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Length of an Increasing Sequence of Ideals

by

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In this paper we shall give some remarks on I-partitions of a set of positive measure and ideals $I|A$ which extend an ideal I.

Let κ be a regular uncountable cardinal number. An ideal over κ is a set I of subsets of κ satisfying the following conditions:

(1) $\phi \in I$,

(2) $\kappa \notin I$,

(3) if $X \in I$ *and* $Y \subseteq X$ *then* $Y \in I$,

(4) if $X \in I$ and $Y \in I$ then $X \cup Y \in I$.

An ideal I over κ is nontrivial if, for all $\alpha < \kappa$, $\{\alpha\} \in I$. And an ideal I over κ is said to be κ-complete, if I satisfies the following condition:

$$\text{If} \quad \lambda < \kappa \quad \text{and} \quad \{X_\alpha \mid \alpha < \lambda\} \subseteq I, \quad \text{then} \quad \bigcup_{\alpha < \lambda} X_\alpha \in I\,.$$

Throughout this paper, an ideal means a nontrivial κ-complete ideal over κ. Let I be an ideal. We set $I^+ = \{X \mid X \subseteq \kappa \text{ and } X \notin I\}$ and say X has a positive measure or is a set of positive measure if $X \in I^+$. A set $Z \subseteq I^+$ is said to be an almost disjoint family with respect to I, if Z satisfies the following condition:

$$\text{If } X,\ Y \in Z \text{ and } X \neq Y \quad \text{then} \quad X \cap Y \in I\,.$$

If the cardinality of every almost disjoint family with respect to I is less than λ, I is said to be λ-saturated. Let A be a set of positive measure. An I-partition of A is a maximal almost disjoint family W of subsets of A. When W_1 and W_2 are I-partitions of A, we say that W_1 is a refinement of W_2, and denote it by $W_1 \leq W_2$, if for every $X \in W_1$ there is $Y \in W_2$ with $X \subseteq Y$. An ideal I is said to be precipitous if whenever $A \in I^+$ and W_n $(n < \omega)$ are I-partitions of A such that

$$W_0 \geq W_1 \geq \cdots \geq W_n \geq \cdots \qquad (n < \omega)\,,$$

there is a sequence of sets

$$X_0 \supseteq X_1 \supseteq \cdots \supseteq X_n \supseteq \cdots \qquad (n < \omega)$$

such that for each $n < \omega$ $X_n \in W_n$ and $\bigcap_{n<\omega} X_n \neq \varnothing$.

It is already known that if I is a κ^+-saturated ideal then I is precipitous. (see [2])

§1. *I*-partitions

LEMMA 1. *Let A be a set of positive measure.*

(1) *If W_1 and W_2 are I-partitions of A, then there is an I-partition W of A such that $W \leq W_1$ and $W \leq W_2$.*

(2) *If W is an I-partition of A and $\alpha < \kappa$, then $W_\alpha = \{X - \alpha \mid X \in W\}$ is also an I-partition of A.*

Proof. I is nontrivial κ-complete, so (2) is obvious. Hence we have only to prove (1). We can easily get the following propositions:

(i) If Z is an I-partition of A and X' is an I-partition of $X \in Z$ then $\bigcup_{X \in Z} X'$ is an I-partition of A.

(ii) If Z is an I-partition of A and X is a subset of A with $X \in I^+$ then $Z_X = \{X \cap Y \mid X \cap Y \in I^+ \text{ and } Y \in Z\}$ is an I-partition of X.

