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Definitive Valuation of Set Theory

by

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In our previous work [3] we defined the definitive valuation of formal languages (first order and second order), and showed the following (*cf.* Theorem 5, §4 of [3]).

Let p be a number satisfying $\frac{1}{2}<p\leq 1$, and let Ξ be a p-definitive structure corresponding to a language L'. Let $\mathscr{A}$ be a set of formulas of L', each of which is p-valid with regards to Ξ, and let $\mathscr{P}(\mathscr{A})$ be the (second order) predicate calculus augmented by the formulas of $\mathscr{A}$ as axioms. Then any formula which is provable in $\mathscr{P}(\mathscr{A})$ is p-valid with regards to Ξ.

In this paper we define a p-definitive structure which interprets ε and $=$ of set theory, and show that every axiom of ZF-set theory is p-valid. By letting $\mathscr{A}$ consist of those axioms and employing the first order case of the theorem above, we conclude that ZF-set theory is definitively valid with regards to our interpretation, presuming that ZFC is consistent.

§1. The universe of sets and the membership relation

We first set some notational conventions.

The closed unit interval [0, 1] will be denoted by I. p will stand for a number in I which is greater than $\frac{1}{2}$; $\frac{1}{2}<p\leq 1$. $\alpha, \beta, \cdots$ will stand for ordinals, and On will denote the class of ordinals.

We assume the standard language of Zermelo-Fraenkel (ZF) set theory; thus, for example, $a\in b$ expresses that a set a belongs to a set b. Although one may refer to any text for notations, we have mostly followed [2].

[Assumption] ZF-set theory augmented by the axiom of choice is assumed throughout; in particular the classical theory of real numbers is taken for granted.

And, we repeat; $\frac{1}{2}<p\leq 1$.

Definition 1.1. Let f be a function (from an appropriate domain, $D\equiv D(f)$) into I.

$$\delta(f)=\inf\{\max(1-f(d), f(d))/d\in D\}\ .$$

If $\delta(f)\geq p$, we say f is p-definitive (*cf.* [3]).

We now define the universe of set theory in p-definitive valuation and two basic relations.

Definition 1.2. $V(p;\ 0)$ is the empty set.

$$V(p;\ \alpha)=\{u \mid u \text{ is a } p\text{-definitive function such that}$$

$$D(u)\subseteq V(p;\ \beta) \text{ for some } \beta<\alpha\}\ .$$

$$V(p)=\bigcup\{V(p;\ \alpha)/\alpha\in On\}\ .$$

rank $(p;\ u)=$the least α such that u belongs to $V(p;\ \alpha)$.

Definition 1.3. We define two functions $M(p;\ u,\ v)$ and $E(p;\ u,\ v)$, where u and v are arbitrary elements in $V(p)$ and those functions assume values in I.

$$M(p;\ u,\ v)=\sup\ \{\min\ (v(y),\ E(p;\ u,\ y))/y\in D(v)\}\ .$$

$$E(p;\ u,\ v)=\min\ (\inf\{\max\ (1-u(x),\ M(p;\ x,\ v))/x\in D(u)\}\ ,$$

$$\inf\ \{\max\ (1-v(y),\ M(p;\ y,\ u))/y\in D(v)\})\ .$$

COROLLARY. *The well-definedness of M and E is established by transfinite induction on* rank $(p;\ u)$ # rank $(p;\ v)$. (*cf.* Remark on Definition 13.3 in [2], for example.)

Note. M and E represent p-definitive valuations of the membership relation and the equality relation respectively. The p in $M(p;\ u,\ v)$ etc. will be omitted when p is fixed.
Henceforth $x,\ y,\ z,\ u,\ v,\ w,\ \cdots$ will stand for the elements in $V(p)$.

LEMMA 1.1. 1) $E(u,\ v)=E(v,\ u)$.
2) *If* $x\in D(u)$, *then*

$$\min\ (u(x),\ E(x,\ x))\leq M(x,\ u).$$

3) $E(u,\ u)\geq p$.
4) *If* $x\in D(u)$ *and* $u(x)\geq p$, *then* $M(x,\ u)\geq p$.

Proof. 2) Let x be in $D(u)$.

$$M(x,\ u)=\sup\ \{\min\ (u(y),\ E(x,\ y))\mid y\in D(u)\}$$

$$\geq\min\ (u(x),\ E(x,\ x))\ .$$

3) By transfinite induction on rank (u).

$$E(u,\ u)=\inf\ \{\max\ (1-u(x),\ M(x,\ u))/x\in D(u)\}$$

$$\geq\inf\ \{\max\ (1-u(x),\ \min\ (u(x),\ E(x,\ x)))/x\in D(u)\}\ ,$$

where $E(x,\ x)\geq p$ by the induction hypothesis. For any x in $D(u)$, we claim

$$(*) \qquad 1-u(x)\geq p \quad \text{or}\ \min\ (u(x),\ E(x,\ x))\geq p\ .$$

If $\min\ (u(x),\ E(x,\ x))=u(x)$, then, since $1-u(x)\geq p$ or $u(x)\geq p$, $(*)$ holds. If $\min(u(x),\ E(x,\ x))=E(x,\ x)$, then the induction hypothesis implies $(*)$.
4) Use 3) and 2).

LEMMA 1.2. *Let r_λ be a number in I indexed by λ, $\lambda \in \Lambda$. If for each λ $r_\lambda \geq p$ or $r_\lambda \leq 1-p$, then* $\sup\{r_\lambda \mid \lambda \in \Lambda\} \geq p$ *if and only if $r_\lambda \geq p$ for some $\lambda \in \Lambda$.*

LEMMA 1.3. 1) *$M(u, v) \geq p$ if and only if* $\min(v(y), E(u, y)) \geq p$ *for some y in $D(v)$.*

2) *$E(u, v) \geq p$ if and only if the following condition holds: for every $x \in D(u)$,* $\max(1-u(x), M(x, v)) \geq p$, *and for every $y \in D(v)$,* $\max(1-v(y), M(y, u)) \geq p$.

3) *$M(u, v) \geq p$ or $M(u, v) \leq 1-p$.*

4) *$E(u, v) \geq p$ or $E(u, v) \leq 1-p$.*

Proof. 1)$\sim$4) are proved together by transfinite induction on rank(u) # rank(v).

1) Suppose $m = \min(v(y), E(u, y)) < p$ for every y in $D(v)$. If $m = v(y)$, then $v(y) \leq 1-p$ implies $m \leq 1-p$. If $m = E(u, y)$, then the induction hypothesis applied to 4) implies $m \leq 1-p$. Thus, $m \leq 1-p$ ($<p$) for every $y \in D(v)$, hence $M(u, v) < p$. The converse is trivial.

3) Suppose $M(u, v) < p$. Then $m < p$ for every $y \in D(v)$. As was seen in the proof of 1), $m \leq 1-p$. So $M(u, v) \leq 1-p$.

