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Bieberbach Conjecture for Univalent Functions

by

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Denote by S the class of functions f which are regular and univalent in the unit disc E and satisfy $f(0)=0$, $f'(0)=1$. Bieberbach conjecture is that, if f: $f(z)=z+\sum_2^\infty a_nz^n\in S$, then $|a_n|\leqq n$ for all n. It has been proved for $n=2, 3, 4, 5$, [5], and 6 [8], [4]. Strong evidence that the conjecture is correct is the remarkable result of Hayman [1] that

$$\lim_{n\to\infty}\frac{|a_n|}{n}=\alpha_f<1\,,$$

except when f is $f(z)=z/(1-ze^{it})^2$, where α_f is a constant depending on f. This shows that, for any fixed $f\in S$, the conjecture is correct for $n>n_0(f)$.

The conjecture is known to be valid for certain special classes of functions in S. It is valid, in particular, if all the a_n are real. it is also valid for the classes, S^* and C of functions which are starlike and convex in E respectively. In fact it is shown [7] that a function g is convex if and only if zg' is starlike. An immediate consequence of this is that if $g: g(z)=z+\sum_2^\infty a_nz^n$ is convex, then $|a_n|\leqq 1$ for all n.

Kaplan [3] introduced a class of univalent function which he called close-to-convex. This class, which we name as K, is the most general subclass of S we have uptill now. Reade [6] has shown that the Bieberbach conjecture is true for this class K.

We shall introduce a class of functions which includes K as a subclass and show that the Bieberbach conjecture is true for this class. First we have the following definitions.

Definition 1 [3]. $f: f(z)=z+\sum_2^\infty a_nz^n\in K$ if there exists a convex function g such that, for $z\in E$,

$$\mathrm{Re}\,\frac{f'(z)}{g'(z)}>0$$

Definition 2 [2]. $f: f(z)=z+\sum_2^\infty a_nz^n\in C^*$ if there exists a convex function g such that, for $z\in E$,

$$\mathrm{Re}\,\frac{(zf'(z))'}{g'(z)}>0$$

It is shown [2] that this class C^* includes convex functions as a proper subclass and itself is included in K. The functions in C^* bear the same relation to the functions of K as convex functions to starlike ones, i.e., $f \in C^*$ if and only if $zf' \in K$. It can be shown [2] easily that for $f: f(z)=z+\sum_2^\infty a_n z^n \in C^*$, $|a_n| \leqq 1$ for all n.

Definition 3. $f: f(z)=z+\sum_2^\infty a_n z^n \in K_1$ if there exists a $g \in C^*$ such that, for $z \in E$,

$$\operatorname{Re}\frac{f'(z)}{g'(z)}>0$$

clearly $K \subseteq K_1$.

But $f \in K_1$ may not be univalent. So we consider the subclass $K_1(U)$ of K_1 which contains univalent functions only. Again

$$K \subseteq K_1(U) \subseteq S .$$

THEOREM 1. *Let $f: f(z)=z+\sum_2^\infty a_n z^n \in K_1(U)$. Then $|a_n| \leqq n$ for all n, and this result is sharp.*

Proof. $f'(z)/g'(z)=h(z)$, $\operatorname{Re} h(z)>0$; $g \in C^*$. Let $f(z)=z+\sum_2^\infty a_n z^n$, $g(z)=z+\sum_2^\infty b_n z^n$, and $h(z)=1+\sum_1^\infty c_n z^n$.
Then

$$(n+1)a_n=c_n+\sum_1^{n-1}(k+1)b_{k+1}c_{n-k}+(n+1)b_{n+1}$$

Since $|C_n| \leqq 2$ and $|b_n| \leqq 1$ for all n, we immediately have $|a_n| \leqq n$ for all n. The function $f_0: f_0(z)=z/(1-e^{it}z)^2$ shows that this result is sharp.

Hence we see that the Bieberbach conjecture is true for a larger subclass of S than K.

Generalization.

Let K_2 be defined as follows: $f \in K_2$ if there exists a function g with $zg' \in K_1$ such that, for $z \in E$,

$$\operatorname{Re}\frac{f'(z)}{g'(z)}>0$$

Let $K_2(U)$ be the subclass of K_2 consisting of univalent functions only then

$$K \subseteq K_1(U) \subseteq K_2(U) \subseteq S .$$

From Theorem 1, it follows that the Bieberbach conjecture is also valid for K_2.

Continuing in the same way, we have

$$K \subseteq K_1(U) \subseteq K_2(U) \subseteq K_3(U) \subseteq \cdots K_n(U) \subseteq S .$$

This result implies that $K_n(U)$ is the most general class of S. Moreover the class $K_n(U)$ contains K as a subclass. It is obvious from Theorem 1 that the Bieberbach conjecture is true for the class $K_n(U)$, thus improving the previous known result for the class K, which is the main aim of this paper.

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