Hence we get that for all $X \in W_1$

$$W_{2X} = \{X \cap Y \mid X \cap Y \in I^+ \text{ and } Y \in W_2\}$$

is an I-partition of X, and $W = \bigcup_{X \in W_1} W_{2X}$ is an I-partition of A. Moreover

$$W = \bigcup_{X \in W_1} W_{2X} = \bigcup_{X \in W_1} \{X \cap Y \mid X \cap Y \in I^+ \text{ and } Y \in W_2\}$$

$$= \{X \cap Y \mid X \cap Y \in I^+ \text{ and } X \in W_1 \text{ and } Y \in W_2\}.$$

This means $W \leq W_1$ and $W \leq W_2$. ■

PROPOSITION 2. *Let I be a κ-*saturated ideal over κ. *Then if $A \in I^+$ and $\{W_n\}_{n<\omega}$ are I-partitions of A such that*

$$W_0 \geq W_1 \geq \cdots \geq W_n \geq \cdots,$$

then there is a sequence of sets with

$$X_0 \supseteq X_1 \supseteq \cdots \supseteq X_n \supseteq \cdots$$

such that $X_n \in W_n$ for each $n < \omega$ and $\left|\bigcap_{n<\omega} X_n\right| = \kappa$. *($|X|$ denotes the cardinality of X)*

Proof. Let $A \in I^+$ and W_n $(n<\omega)$ be I-partititons of A such that $W_0 \geq W_1 \geq \cdots \geq W_n \geq \cdots$. By Lemma 1. (2), we have for each $\alpha < \kappa$ $W_{n\alpha}$ $(n<\omega)$ are I-partitions of A such that $W_{0\alpha} \geq W_{1\alpha} \geq \cdots \geq W_{n\alpha} \geq \cdots$. Since I is κ-saturated, I is precipitous. Hence there is a sequence $X_{0\alpha} \supseteq X_{1\alpha} \supseteq \cdots \supseteq X_{n\alpha} \supseteq X_{n+1\alpha} \supseteq \cdots$ such that for each $n<\omega$ $X_{n\alpha} \in W_{n\alpha}$ and $\bigcup_{n<\omega} X_{n\alpha} \neq \varnothing$. By the definition of $W_{n\alpha}$ there is an $X_n \in W_n$ with $X_{n\alpha} \subseteq X_n$, and such and X_n is unique. Let $X_n \in W_n$, $X_{n\alpha} \subseteq X_n$ and $X_{n+1} \in W_{n+1}$, $X_{n+1\alpha} \subseteq X_{n+1}$. Then we have $X_n \supseteq X_{n+1}$. Because of $W_n \geq W_{n+1}$, there is a $Y \in W_n$ such that $Y \supseteq X_{n+1}$. But since $X_n \supseteq X_{n+1\alpha}$ and $Y \supseteq X_{n+1\alpha}$, we get $X_n = Y$. Hence $X_n \supseteq X_{n+1}$. Thus we get for each $\alpha < \kappa$ there is a sequence $X_0 \supseteq X_1 \supseteq \cdots \supseteq X_n \supseteq \cdots$ such that for each $n<\omega$ $X_n \in W_n$ and

$(\bigcap_{n<\omega} X_n)\cap(\kappa-\alpha)\neq\varnothing$. Since I is κ-saturated, we have $|W_n|<\kappa$ for all $n<\omega$. Set Y be the set of all sequences $\{Y_n\}_{n<\omega}$ such that $Y_0\supseteq Y_1\supseteq\cdots\supseteq Y_n\supseteq\cdots$ $(Y_n\in W_n)$, then $|Y|<\kappa$, because κ is a regular cardinal. Assume that for each $\{Y_n\}_{n<\omega}\in Y$, $|\bigcap_{n<\omega} Y_n|<\kappa$. Then there is a $\beta<\kappa$ such that for all $\{Y_n\}_{n<\omega}\in Y$

$$(\bigcap_{n<\omega} Y_n)\cap(\kappa-\beta)=\varnothing$$

a contradiction. ■

Let I be an ideal over κ. Then we say f is an I-function if $\mathrm{dom}(f)\in I^+$. Let $A\in I^+$. Then we say $f:A\to\kappa$ be an unbounded I-function if $\{\alpha \mid f(\alpha)\leq\gamma\}\in I$ for all $\gamma<\kappa$. An ideal I is said to be weakly normal if for each $A\in I^+$ there is a minimal unbounded I-function f with $\mathrm{dom}(f)\subseteq A$, where a minimal unbounded I-function means an unbounded I-function f such that there is no unbounded I-function g with $\mathrm{dom}(g)\subseteq\mathrm{dom}(f)$ and $g(\alpha)<f(\alpha)$ for all $\alpha\in\mathrm{dom}(g)$. A collection F of I-functions is said to be closed under restrictions if for each I-function g with $g\subseteq f\in F$ we have $g\in F$.