4) Suppose $E(u, v) < p$, and suppose that this is caused by

$$\max(1-u(x), M(x, v)) < p$$

for some $x \in D(u)$ (*cf.* 2)). Then $1-u(x) \leq 1-p$ and $M(x, v) \leq 1-p$ (*cf.* 3)), hence

$$\inf\{\max(1-u(x), M(x, v))/x \in D(u)\} \leq 1-p,$$

which implies $E(u, v) \leq 1-p$. Then other case can be dealt with similarly.

We next list some properties about M and E.

PROPOSITION 1.1 (*cf.* Theorem 13.5 of [2]). *Let u, u', v, v', w be in $V(p)$ and let α be an ordinal.*

1) *Suppose* rank$(u) < \alpha$, rank$(u') < \alpha$ *and* rank$(v) \leq \alpha$. *Then $E(u, u') \geq p$ and $M(u, v) \geq p$ imply $M(u', v) \geq p$.*

2) *Suppose* rank$(u) < \alpha$, rank$(v) \leq \alpha$ *and* rank$(v') \leq \alpha$. *Then $M(u, v) \geq p$ and $E(v, v') \geq p$ imply $M(u, v') \geq p$.*

3) *Suppose* rank$(u) \leq \alpha$, rank$(v) \leq \alpha$ *and* rank$(w) \leq \alpha$. *If $E(u, v) \geq p$ and $E(v, w) \geq p$, then $E(u, w) \geq p$.*

Proof. By transfinite induction on α.

1) From $M(u, v) \geq p$ follows that

$$v(y) \geq p \quad \text{and} \quad E(u, y) \geq p$$

for some y in $D(v)$ (*by 1) of Lemma 1.3*). *3) applied to* $E(u, y)$ and $E(u, u')$ yields $E(u', y) \geq p$ (by the induction hypotheses). Thus,

$$\min(v(y), E(u', y)) \geq p$$

for some y in $D(v)$, which implies $M(u', v) \geq p$.

2) Apply 1) to $E(u, y)$ and $M(y, v')$ for every y in $D(v)$.

3) For any $x \in D(u)$, suppose $1-u(x)<p$, or $u(x)\geq p$. Then $M(x, u)\geq p$ by 4) of Lemma 1.1. 2) applied to M(x, u) and $E(u, v)$ yields $M(x, v)\geq p$. 2) applied to $M(x, v)$ and $E(v, w)$ yields $M(x, w)\geq p$. Thus,

$$\inf\{\max(1-u(x),\ M(x, w))/x \in D(u)\} \geq p.$$

Similarly we obtain

$$\inf\{\max(1-w(y),\ M(y, u))/y \in D(w)\} \geq p\,.$$

Those two relations imply $E(u, w)\geq p$.

As an immediate consequence of the proposition above, we obtain

PROPOSITION 1.2 (*cf.* Corollary 13.6, [2]). *Let* u, u', v, v', w *be in* $V(p)$.

1) *If* $E(u, u')\geq p$ *and* $M(u, v)\geq p$, *then* $M(u', v)\geq p$.
2) *If* $M(u, v)\geq p$ *and* $E(v, v')\geq p$, *then* $M(u, v')\geq p$.
3) *If* $E(u, v)\geq p$ *and* $E(v, w)\geq p$, *then* $E(u, w)\geq p$.

§2. Definitive valuation

Let p be a real number such that $\frac{1}{2}<p\leq 1$.

Let D be either $V(p, \alpha)$ for some α an ordinal or $V(p)$ (the latter of which is a proper class).

Let L(ZF) be the language of ZF with ε and $=$.

Definition 2.1. When p and D are fixed as above, we define the p-definitive valuation of L(ZF)-formulas. For a formula A and an assignment ψ of a elements of D to variables, we write $|A; D, \psi|$ for the value of A with regards to ψ in the domain D. a, b, $\cdots$ denote variables.

$$|a \in b;\ D, \psi| = M(\psi(a), \psi(b))\,.$$

$$|a=b;\ D, \psi| = E(\psi(a), \psi(b))\,.$$

For the compound formulas, the definition is exactly the same as the first order valuation in Definition 1.1 of [3]; we repeat the definition for the reader's convenience. We omit the D in $|A; D, \psi|$ unless D needs to be explicitly expressed.

$$|A \wedge B;\ \psi| = \min(|A;\ \psi|,\ |B;\ \psi|)\,.$$

$$|A \vee B;\ \psi| = \max(|A;\ \psi|,\ |B;\ \psi|)\,.$$

$$|\neg A;\ \psi| = 1-|A;\ \psi|\,.$$

$$|A \supset B;\ \psi| = \max(1-|A;\ \psi|,\ |B;\ \psi|)\,.$$

$$|\forall a A(a);\ \psi| = \inf\{|A(a);\ \psi[a/d]|/d \in D\}\,,$$

where $\psi[a/d]$ represents the assignment obtained from ψ by assigning d to a.

$$|\exists a A(a);\ \psi| = \sup\{|A(a);\ \psi[a/d]|/d \in D\}\,.$$

Proposition 1.1 in [3] holds for our valuation.

Remark. For a closed formula A, the value of A is determined independent of assignments. Thus, we can write $|A|$ for that value.

We shall abbreviate $|A;\ V(p, \alpha),\ \psi|$ to $|A;\ p, \alpha, \psi|$, or even to $|A;\ \alpha, \psi|$ when p is fixed, and $|A;\ V(p),\ \psi|$ to $|A;\ p, \psi|$.

By virtue of 3) and 4) of Lemma 1.3, a parallel to Propositions 4.1 and 4.3 in [3] holds:

PROPOSITION 2.1. 1) *For every A a formula and ψ an assignment, either $|A;\ \psi|\geq p$ or $|A;\ \psi|\leq 1-p$.*

2) *Let $(A_\lambda, \psi_\lambda)$ be a pair of a formula and an assignment indexed by $\lambda\in\Lambda$. Then*

$$\sup\{|A_\lambda;\ \psi_\lambda|/\lambda\in\Lambda\}\geq p$$

if and only if $|A_\lambda;\ \psi_\lambda|\geq p$ for some $\lambda\in\Lambda$. Thus, in particular, $|\exists aA(a);\ \psi|\geq p$ if and only if $|A(a);\ \psi[a/d]|\geq p$ for some d in D.

PROPOSITION 2.2 *For any formula A with n arguments, $a_1, \cdots, a_n$, and for any two assignments from D, say ψ and ϕ, if $E(\psi(a_i), \phi(a_i))\geq p$, $i=1, \cdots, n$, and if $|A;\ \psi|\geq p$, then $|A;\ \phi|\geq p$.*

We shall refer to this as the equality axiom.

