^{n+1}G$, for some $w \in H[p]$.

If however, no representative of $g + pH$ is in $p^\omega G$, we may assume $h_G(g) \geqq n+2$, and take $y \in G$ with $py = g$ and $h_G(y) \geqq n+1$. Let $K = \langle H, y+x \rangle$, and let L be a neat hull of H containing K. We can find elements $g_s \in H$ with $h_G(g+pg_s) \geqq s$, for $s > 0$. If $h_G(g_s + g_s') \geqq n+1$, for some $g_s' \in H[p]$ and all $s > h_G(g)$, then

$$h_G(g_s + g_s' + x + y + h) \leqq n$$

for all $h \in H[p]$. But then the sequence $\{g + pg_s\}$ in L satisfies Case I, so $L \nless_p G$. If however $h_G(g_{s_0} + g) \leqq n$, for some $s_0 > h_G(g)$ and all $h \in H[p]$, then we note that $h_G(pg_{s_0}) = h_G(g)$. Set $z_1 = pg_{s_0}$ and choose another representative g_2 of $g + pH$ with $h_G(g_2) > h_G(g)$. Repeating the above argument, we obtain a sequence $\{z_i\}$ in H satisfying Case I.

Subcase 2: Suppose $(H/pH) \cap p^{\omega}(G/pH) = 0$. Choose a sequence of elements $\{g_s\}_{s \geqq n+2}$ in H such that $h_G(g_s) \geqq s$ and $\{g_s + pH\}_{s \geqq n+2}$ is independent in G/pH. Let $y_s \in G$ satisfy $py_s = g_s$ and $h_G(y_s) = h_G(g_s) - 1$. Let $K = \langle H, y_s + x | s \geqq n+2 \rangle$, and L a neat hull of K. Again by Case I, $K \nless_p G$.

Case III: Finally, suppose that $py_k = p^k g_k$, for all k, and that for every t, there is an integer k_t with $h_G(y_{k_t}) > t$. We may assume $h(y_k) \geqq k$. Then $y_k \in p^i H$, for all k. If $\{y_k\}$ does not satisfy either Case I or Case II, we may continue the process, and find a sequence $\{y_{k,i-1}\}$ in H with $h_G(y_{k,i-1}) \geqq k$ and $y_{k,i-1} \notin p^{i-1}H$, for all k. We finally arrive at a sequence $\{y_{k,1}\}$ with $y_{k,1} \notin pH$ and $h_G(y_{k,1}) \geqq k$, which is just Case I.

5. Imbedded-complete groups

In this section, we restrict our attention to primary groups which are separable, i.e. $p^{\omega}G = 0$. We are concerned with those subsocles which support imbedded subgroups.

Definition 5.1. A subgroup H of a p-group G is *regularly imbedded* if there is an integer k such that $H \cap p^{k+n}G \subset p^n(H \cap p^k G)$, for $n \in Z^+$.

THEOREM 5.2. *Let G be a separable p-group, and S a subsocle of G. The following are equivalent*:

1) *S supports a regularly imbedded subgroup of G*
2) *S is an l_p-center in a pure subgroup of G, for some function l_p.*
3) *S supports a pure subgroup of G.*

Proof. $(1 => 2)$: Let $H[p] = S$, where H is regularly imbedded in G. Then by Thm. 5.4 of [8], there is a pure subgroup P of G with $p^k P \subset H \subset P$, for some k. Clearly, $(p^k P)[p] \subset S$.

$(2 => 3)$: If P is pure in G and $(p^k P)\,[p] \subset S$, then all S-high subgroups of P are bounded. Thus by Thm. 1.2 of [1], S supports a pure subgroup of P, which is then pure in G.

$(3 => 1)$: Trivial.

It is tempting at this point to conjecture that if S supports an imbedded subgroup, it is a p-adic center in a pure subgroup, and is therefore purifiable. That this is not true is seen in the following example, due to Hill [3].

Let $\langle a_i \rangle$ be cyclic of order p^{2i}, and let

$$B_1 = \bigoplus_1 \langle a_i \rangle, \qquad B_2 = \bigoplus_2 \langle a_i \rangle ,$$

where 1 and 2 denote summation over the odd and even integers. Let

$$H_1 = T\Big(\prod_1 \langle a_i \rangle\Big),\ H_2 = T\Big(\prod_2 \langle a_i \rangle\Big),\ H = H_1 \oplus H_2,$$

$$K = B_1 \oplus H_2,\ G_2 = T\Big(\prod_2 \langle a_{i-1} + pa_i \rangle\Big),$$

and $G = \langle G_2, B_2 \rangle$. Megibben [6] has shown that $H_2[p]$ does not support a pure subgroup of G. However, $G_2[p] = H_2[p]$, and clearly $G_2 <_{l_p} G$, where $l_p(i) = i+1$.

Definition 5.3. A separable p-group is *imbedded-complete* if every subsocle supports an imbedded subgroup.

We conclude by showing that the class of imbedded-complete groups properly contains the pure-complete groups.

THEOREM 5.4. *There exist imbedded-complete p-groups which are not pure-complete.*

Proof. In the example above, let S be a subgroup of $G[p] = B_1[p] \oplus H_2[p]$. Then $S = (H_2[p] \cap S) \oplus T$. Now G_2 is torsion-complete, and hence pure-complete, so $H_2[p] \cap S$ supports a pure subgroup M of G_2, which is then imbedded in G. Also, $(T+M)/M \subset (G/M)[p]$, and $(T+M)/M \simeq T$ is countable, so supports a pure subgroup of G/M, say M'/M. Then $M' < G$. But $M'[p] = S$, since if $x \in M'[p]$, then $x + M \in (M'/M)[p]$, so $x + M = t + M$, for some $t \in T$, and $x - t \in M[p] = H_2[p] \cap S$. Therefore $x = t + h$, some $h \in H_2[p] \cap S$ and $x \in S$.

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