If I is precipitous, then every nonempty collection of I-functions which is closed under restrictions has a minimal element (see [2]).

PROPOSITION 3 ([2]). *If I if a precipitous ideal, then I is weakly normal.*

Proof. We show that, if I is an ideal such that every nonempty collection of I-functions which is closed under restrictions has a minimal element, then I is weakly normal.

Let A be a set of positive measure. Set

$$F_A=\{f \mid f \text{ is an unbounded } I\text{-function with } \mathrm{dom}(f)\subseteq A\}\,.$$

Let g be an I-function such that $g\subseteq f$ for some $f\in F_A$. Assume that g is not unbounded. Then there is $\gamma<\kappa$ with $\{\alpha|g(\alpha)\leq\gamma\}\notin I$. Hence $\{\alpha|f(\alpha)\leq\gamma\}\notin I$. This contradicts f is unbounded. Thus we get F_A is a collection of I-functions closed under restrictions. Hence F_A has a minimal element. This means I is weakly normal. ■

§2. Length of an increasing sequence of ideals

Let I be an ideal over κ and $A\in I^+$. We set

$$I|A=\{X \mid X\subseteq\kappa \quad\text{and}\quad X\cap A\in I\}\,.$$

$I|A$ is an ideal which extends I. It is known that if I is λ-saturated, then $I|A$ is also λ-saturated.

PROPOSITION 4. (1) *If I is precipitous, then so is $I|A$.*

(2) *If I is an ideal such that every nonempty collection of I-functions closed under restrictions has a minimal element, then so is $I|A$.*

(3) *If I is weakly normal, then so is $I|A$.*

Proof. (1): Let I be a precipitous ideal and $A\in I^+$. If $S\in(I|A)^+$, then $S\cap A\notin I$. Hence $S\cap A\in I^+$. Let W be $I|A$-partition of $S\in(I|A)^+$. Then for any distinct elements X, Y of W, we get $X\cap A\notin I$, $Y\cap A\notin I$ and $(X\cap A)\cap(Y\cap A)=(X\cap Y)\cap A\in I$. Set

$W_A=\{X\cap A \mid X\in W\}$. Then we get W_A is an I-partition of $S\cap A$. Assume W_A is not an I-partition of $S\cap A$. Then there is a $T\notin I$ such that $T\subseteq S\cap A$ and $(X\cap A)\cap T=(T\cap X)\cap A\in I$ for all $X\in W$. Since $T\notin I$ and $T=T\cap A$, we get $T\notin I|A$. Hence there is a T such that $T\subseteq S$, $T\in(I|A)^+$ and $T\cap X\in I|A$ for all $X\in W$. But this contradicts W is an $I|A$-partition of S. Now let $\{W_n\}_{n<\omega}$ be $I|A$-partitions of $S\in(I|A)^+$ such that $W_0\geq W_1\geq \cdots\geq W_n\geq\cdots$. Then $\{W_{nA}\}_{n<\omega}$ are I-partitions of $S\cap A$ such that $W_{0A}\geq W_{1A}\geq\cdots\geq W_{nA}\geq\cdots$. Since I is precipitous, there is a sequence of sets $X_0\cap A\supseteq X_1\cap A\supseteq\cdots\supseteq X_n\cap A\supseteq\cdots$ with $X_n\cap A\in W_{nA}$ for each $n<\omega$ and $\bigcap_{n<\omega}(X_n\cap A)\neq\emptyset$. Thus if $X_0\supseteq X_1\supseteq\cdots\supseteq X_n\supseteq\cdots$, then the proof is complete. Let $X_n\not\supseteq X_{n+1}$ for some $n<\omega$. By hypothesis there is a $Y\in W_n$ with $Y\supseteq X_{n+1}$. So we have $X_n\supseteq X_{n+1}\cap A$, $Y\supseteq X_{n+1}\cap A\notin I|A$ and $X_n\neq Y$. But this contradicts X_n, $Y\in W_n$.