Proof. For an atomic A, the proposition follows from Proposition 1.2. For a compound A, apply the induction hypothesis and Proposition 2.1.

From Definition 2.1 it follows immediately:

LEMMA 2.1. *Let $A(a, b)$ be either $a\in b$ or $a=b$. Let D be $V(p, \alpha)$ for an α, and let E be either $V(p, \beta)$ for a $\beta>\alpha$ or $V(p)$. Let ψ be an assignment from D. Then*

$$|A(a, b);\ D, \psi|=|A(a, b);\ E, \psi|\ .$$

In the subsequent development, we follow to a great deal §9 and §13 of [2], although we do not always refer to them.

Definition 2.2. Let u be in $V(p)$. u is said to be defined over $V(p, \alpha)$ if for every v in $V(p)$,

$$M(v, u)\geq p \quad \text{if and only if} \quad E(v, x)\geq p \quad \text{and} \quad M(x, u)\geq p$$
$$\text{for some} \quad x \quad \text{in } V(p, \alpha)\ .$$

(*cf.* §9 of [2])

LEMMA 2.2. *Every element of $V(p, \alpha+1)$ is defined over $V(p, \alpha)$.*

Proof. By 1) of Lemma 1.3, $M(v, u)\geq p$ is equivalent to

$$\min(u(x),\ E(v,\ x)) \geq p \qquad \text{for some} \quad x \quad \text{in } D(u)\,,$$

or

$$u(x) \geq p \quad \text{and} \quad E(v,\ x) \geq p \quad \text{for some} \quad x \in D(u)\,.$$

$u(x) \geq p$ implies $M(x, u) \geq p$ (4) of Lemma 1.1). Notice that $D(u) \subseteq V(p, \alpha)$. Thus follows the "only if" part.

Next assume $E(v, x) \geq p$ and $M(x, u) \geq p$ for some x in $V(p, \alpha)$. By Proposition 1.2 it follows $M(v, u) \geq p$, which proves the "if" part.

LEMMA 2.3. *Let u and v be in $V(p, \alpha)$, and let ψ be an assignment such that $\psi(a)=u$ and $\psi(b)=v$. Then*

(1) $|\forall c(c \in a \equiv c \in b);\ p,\ \alpha,\ \psi| \geq p$

implies $|a=b;\ p,\ \alpha,\ \psi| \geq p$.

Proof. Assume (1).

$$|a=b;\ p,\ \alpha,\ \psi| = E(u,\ v)\,.$$

Thus it suffices to show that

(2) $\inf\{\max(1-u(x),\ M(x,\ v))/x \in D(u)\} \geq p$

and

(3) $\inf\{\max(1-v(y),\ M(y,\ u))/y \in D(v)\} \geq p$.

(1) can be rewritten as follows: for every x in $V(p, \alpha)$,

(4) $\max(1-M(x,\ u),\ M(x,\ v)) \geq p$

and

(5) $\max(1-M(x,\ v),\ M(x,\ u)) \geq p$.

Let x be in $D(u)$ ($\subseteq V(p, \alpha)$).

$$\begin{aligned} 1-M(x,\ u) &\leq 1-\min(u(x),\ E(x,\ x)) \\ &= \max(1-u(x),\ 1-E(x,\ x)) \end{aligned}$$

by 2) of Lemma 1.1. So, if $1-M(x, u) \geq p$, then $\max(1-u(x), 1-E(x, x)) \geq p$. But $1-E(x, x) < p$ (*cf.* 3) of Lemma 1.1). So this means $1-u(x) \geq p$. From this and (4) follows (2). (3) can be established in a similar manner, using (5).

LEMMA 2.4. *Let A be a formula with arguments $a, a_1, \cdots, a_n$. For any $u_1, \cdots, u_n$ in $V(p, \alpha)$, there exists a v in $V(p, \alpha+1)$ such that if ψ is an assignment from $V(p, \alpha)$, where $\psi(a_i)=u_i$, $i=1, \cdots, n$, then $|A;\ p,\ \alpha\ \psi| \geq p$ if and only if $M(\psi(a), v) \geq p$.*

Proof. Define $v: V(p, \alpha) \rightarrow I$ by

$$v(u) = |A;\ p,\ \alpha,\ \psi[a/u]|$$

for every u in $V(p, \alpha)$, ψ as above. (It is obvious that the value is uniquely determined by u when $u_1, \cdots, u_n$ are fixed, independent of the values of ψ at other variables than a, $a_1, \cdots, a_n$.) $v \in V(p, \alpha+1)$, and $M(u, v) \geq p$ is equivalent to

$$|A;\, p,\, \alpha,\, \psi[a/y]| \geq p \quad \text{and} \quad E(u,\, y) \geq p$$

$$\text{for some} \quad y \quad \text{in} \quad V(p,\, \alpha)\,.$$

Then, by the equality axiom (Proposition 2.2), $|A; p, \alpha, \psi[a/u]| \geq p$. On the other hand, $|A; p, \alpha, \psi[a/u]| \geq p$ implies $\min(|A; p, \alpha, \psi[a/u]|, E(u, u)) \geq p$ since $E(u, u)) \geq p$. So $M(u, v) \geq p$.

PROPOSITION 2.3 (*cf.* Theorems 9.6 and 9.7 of [2]). *For any u in $V(p, \alpha)$,*
1) $|\exists b \in a A(b); p, \psi| \geq p$ *if and only if*

$$\sup\{\min(M(y, u),\, |A(b); p, \psi[b/y]|)/y \in V(p, \alpha)\} \geq p\,,$$

where ψ is any assignment from $V(p)$ satisfying $\psi(a) = u$;
2) $|\forall b \in a A(b); p, \psi| \geq p$ *if and only if*

$$\inf\{\max(1 - M(y, u),\, |A(b); p, \psi[b/y]|)/y \in V(p, \alpha)\} \geq p\,.$$

Proof. Let ψ be as above.
$|\exists b \in a A(b); p, \psi| \geq p$ if and only if
(1) $M(x, u) \geq p$ and $|A(b); p, \psi[b/x]| \geq p$ for some x in $V(p)$ (*cf.* 2) of Proposition 2.1). $u \in V(p, \alpha) \subseteq V(p, \alpha+1)$, so u is defined over $V(p, \alpha)$ (*cf.* Lemma 2.2). Thus, $M(x, u) \geq p$ is equivalent to
(2) $E(x, y) \geq p$ and $M(y, u) \geq p$ for some y in $V(p, \alpha)$. (1), (2) and the equality axiom yield
(3) $|A(b); p, \psi[b/y]| \geq p$.
$M(y, u) \geq p$ in (2) and (3) yield the necessary condition in 1). The converse is trivial.