There are some equivalent definitions of precipitous ideals. We can see another proof based on one of them in [3].

(2): Let I be an ideal satisfying the assumption of (2), $A\in I^+$, and F be a nonempty collection of $I|A$-functions closed under restrictions. Set

$$F\restriction A=\{f\restriction A \mid f\in F\},$$

where $f\restriction A$ is a function with $\mathrm{dom}(f\restriction A)=\mathrm{dom}(f)\cap A$ and $f\restriction A\subseteq f$. Since for each $f\in F$ $\mathrm{dom}(f)\notin I|A$, we have $\mathrm{dom}(f\restriction A)=\mathrm{dom}(f)\cap A\notin I$. Hence $f\restriction A$ is an I-function. Let g be an I-function with $g\subseteq f\restriction A$ for some $f\in F$. Then g is an $I|A$-function, because $\mathrm{dom}(g)\cap A=\mathrm{dom}(g)\notin I$ implies $\mathrm{dom}(g)\notin I|A$. Hence we get $g\in F$, because of $g\subseteq f\restriction A\subseteq f\in F$. It is clear that $g=g\restriction A$. So $g\in F\restriction A$. Hence $F\restriction A$ is a nonempty collection of I-functions closed under restrictions. Therefore, by assumption, $F\restriction A$ has a minimal element h, which is clearly also a minimal element of F.

(3): Let I be weakly normal and $A\in I^+$. Assume that $I|A$ is not weakly normal. Then there is a sequence of unbounded $I|A$-functions

$$f_0>f_1>\cdots>f_n>\cdots \qquad (n<\omega),$$

where $f_n>f_{n+1}$ means $\mathrm{dom}(f_n)\supseteq\mathrm{dom}(f_{n+1})$ and $f_n(\alpha)>f_{n+1}(\alpha)$ for all $\alpha\in\mathrm{dom}(f_{n+1})$. Then we have a sequence of I-functions

$$f_0\restriction A>f_1\restriction A>\cdots>f_n\restriction A>\cdots \qquad (n<\omega).$$

For each $n<\omega$, f_n is an unbounded $I|A$-function, so we get $\{\alpha \mid f_n(\alpha)\leq\gamma\}\in I|A$ *for all* $\gamma<\kappa$. Hence we have

$$\{\alpha \mid (f_n\restriction A)(\alpha)\leq\gamma\}=\{\alpha \mid f_n(\alpha)\leq\gamma\}\cap A\in I$$

for all $\gamma<\kappa$. This means $f_n\restriction A$ $(n<\omega)$ are unbounded I-functions, a contradiction. ■

Next we shall give a remark on the length of the increasing sequence of ideals which extends I.

LEMMA 5. *Let I be an ideal and be $A\in I^+$, $B\in I^+$. Then we have*

(1) *$I|A=I|B$ iff $A\cap B\notin I$, $A-B\in I$ and $B-A\in I$.*

(2) $I|B \subsetneq I|A$ *iff there is a* $C \subseteq B$ *such that* $C \notin I$, $B-C \notin I$ *and* $I|A=I|C$.

Proof. (1): Let assume $I|A=I|B$. If $A-B \notin I$, then $A-B \notin I|A$ and $A-B \in I|B$, a contradiction. Hence we have $A-B \in I$, and then $A \cap B \notin I$. Similarly $B-A \in I$.