LEMMA 2.5. *If A is a formula with arguments at most $a_1, \cdots, a_n$, and if ψ and ϕ are assignments from D which agree at $a_1, \cdots, a_n$, then $|A; D, \psi| = |A; D, \phi|$.*

§3. Definitive validity of set theory

Assume again that $\frac{1}{2} < p \leq 1$.

Definition 3.1. A formula A of $L(\mathrm{ZF})$ is said to be p-valid if $|A; p, \psi| \geq p$ for every assignment ψ. A is said to be definitively valid if it is p-valid for every p, $\frac{1}{2} < p \leq 1$.

THEOREM. *ZF-set theory is defnitively valid; namely, every ZF-provable formula is definitively valid.*

MAIN PROPOSITION. *For every p, each axiom of ZF is p-valid.*

Proof of Theorem. As a special case of Theorem 5 in [3] we have:

(*) if $\mathscr{A}$ is a set of formulas (of a fixed first order language) each of which is p-valid with regards to Ξ, where Ξ stands for a triple of a domain D, a family of p-definitive functions defined on D and a specification of constants, then any formula provable in the first order predicate calculus augmented by $\mathscr{A}$ as axioms is p-valid with regards to Ξ.

Now the theorem is an immediate consequence of (∗) and the Main Proposition.

In establishing the Main Proposition, we more or less follow § 9 of [2]. Lemmas and propositions in the preceding section will play important roles.

Since p will be fixed throughout, we write $|A; \psi|$ for $|A; V(p), \psi|$ and $|A; \alpha, \psi|$ for $|A; V(p, \alpha), \psi|$.

PROPOSITION 3.1. *The axiom of extensionality is p-valid; for any u and v (in V(p)), and for any ψ such that $\psi(a)=u$ and $\psi(b)=v$,*

$$|\forall c(c\in a\equiv c\in b)\supset a=b;\ \psi|\geq p\,.$$

Proof. $u, v\in V(p, \alpha)$ for some α. For any ϕ an assignment from $V(p, \alpha)$ which agrees with ψ at a and b, $|\forall c(c\in a\equiv c\in b); \psi|\geq p$ implies $|\forall c(c\in a\equiv c\in b); \alpha, \phi|\geq p$ (*cf.* Lemma 2.5); then by Lemma 2.3 $|a=b; \alpha, \psi'|\geq p$, or $E(u, v)\geq p$, which implies $|a=b; \psi|\geq p$.

PROPOSITION 3.2. *The axiom of pairing is p-valid; for any u and v, and any ψ such that $\psi(a)=u$ and $\psi(b)=v$,*

(1) $|\exists c\forall d(d\in c\equiv(d=a\vee d=b)); \psi|\geq p$.

Proof. Let u and v be in $V(p, \alpha)$. Define

$$w(z)=\max(E(z, u), E(z, v))$$

for every z in $V(p, \alpha)$. w is p-definitive (Lemma 1.3). The proof that w acts as the c in (1) goes much the same as the proof of the axiom of the sum set, which is to be presented below. So we proceed to the next proposition.

PROPOSITION 3.3. *The axiom of the sum set is p-valid; for any u and for any ψ such that $\psi(a)=u$,*

$$|\exists b\forall c(c\in b\equiv\exists d\in a(c\in d)); \psi|\geq p\,.$$

Proof. Let u be in $V(p, \alpha)$ for an α. Define v by

$$v(z)=|\exists d\in a(c\in d);\ \alpha,\ \psi[c/z]|$$

for every z in $V(p, \alpha)$, where $\psi(a)=u$. (The uniqueness of the value $v(z)$ is guaranteed by Lemma 2.5.) $D(v)=V(p, \alpha)$ and v is p-definitive (*cf.* 1) of Proposition 2.1). So, $v\in V(p, \alpha+1)$. We first claim that

(1) for every $z\in V(p, \alpha)$, $M(z, v)\geq p$ if and only if $v(z)\geq p$.

Suppose $M(z, v)\geq p$. Then by 1) of Lemma 1.3, $v(y)\geq p$ and $E(z, y)\geq p$ for some $y\in V(p, \alpha)$. But

$$v(y)=|\exists d\in a(c\in d);\ \alpha,\ \psi[c/y]|\geq p\,.$$

This, $E(z, y)\geq p$ and the equality axiom yield $v(z)\geq p$.

Suppose $v(z)\geq p$. $E(z, z)\geq p$. So $M(z, v)\geq p$.

Having proved (1), we proceed to showing:

(2) for any z in $V(p)$, $M(z, v) \geq p$ if and only if $|\exists d \in a(c \in d); \psi[c/z]| \geq p$, ψ being any assignment satisfying $\psi(a) = u$,

which proves the existence of the sum set.

Suppose $M(z, v) \geq p$. By Lemma 2.2,

(3) $E(z, x) \geq p$ and $M(x, v) \geq p$ for some x in $V(p, \alpha)$.

(1) applied to $M(x, v)$ and (3) then yield

$$E(z, x) \geq p \quad \text{and} \quad |\exists d \in a(c \in d);\ \alpha,\ \psi[c/x]| \geq p\,,$$

or

(4) $E(z, x) \geq p$ and $\min(M(y, u), M(x, y)) \geq p$ for some y in $V(p, \alpha)$.

Lemma 2.2 applied to $E(z, x)$ and $M(x, y)$ yields $M(z, y) \geq p$. So by (4)

(5) $M(y, u) \geq p$ and $M(z, y) \geq p$.

This implies $|\exists d \in a(c \in d); \psi[c/z]| \geq p$. The opposite direction can be established by tracing the argument above backwards, making use of Lemma 2.2 and Proposition 2.3.

PROPOSITION 3.4. *The axiom of regularity is p-valid; for every u in V(p) and every ψ such that $\psi(a) = u$,*

$$|\exists b \in a \supset \exists b \in a \forall c \in b(\neg c \in a);\ \psi| \geq p\,.$$

Proof. Let u be in $V(p)$. We shall show:

(1) $|\exists b \in a; \psi| \geq p$ implies $|\exists b \in a \forall c \in b(\neg c \in a); \psi| \geq p$.

Assume $|\exists b \in a; \psi| \geq p$, and suppose

(2) $|\exists b \in a \forall c \in b(\neg c \in a); \psi| < p$, hence $\leq 1 - p$ (*cf.* Proposition 2.1).

Define α by:

$$\alpha = \mu\gamma\{\exists x_0 \in V(p, \gamma)\ (M(x_0, u) \geq p \wedge |\exists b \in a \forall c \in b(\neg c \in a); \psi| < p\}\,.$$

(Such an α is well-defined.) Then, for an x_0 in $V(p, \alpha)$,

$$M(x_0, u) \geq p \quad \text{and} \quad |\exists b \in a \forall c \in b(\neg c \in a);\ \psi| < p\,.$$

The latter relation implies that for any x, if $M(x, u) \geq p$ then $|\forall c \in b(\neg c \in a); \psi[b/x]| < p$. Let x be x_0. Then $M(x_0, u) \geq p$ holds, so

$$|\forall c \in b(\neg c \in a);\ \psi[b/x_0]| < p\,,$$

i.e., for some y_0,

$$M(y_0, x_0) \geq p \quad \text{and} \quad M(y_0, u_0) \geq p\,.$$

From the first relation follows that for some z_0 in $D(x_0)$,

$$x_0(z_0) \geq p \quad \text{and} \quad E(y_0, z_0) \geq p\,.$$

$\text{rank}(z_0) < \text{rank}(x_0) = \alpha$ (by the definition of α). $M(y_0, x_0) \geq p$ and $E(y_0, z_0) \geq p$ imply $M(z_0, x_0) \geq p$ (Proposition 1.2). Also, $E(z_0, y_0) \geq p$ and $M(y_0, u) \geq p$ imply $M(z_0, u) \geq p$. Thus, we have obtained

$$M(z_0, u) \geq p \wedge |\exists b \in a \forall c \in b(\neg c \in a); \psi| < p ,$$

where rank $(z_0) < \alpha$, contradicting the choice of α. So, (2) cannot hold; hence (1).

The p-validity of the axiom of separation requires some preliminaries.

LEMMA 3.1 (*cf.* §9 of [2]). 1) *The function* $F : On \rightarrow I$ *defined by*

$$F(\beta) = \sup \{|A(a); \psi[a/u]| / u \in V(p, \beta)\}$$

(for a fixed ψ*) is non-decreasing with respect to the natural order of* I*, and it is continuous.*

2) *If a function* $F : On \rightarrow I$ *is non-decreasing, then* F *is eventually constant*;

$$\exists \beta \forall \alpha \geq \beta (F(\alpha) = F(\beta)) .$$

3) *Given* α, *there exists a* $\beta \geq \alpha$ *such that*

$$|\exists a A(a); \psi| \geq p \quad \textit{if and only if} \quad \sup \{|A(a); \psi[a/u]| / u \in V(p, \beta)\} \geq p .$$

4) *For every* α *there exists a* $\beta_0 \geq \alpha$ *such that given* $u_1, \cdots, u_n$ *in* $V(p, \alpha)$,

$$|\exists a A(a_1, \cdots, a_n, a); \psi| \geq p \quad \textit{if and only if}$$

$$\sup \{|A(a_1, \cdots, a_n, a); \psi[a/u]| / u \in V(p, \beta_0)\} \geq p ,$$

where $\psi(a_i) = u_i$, $i = 1, \cdots, n$.

5) *For any formula* $A(a_1, \cdots, a_n, a)$, *there exists a semi-normal function* G *which satisfies the following condition*; *for every* α, *given* $u_1, \cdots, u_n$ *in* $V(p, \alpha)$,

$$|\exists a A(a_1, \cdots, a_n, a); \psi| \geq p \quad \textit{if and only if}$$

$$\sup \{|A(a_1, \cdots, a_n, a); \psi[a/u]| / u \in V(p, G(\alpha))\} \geq p ,$$

where $\psi(a_i) = u_i$, $i = 1, \cdots, n$.

Note. According to [2], a function $G : On \rightarrow On$ is said to be semi-normal if the following conditions are fulfilled (*cf.* Definition 7.13 of [2]).

1. $\forall \alpha (\alpha \leq G(\alpha))$.
2. $\forall \alpha, \beta (\alpha < \beta \supset G(\alpha) \leq G(\beta))$.
3. $\forall \alpha \in K_{II} (G(\alpha) = \bigcup \{G(\beta) / \beta < \alpha\})$.

6) *For any* $A(a_1, \cdots, a_n, a)$ *there exists a semi-normal function* G *such that if* β *is a fixed point of* G *and if* $u_1, \cdots, u_n$ *are in* $V(p, \beta)$, *then*

$$|\exists a A(a_1, \cdots, a_n, a); \psi| \geq p \quad \textit{if and only if}$$

$$\sup \{|A(a_1, \cdots, a_n, a); \psi[a/u]|; u \in V(p, \beta)\} \geq p ,$$

where $\psi(a_i) = u_i$, $i = 1, \cdots, n$.

7) *For any* A *with arguments* $a_1, \cdots, a_n$, *there exist some semi-normal functions* $G_1, \cdots, G_m$ *such that if* β *is a common fixed point of* $G_1, \cdots, G_m$ *and if* $u_1, \cdots, u_n$ *are in*

$V(p, \beta)$, then $|A; \psi| \geq p$ if and only if $|A; \beta, \psi| \geq p$ for any ψ such that $\psi(a_i)=u_i$, $i=1, \cdots, n$.

Those lemmas can be proved successively.

PROPOSITION 3.5. *The axiom of separation is p-valid; given $u_1, \cdots, u_n, u$ in $V(p)$, and given $\psi(a_i)=u_i$ and $\psi(a)=u$,*

$$|\exists b \forall c(c \in b \equiv c \in a \wedge A(c, a_1, \cdots, a_n)); \psi| \geq p.$$

Proof. Given $u_1, \cdots, u_n, u$ in $V(p)$. Let the A in 1) of Lemma 3.1 be $c \in a \wedge A$ $(c, a_1, \cdots, a_n)$. There exist semi-normal functions $G_1, \cdots, G_m$ satisfying the condition in 7) of Lemma 3.1 for this formula. It is a theorem in set theory that for any α there exists a common fixed point of $G_1, \cdots, G_m$ greater than α. For an α such that $u_1, \cdots, u_n, u$ belong to $V(p, \alpha)$, let β be a fixed point as above. Then,

(1) for any v in $V(p, \beta)$,

$$|c \in a \wedge A\ (c, a_1, \cdots, a_n); \psi[c/v]| \geq p$$

if and only if

$$|c \in a \wedge A\ (c, a_1, \cdots, a_n); \beta, \psi[c/v]| \geq p,$$

for any ψ satisfying $\psi(a_i)=u_i$ and $\psi(A)=u$. Lemma 2.4 applied to $c \in a \wedge A$ $(c, a_1, \cdots, a_n)$ yields that there exists a w in $V(p, \beta+1)$ such that if ψ is an assignment from $V(p, \beta)$ where $\psi(a_i)=u_i$ and $\psi(a)=u$, then

(2) $|c \in a \wedge A\ (c, a_1, \cdots, a_n); \beta, \psi[c/v]| \geq p$ if and only if $M(v, w) \geq p$.