Assume $A \cap B \notin I$, $A-B \in I$ and $B-A \in I$. Then $X \in I|A$ iff $X \cap A \in I$. But $X \cap A = X \cap ((A \cap B) \cup (A-B)) = (X \cap (A \cap B)) \cup (X \cap (A-B))$. So $X \in I|A$ iff $X \cap (A \cap B) \in I$. Similarly $X \in I|B$ iff $X \cap (A \cap B) \in I$. Hence we get $X \in I|A$ iff $X \in I|B$. Thus $I|A = I|B$.

(2): Assume $C \subseteq B$, $C \notin I$, $B-C \notin I$ and $I|A=I|C$. Then we have $I|B \subseteq I|C$, because $X \cap B \in I$ implies $X \cap C \in I$. From (1) we get $I|B \neq I|C$. Hence $I|B \subsetneq I|A$. Next let us assume $I|B \subsetneq I|A$. It suffices to observe the following 4 cases:

(i) $A \cap B \in I$,

(ii) $A \cap B \notin I$, $A-B \in I$ and $B-A \in I$,

(iii) $A \cap B \notin I$ and $A-B \notin I$,

(iv) $A \cap B \notin I$, $A-B \in I$ and $B-A \notin I$.

Assume the case (iv) occurs. Then if we set $C=A \cap B$, then we have $C \subseteq B$, $C \notin I$, $B-C=B-(A \cap B)=B-A \notin I$. And we get $A \cap C = A \cap (A \cap B) = A \cap B \notin I$, $A-C = A-(A \cap B) = A-B \in I$ and $C-A=\phi \in I$. Hence from (1) we get $I|A=I|C$. Next we shall prove that only the case (iv) occurs. If (i) occurs, then we have $A \in I|B$ and $A \notin I|A$, a contradiction. If (ii) occurs, then from (1) $I|A=I|B$, a contradiction. If (iii) occurs, then we have $A-B \notin I|A$, because $(A-B) \cap A = A-B \notin I$. But $A-B \in I|B$, a contradiction. Hence only (iv) occurs. ■

If I is an ideal over κ and Z is an almost disjoint family with respect to I such that $\omega \leq |Z| < \kappa$, then we can construct a sequence of ideals of the form $I|A$ such that

$$I \subsetneq I|A_0 \subsetneq \cdots \subsetneq I|A_\alpha \subsetneq \cdots \qquad (\alpha < |Z|)$$

in the following way:
Set $\gamma=|Z|$, $Z=\{X_\alpha | \alpha<\gamma\}$ and

$$A_\alpha = \bigcup_{\beta<\gamma} X_\beta - \bigcup_{\delta<\alpha} X_\delta$$

for each $\alpha<\gamma$. Since

$$A_\alpha \supseteq X_\alpha - \bigcup_{\delta<\alpha} X_\delta = X_\alpha - \bigcup_{\delta<\alpha} (X_\alpha \cap X_\delta) \notin I, \quad \text{then } A_\alpha \notin I.$$

And

$$A_\alpha - A_{\alpha+1} = \left(\bigcup_{\beta<\gamma} X_\beta - \bigcup_{\delta<\alpha} X_\delta\right) - \left(\bigcup_{\beta<\gamma} X_\beta - \bigcup_{\delta<\alpha+1} X_\delta\right)$$

$$= \left(\bigcup_{\beta<\gamma} X_\beta\right) \cap X_\alpha - \bigcup_{\delta<\alpha} X_\delta = X_\alpha - \bigcup_{\delta<\alpha} X_\delta \notin I.$$

Hence from Lemma 5.(2) we get $I|A_\alpha \subsetneq I|A_{\alpha+1}$.

To the contrary we have

PROPOSITION 6. *Let I be a λ-saturated ($\lambda\leq\kappa$) ideal over κ. If there is a sequence of ideals such that*

$$I\subsetneq I|A_0\subsetneq\cdots\subsetneq I|A_\alpha\subsetneq\cdots\qquad(\alpha<\mu),$$

then we have $\mu<\lambda$.