For the axiom of separation, it suffices to establish that

(3) for any x in $V(p)$, $M(x, w) \geq p$ if and only if

$$|c \in a \wedge A\ (c, a_1, \cdots, a_n); \psi[c/x]| \geq p.$$

$M(x, w) \geq p$ is equivalent to:

for some v in $V(p, \beta)$, $E(x, v) \geq p$ and $M(v, w) \geq p$ (Lemma 2.2). $M(v, w) \geq p$ is equivalent to

$$|c \in a \wedge A\ (c, a_1, \cdots, a_n); \beta, \psi[c/v]| \geq p$$

by (2). By (1) this is equivalent to

$$|c \in a \wedge A\ (c, a_1, \cdots, a_n); \psi[c/v]| \geq p.$$

$E(x, v) \geq p$ then yields

$$|c \in a \wedge A\ (c, a_1, \cdots, a_n); \psi[c/x]| \geq p$$

by the equality axiom.

Conversely, if the necessary condition of (3) holds, then $M(x, u) \geq p$ and $|A(c, a_1, \cdots, a_n); \psi[c/x]| \geq p$.

$M(x, u) \geq p$ implies

$$E(x, v) \geq p \quad \text{and} \quad M(v, u) \geq p$$

for some v in $V(p, \beta)$. So, by the equality axiom,

$$|A(c, a_1, \cdots, a_n); \psi[c/v]| \geq p .$$

Thus, we have obtained $E(x, v) \geq p$ and $M(v, w) \geq p$ for some v in $V(p, \beta)$, which is equivalent to $M(x, w) \geq p$.

The axiom of the power set requires some preliminaries (*cf.* §9 of [2]).

LEMMA 3.2. *If u is in $V(p, \alpha+1)$ and v is in $V(p)$, and if*

(1) $|\forall c(c \in b \equiv c \in a \wedge A(c, a_1, \cdots, a_n)); \psi| \geq p$

for a ψ where $\psi(a)=u$ and $\psi(b)=v$, then v is defined over $V(p, \alpha)$; namely,

(2) *for every x in $V(p)$, $M(x, v) \geq p$ if and only if*

$$E(x, y) \geq p \text{ and } M(y, v) \geq p \text{ for some } y \text{ in } V(p, \alpha).$$

Proof. By (1), $M(x, v) \geq p$ if and only if $M(x, u) \geq p$ and $|A(c, a_1, \cdots, a_n); \psi[c/x]| \geq p$. Since u belongs to $V(p, \alpha+1)$, $M(x, u) \geq p$ if and only if $E(x, y) \geq p$ and $M(y, u) \geq p$ for some y in $V(p, \alpha)$ (Lemma 2.2). But then by the equality axiom $|A(c, a_1, \cdots, a_n); \psi[c/x]| \geq p$ if and only if $|A(c, a_1, \cdots, a_n); \psi[c/y]| \geq p$. Applying (1) again, the latter is equivalent to $M(y, v) \geq p$. Thus follows (2).

LEMMA 3.3. *Let u_1 and u_2 be defined over $V(p, \alpha)$. For a ψ satisfying $\psi(b_1)=u_1$ and $\psi(b_2)=u_2$,*

(1) *if $|c \in b_1 \equiv c \in b_2; \psi[c/y]| \geq p$ for every y in $V(p, \alpha)$, then $E(u_1, u_2)$.*

Proof. By virtue of the axiom of extensionality (Proposition 3.1), it suffices to show that, from the premise in (1) follows

$$|c \in b_1 \equiv c \in b_2; \psi[c/z]| \geq p \qquad \text{for every } z \text{ in } V(p),$$

or

(2) $M(z, u_1) \geq p$ if and only if $M(z, u_2) \geq p$ for every z.

This can be established by making use of the fact that u_1 and u_2 are defined over $V(p, \alpha)$.

LEMMA 3.4. *For any α there exists a β such that, if u is defined over $V(p, \alpha)$, then $E(u, v) \geq p$ for some v in $V(p, \beta)$.*

Proof. $J(p, \alpha)=\{s \mid s \text{ is a map from } V(p, \alpha) \text{ to } I\}$ is a set. For any s in $J(p, \alpha)$, define

$$f(s)=\mu\beta(\exists u \in V(p, \beta)\ (u \text{ is defined over } V(p, \alpha) \text{ and}$$
$$s=\{\langle x, M(x, u)\rangle / x \in V(p, \alpha)\})) .$$
$$\beta=\sup\{f(s)/s \in J(p, \alpha)\}$$

is well-defined. Suppose u is defined over $V(p, \alpha)$, and put

$$s=\{\langle x, M(x, u)\rangle / x \in V(p, \alpha)\} .$$

s belongs to $J(p, \alpha)$. By the definition of $f(s)$, there exists a v in $V(p, f(s))$ such that v is defined over $V(p, \alpha)$ and $s=\{\langle x, M(x, v)\rangle / x\in V(p, \alpha)\}$. So $M(x, u)=M(x, v)$ for every x in $V(p, \alpha)$. By Lemma 3.3 applied to u and v, this implies $E(u, v)\geq p$.

LEMMA 3.5. *For any α, there exists a v such that for every u in $V(p, \alpha)$, $M(u, v)\geq p$.*

Proof. Apply Lemma 2.4 to the formula $a=a$; recall that $|a=a;\ \alpha,\ \psi|\geq p$.

PROPOSITION 3.6. *The axiom of the power set is p-valid; for every u in $V(p)$ and for every ψ such that $\psi(a)=u$,*

$$|\exists b(b=P(a));\ \psi|\geq p\,.$$

Notice that $b=P(a)$ is an abbreviation of $\forall c(c\in b\equiv c\subseteq a)$, or $\forall c(c\in b\equiv\forall d(d\in c\supset d\in a))$.

Proof. Let u be in $V(p, \alpha)$. By Lemma 3.4,

(1) for a β, $\forall v$ (if v is defined over $V(p, \alpha)$, *then*

$$\exists v'\in V(p, \beta)\ (E(v, v')\geq p))\,.$$

Lemma 3.5 applied to a β as in (1) yields

(2) $\exists w\forall v'\in V(p, \beta)\ (M(v, w)\geq p)$.

For such a w we shall show that

(3) $|b=P(a);\ \psi[b/w]|\geq p$.

Suppose temporarily that

(4) $|P(a)\subseteq b;\ \psi[b/w]|\geq p$.

Letting the A in Proposition 3.5 be $d\subseteq a$, we obtain

$$|\exists e\forall d(d\in e\equiv d\in b\wedge d\subseteq a);\ \psi[b/w]|\geq p\,,$$

or

(5) $\exists x\forall y(M(y, x)\geq p$ if and only if $M(y, w)\geq p$ and $|d\subseteq a;\ \psi[b/w][d/y]|\geq p$.

But (4) means that for any y, $|d\subseteq a;\ \psi[b/w][d/y]|\geq p$ implies $M(y, w)\geq p$. So, (5) can be reduced to

$$M(y, x)\geq p \text{ if and only if } |d\subseteq a;\ \psi[b/w][d/y]|\geq p\,,$$

or

$$|\exists e\forall d(d\in e\equiv d\in P(a));\ \psi[b/w]|\geq p\,,$$

which is $|\exists b(b=P(a));\ \psi|\geq p$. So, it suffices to establish (4), which is

(6) $|\forall c(\forall d(d\in c\supset d\in a)\supset c\in b);\ \psi[b/w]|\geq p$.

Consider an x in $V(p)$. By the axiom of separation applied to $f\in c$,

(7) $\exists v(|\forall f(f\in b\equiv f\in c\wedge f\in a);\ \psi[b/v][c/x]|\geq p)$.

Since $u\in V(p, \alpha)\subseteq V(p, \alpha+1)$, Lemma 3.2 applied to (7) yields that v is defined over $V(p, \alpha)$. We can apply (1) to this v, hence we obtain

(8) $\exists v'\in V(p, \beta)\ (E(v, v')\geq p)$

for a β as in (1).

Now, by virtue of (7),

$$|c\subseteq a;\ \psi[c/x]|\geq p \text{ if and only if } \min(|\forall d(d\in c\supset c\in a);\ \psi[c/x]|,$$

$$|\forall f(f\in b\equiv f\in c\wedge f\in a);\ \psi[b/v][c/x]|)\geq p\ .$$

But this implies

$$|\forall f(f\in b\equiv f\in c);\ \psi[b/v][c/x]|\geq p\ .$$

Then the axiom of extensionality implies $E(v, x)\geq p$. This and (8) imply $E(v', x)\geq p$ for a v' in $V(p, \beta)$. Applying (2), $M(v, w)\geq p$ for a w, and hence $E(x, w)\geq p$. Namely, we have shown that $|c\subseteq a;\ \psi[c/x]|\geq p$ implies $|c\in b;\ \psi[b/w][c/x]|\geq p$. Since x was arbitrary in $V(p)$, this means $|\forall c(c\subseteq a\supset c\in b);\ \psi[b/w]|\geq p$, which is (6).

PROPOSITION 3.7. *The axiom of replacement is p-definitive; for any* ψ,

$$|\exists b\forall c\in a\exists d\in b(\exists eA(c, e)\supset A(c, d));\ \psi|\geq p\ .$$

Proof. Let $\psi(a)$ be u. Abbreviate $\exists eA(c, e)\supset A(c, d)$ to $B(c, d)$.

(1) $|\exists dB(c, d);\ \psi|\geq p$

for any ψ, for $\exists dB(c, d)$ is provable in the first order predicate calculus. We may assume u belongs to $V(p, \alpha)$. By virtue of (1) and 4) of Lemma 3.1, there exists a $\beta\geq\alpha$ such that for any x in $V(p, \alpha)$,

(2) $\sup\{|B(c, d);\ \psi[c/x][d/y]|\,|\,y\in V(p, \beta)\}\geq p$.

Also for this β there exists a v such that

(3) $M(y, v)\geq p$ for every y in $V(p, \beta)$ (*cf.* Lemma 3.5).

It trivially follows from (2) and (3) that

$$\sup\{\min(M(y, v), |B(c, d);\ \psi[c/x][d/y]|)/y\in V(p, \beta)\}\geq p\ ,$$

or

(4) $|\exists d\in bB(c, d);\ \psi[b/v][c/x]|\geq p$,

for any x in $V(p, \alpha)$.

Suppose $|c\in a;\ \psi[c/x][a/u]|\geq p$, or $M(x, u)\geq p$, for an arbitrary x in $V(p)$. Since $u\in V(p, \alpha+1)$, Lemma 2.2 implies that $E(x, z)\geq p$ and $M(z, u)\geq p$ for some z in $V(p, \alpha)$. Applying (4) to this z, we obtain

$$|\exists d\in bB(c, d);\ \psi[b/v][c/z]|\geq p\ .$$

This and $E(x, z)\geq p$ imply

$$|\exists d\in bB(c, d);\ \psi[b/v][c/x]|\geq p$$

for any x satisfying $M(x, u)\geq p$. Thus,

$$|\forall c\in a\exists d\in bB(c, d);\ \psi[b/v]|\geq p\ , \qquad \text{q.e.d.}$$

In order to show the p-validity of the axiom of infinity, we shall develop a theory of relations between "intuitive sets" (*cf.* [1]) and $V(p)$.

PROPOSITION 3.8. *For any* ψ *if* $\psi(a)=u$,

1) $|\exists b\in aA(b);\ \psi|\geq p$ *if and only if*

$$\sup\{\min(u(x),\ |A(b);\ \psi[b/x]|)/x\in D(u)\}\geq p\ .$$

2) $|\forall b\in aA(b);\ \psi|\geq p$ *if and only if*

$$\inf\{\max(1-u(x),\ |A(b);\ \psi[b/x]|)/x\in D(u)\}\geq p\ .$$

PROPOSITION 3.9. *Suppose* $\frac{1}{2}<q\leq p\leq 1$.

1) $V(p)\subseteq V(q)$.

2) *If* u *and* v *are in* $V(p)$, *then* $M(u, v; p)=M(u, v; q)$ *and* $E(u, v; p)=E(u, v; q)$.

Proof. 1) Show $V(p, \alpha)\subseteq V(q, \alpha)$ by transfinite induction on α.

2) By transfinite induction on rank$(p; u)$ # rank$(p; v)$.

PROPOSITION 3.10. *Assume* $\frac{1}{2}<q\leq p\leq 1$. *Let* A *be a bounded formula with arguments* $a_1, \cdots, a_n$. *Let* $\psi(a_i)=u_i$ *be in* $V(p)$, $i=1, \cdots, n$. *Then* $|A;\ V(p),\ \psi|\geq p$ *if and only if* $|A;\ V(q),\ \psi|\geq q$.

Proof. By induction on the construction of A. When A is atomic, this follows from Proposition 3.9 and Lemma 1.3. Suppose A is of the form $\exists b\in aA(a, b)$ and $\psi(a)=u$, where $u\in V(p)$. Notice that $u(x)\geq p$ if and only if $u(x)\geq q$ when $x\in D(u)$. By Proposition 3.8,

$$|A;\ p,\ \psi|\geq p \quad \text{if and only if}$$

$$\sup\{\min(u(x),\ |A(a, b);\ p,\ \psi[b/x]|/x\in D(u)\}\geq p\ ,$$

or

(1) $u(x)\geq p$ and $|A(a, b); p, \psi[b/x]|\geq p$ for some $x\in D(u)$ by Proposition 2.1 and p-definitiveness of u. By the induction hypothesis,

$$|A(a, b);\ p,\ \psi[b/x]|\geq p \quad \text{if and only if} \quad |A(a, b);\ q,\ \psi[b/x]|\geq q\ .$$

Noticing this and the remark above we obtain that (1) is equivalent to

$$u(x)\geq q \quad \text{and} \quad |A(a, b);\ q,\ \psi[b/x]|\geq q \quad \text{for some} \quad x\in D(u)\ ,$$

which is $|A;\ q,\ \psi|\geq q$ by Proposition 3.8.