Proof. Assume $\mu\geq\lambda$. We shall construct a sequence of sets of positive measure such that

$$B_0\supseteq B_1\supseteq\cdots\supseteq B_\alpha\supseteq\cdots\qquad(\alpha<\lambda)$$

and $B_\beta-B_\alpha\notin I$ for all $\beta<\alpha$ and $I|A_\alpha=I|B_\alpha$ for all $\alpha<\lambda$. Then we can easily construct a set of cardinality λ and of pairwise disjoint sets of positive measure from $\{B_\alpha|\alpha<\lambda\}$. But this contradicts I is λ-saturated.

We shall construct B_α ($\alpha<\lambda$) by induction.

Let $\alpha=\gamma+1$. By hypothesis of induction, we already get $\{B_\delta|\delta\leq\gamma\}$ such that $I|A_\gamma=I|B_\gamma\subsetneq I|A_{\gamma+1}$. Hence from Lemma 5.(2) there is $B_{\gamma+1}\subseteq B_\gamma$ with $B_{\gamma+1}\notin I$, $B_\gamma-B_{\gamma+1}\notin I$ and $I|A_{\gamma+1}=I|B_{\gamma+1}$.

Let α be limit. We already get $\{B_\beta|\beta<\alpha\}$ such that $I|A_\beta=I|B_\beta\subsetneq I|A_\alpha$ and $B_\beta-B_{\beta+1}\notin I$ for each $\beta<\alpha$. Hence for each $\beta<\alpha$ there is a C_β such that $C_\beta\subseteq B_\beta$, $B_\beta-C_\beta\notin I$ and $I|A_\alpha=I|C_\beta$. Then from Lemma 5.(1) we have $A_\alpha\cap C_\beta\notin I$, $A_\alpha-C_\beta\in I$ and $C_\beta-A_\alpha\in I$. Hence $\bigcup_{\beta<\alpha}(A_\alpha-C_\beta)\in I$, because of $\alpha<\lambda\leq\kappa$. Let us define $D_\beta=A_\alpha\cap C_\beta$. Then from Lemma 5.(1) we have $I|D_\beta=I|C_\beta$. Now define $B_\alpha=\bigcap_{\beta<\alpha}D_\beta$. Then we get

$$B_\alpha=\bigcap_{\beta<\alpha}D_\beta=\bigcap_{\beta<\alpha}(A_\alpha\cap C_\beta)=\bigcap_{\beta<\alpha}(A_\alpha-(A_\alpha-C_\beta))=A_\alpha-\bigcup_{\beta<\alpha}(A_\alpha-C_\beta)\notin I\,.$$

And $A_\alpha-B_\alpha\subseteq\bigcup_{\beta<\alpha}(A_\alpha-C_\beta)\in I$, so $A_\alpha-B_\alpha\in I$. Moreover $B_\alpha-A_\alpha=\phi\in I$. Therefore we have $I|A_\alpha=I|B_\alpha$. Let be $\beta<\alpha$. Since $B_\alpha=\bigcap_{\delta<\alpha}(A_\alpha\cap C_\delta)$, we get $B_\alpha\subseteq C_\beta$. And $B_\beta-B_\alpha\supseteq B_\beta-C_\beta\notin I$ implies $B_\beta-B_\alpha\notin I$. Hence for all $\beta<\alpha$ $B_\beta-B_\alpha\notin I$. This completes the proof. ■

From the remark of [1] we have that an ideal I over κ is κ-saturated iff every ideal over κ extending I is the ideal of form $I|A$. Hence we have

COROLLARY 7. *Let I be a κ-saturated ideal over κ. If there is a sequence of ideals such that*

$$I\subsetneq I_0\subsetneq\cdots\subsetneq I_\alpha\subsetneq\cdots\qquad(\alpha<\mu)$$

then we have $\mu<\kappa$.

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