Other cases can be dealt with in a similar manner.

Definition 3.2. Let y be an intuitive set. Then define $\iota(y)$ by

$$\iota(y)=\{<\iota(x),\ 1>/x\in y\}\ .$$

COROLLARY. $\iota(y)\in V(1)$, *hence* $\iota(y)\in V(p)$ *for any* p, $\frac{1}{2}<p\leq 1$.

PROPOSITION 3.11 (*cf.* Theorem 13.17 of [2]). *Let* u, v *be intuitive sets.*

1) $u\in v$ *if and only if* $M(\iota(u), \iota(v))=1$;

$\neg u\in v$ *if and only if* $M(\iota(u), \iota(v))=0$.

2) $u=v$ if and only if $E(\iota(u), \iota(v))=1$;

$\neg u = v$ *if and only if* $E(\imath(u), \imath(v)) = 0$.

3) *For any* u *in* $V(1)$, *there exists a unique intuitive set* v *such that* $E(u, \imath(v)) = 1$.

Note. We abbreviated $M(p; x, y)$ to $M(x, y)$ for any p; this is permitted due to 2) of Proposition 3.9 (since $x, y \in V(p)$ is assumed).

Proof. 1) and 2) are proven simultaneously by transfinite induction on rank (u) # rank (v).

3) Let u be in $V(1)$, and assume as the induction hypothesis

(1) $\forall x \in D(u) \exists ! z (E(x, \imath(z)) = 1)$,

where z stands for an intuitive set. (Note. $u \in V(1)$ implies $D(u) \subseteq V(1)$.)

Define

$$y = \{z; \exists x \in D(u)(E(x, \imath(z)) = 1 \wedge M(x, u) = 1)\}$$

(which is a set by (1)). We claim

(2) $E(u, \imath(y)) = 1$.

Recall that (2) is equivalent to the following two conditions (3) and (4).

(3) $\forall x \in D(u)$, $u(x) = 0$ or $M(x, \imath(y)) = 1$.

(4) $\forall w \in D(\imath(y))$, $\imath(y)(w) = 0$ or $M(w, u) = 1$.

For (3), let x be in $D(u)$ and suppose $u(x) = 1$. Then, since $E(x, x) = 1$, $M(x, u) = 1$. By (1), $\exists ! z (E(x, \imath(z)) = 1)$. Such a z belongs to y, hence $M(\imath(z), \imath(y)) = 1$ by 1) above. This and $E(x, \imath(z)) = 1$ imply $M(z, \imath(y)) = 1$; thus (3).

For (4), let w be in $D(\imath(y))$. Then $w = \imath(z)$ for a z such that $z \in y$, hence by definition

$$\exists x \in D(u)(E(x, \imath(z)) = 1 \wedge M(x, u) = 1) .$$

Then $M(w, u) = 1$. ($\imath(y)(\imath(z)) = 1$ by definition.) So follows (4).

Suppose there is a V satisfying also $E(u, \imath(v)) = 1$. By the equality axiom $E(\imath(y), \imath(v)) = 1$, applying to which 2) above, we obtain $y = v$ as intuitive sets.

PROPOSITION 3.12. *Let* $A(a_1, \cdots, a_n)$ *be a bounded formula with the indicated arguments, and let* $u_1, \cdots, u_n$ *be intuitive sets. If* ψ *is an assignment from* $V(1)$ *satisfying* $\psi(a_i) = \imath(u_i)$, $i = 1, \cdots, n$.

$$A(u_1, \cdots, u_n) \text{ if and only if } |A(a_1, \cdots, a_n); 1, \psi| = 1 .$$

Proof. By induction on the construction of A. For atomic A apply 1) and 2) of Proposition 3.11. Suppose A is of the form $\exists b \in a_1 B(b, a_1, \cdots, a_n)$. $A(u_1, \cdots, u_n)$ if and only if

(1) $B(u, u_1, \cdots, u_n)$ for a $u \in u_1$.

By Proposition 3.11 and the induction hypothesis, (1) is equivalent to

(2) $|B(b, a_1, \cdots, a_n); 1, \psi[b/\imath(u)]| = 1$ for a $u \in u_1$.

On the other hand, $|A(a_1, \cdots, a_n); 1, \psi| = 1$ if and only if

(3) $\imath(u_1)(x) = 1$ and $|B(b, a_1, \cdots, a_n); 1, \psi[b/x]| = 1$ for an $x \in D(\imath(u_1))$

(*cf.* Proposition 3.8).

Suppose first (2) holds. $u \in u_1$ implies $\iota(u) \in D(\iota(u_1))$ and $\iota(u_1)(\iota(u)) = 1$ by definition. So, taking x to be $\iota(u)$, (3) holds. Suppose next (3) holds. By 3) of Proposition 3.11, there exists a unique u such that $E(x, \iota(u)) = 1$. Then $|B(b, a_1, \cdots, a_n); 1, \psi[b/\iota(u)]| = 1$ by the equality axiom. $x \in D(\iota(u_1))$ means $x = \iota(w)$ for some w, and $w \in u_1$. So $E(x, \iota(w)) = 1$. But then by the uniqueness of u, $w = u$. Thus, $u \in u_1$, and we have established (2).

Other cases are dealt with in a similar manner.

PROPOSITION 3.13. *Let $A(a_1, \cdots, a_n)$, $u_1, \cdots, u_n$ and ψ be as in Proposition* 3.12. *Then*

$$A(u_1, \cdots, u_n) \text{ if and only if } |A(a_1, \cdots, a_n); p, \psi| \geq p .$$

Proof. Apply Propositions 3.10 (where p:1 and q:p) and 3.12.

At last, we are in the position to prove the p-validity of the axiom of infinity.

PROPOSITION 3.14. *The axiom of infinity is p-valid*; $|\exists a(\exists b \in a \wedge \forall b \in a \exists c \in a(b \in c)); p| \geq p$.

Proof. Let the $A(a)$ in Proposition 3.13 be $\exists b \in a \wedge \forall b \in a \exists c \in a(b \in c)$. $A(a)$ is bounded. Let ω be the intuitive set which is the first infinite ordinal. By Proposition 3.13, $A(\omega)$ if and only if $|A(a); p, \psi| \geq p$ for a ψ such that $\psi(a) = \iota(\omega)$. But then $|\exists a A(a); p, \psi| \geq p$, or $|\exists a A(a); p| \geq p$.

This completes the proof of the Main Proposition.

There are various properties and examples of the Boolean version of set-theoretical statements seen in § 13 of [2]. It is an easy exercise to give them a definitive version, the practice of which is omitted here.

References

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