COMMENTARII MATHEMATICI
UNIVERSITATIS SANCTI PAULI Vol. 30, No. 2, 1981

Definite Integrals Associated with the H-Function of Several Variables

by

Rekha SRIVASTAVA

(Received September 11, 1980)

In the present note the author evaluates two definite integrals involving the product of the H-function of several variables (which was introduced and studied elsewhere by us; see, for example, [5] through [13]) and either the associated Legendre function or Gauss's hypergeometric function. These results would provide interesting extensions of the corresponding integrals associated with the generalized Lauricella function of several variables, which were given earlier by the author [4].

1. Introduction

Following the various notations already explained fairly fully in our earlier papers [9] and [10], let

$$H_{A,\,C:\ [B',\,D'];\cdots;[B^{(n)},\,D^{(n)}]}^{0,\,\lambda:\ (\mu',\,\nu');\cdots;(\mu^{(n)},\,\nu^{(n)})}\begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix} \tag{1.1}$$

denote the H-function of n complex variables $z_1, \cdots, z_n$ (see also [8], p. 271 *et seq.*). Also, let the associated positive numbers

$$\begin{cases}\theta_j^{(i)},\ j=1,\cdots,A;\ \phi_j^{(i)},\ j=1,\cdots,B^{(i)};\\ \psi_j^{(i)},\ j=1,\cdots,C;\ \delta_j^{(i)},\ j=1,\cdots,D^{(i)};\ i=1,\cdots,n,\end{cases} \tag{1.2}$$

be constrained by the following inequalities:

$$\begin{aligned} \Delta_i \equiv -\sum_{j=\lambda+1}^{A}\theta_j^{(i)}+\sum_{j=1}^{\nu(i)}\phi_j^{(i)}-\sum_{j=\nu(i)+1}^{B(i)}\phi_j^{(i)} \\ -\sum_{j=1}^{C}\psi_j^{(i)}+\sum_{j=1}^{\mu(i)}\delta_j^{(i)}-\sum_{j=\mu(i)+1}^{D(i)}\delta_j^{(i)}>0 \end{aligned} \tag{1.3}$$

and

$$\Omega_i \equiv \sum_{j=1}^{A}\theta_j^{(i)}+\sum_{j=1}^{B(i)}\phi_j^{(i)}-\sum_{j=1}^{C}\psi_j^{(i)}-\sum_{j=1}^{D(i)}\delta_j^{(i)}<0, \qquad \forall i\in\{1,\cdots,n\}. \tag{1.4}$$

(*AMS*) 1980 *Mathematics subject classifications*: Primary 33A30.

Then it is well known that the multiple Mellin-Barnes contour integral defining the H-function (1.1) would converge absolutely when

(1.5) $$|\arg (z_i)| < \frac{1}{2}\Lambda_i \pi, \qquad i=1, \cdots, n,$$

it being understood that the points $z_i=0$, $i=1, \cdots, n$, are excluded, and that (*cf.*, *e.g.*, [9], p. 131, Eq. (1.9)):

(1.6) $$H\begin{matrix} 0, \lambda: & (\mu', \nu');\cdots;(\mu^{(n)}, \nu^{(n)}) \\ A, C: & [B', D'];\cdots;[B^{(n)}, D^{(n)}] \end{matrix}\begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix}$$
$$=\begin{cases} O(|z_1|^{\alpha_1}\cdots|z_n|^{\alpha_n}), \max\{|z_1|, \cdots, |z_n|\}\to 0, \\ O(|z_1|^{-\beta_1}\cdots|z_n|^{-\beta_n}), \lambda\equiv 0, \min\{|z_1|, \cdots, |z_n|\}\to\infty, \end{cases}$$

where, with $i=1, \cdots, n$,

(1.7) $$\begin{cases} \alpha_i=\min\{\mathrm{Re}\,(d_j^{(i)})/\delta_j^{(i)}\}, & j=1, \cdots, \mu^{(i)}, \\ \beta_i=\min\{\mathrm{Re}\,(1-b_j^{(i)})/\phi_j^{(i)}\}, & j=1, \cdots, \nu^{(i)}. \end{cases}$$

The main results of the present paper are the following two definite integrals:

(1.8) $$\int_0^1 t^{\xi-1}(1-t^2)^{-s/2}P_k^s(t)$$
$$\times H\begin{matrix} 0, \lambda: & (\mu', \nu');\cdots;(\mu^{(n)}, \nu^{(n)}) \\ A, C: & [B', D'];\cdots;[B^{(n)}, D^{(n)}] \end{matrix}\begin{pmatrix} z_1t^{2\sigma_1} \\ \vdots \\ z_nt^{2\sigma_n} \end{pmatrix}dt$$
$$=2^{s-1}H\begin{matrix} 0, 2+\lambda: & (\mu', \nu');\cdots;(\mu^{(n)}, \nu^{(n)}) \\ 2+A, C+2: & [B', D'];\cdots;[B^{(n)}, D^{(n)}] \end{matrix}$$
$$\begin{pmatrix} [1-\xi/2: \sigma_1, \cdots, \sigma_n], & [(1-\xi)/2: \sigma_1, \cdots, \sigma_n], & [(a): \theta', \cdots, \theta^{(n)}]: \\ [(c): \psi', \cdots, \psi^{(n)}], & [(s-k-\xi)/2: \sigma_1, \cdots, \sigma_n], & [(1+s+k-\xi)/2: \sigma_1, \cdots, \sigma_n]: \end{pmatrix}$$
$$\left.\begin{matrix} [(b'): \phi'];\cdots;[(b^{(n)}): \phi^{(n)}]; \\ [(d'): \delta'];\cdots;[(d^{(n)}): \delta^{(n)}]; \end{matrix}\; z_1, \cdots, z_n\right),$$

provided that $\sigma_1, \cdots, \sigma_n>0$, $\mathrm{Re}(s)<1$, and

(1.9) $$\mathrm{Re}\left(\xi+2\sum_{i=1}^n \sigma_i\alpha_i\right)>0;$$

(1.10) $$\int_0^1 t^{\eta-1}(1-t)^{\gamma-1}{}_2F_1(\alpha, \beta; \gamma; 1-t)$$

$$\times H_{A,\, C:\, [B',\, D'];\cdots;\, [B^{(n)},\, D^{(n)}]}^{0,\, \lambda:\, (\mu',\, \nu');\cdots;\, (\mu^{(n)},\, \nu^{(n)})}\begin{pmatrix} z_1 t^{\sigma_1} \\ \vdots \\ z_n t^{\sigma_n}\end{pmatrix} dt$$

$$= \Gamma(\gamma) H_{2+A,\, C+2:\, [B',\, D'];\cdots;\, [B^{(n)},\, D^{(n)}]}^{0,\, 2+\lambda:\, (\mu',\, \nu');\cdots;\, (\mu^{(n)},\, \nu^{(n)})}$$

$$\left(\begin{matrix} [1-\eta:\sigma_1,\cdots,\sigma_n],\, [1-\eta+\alpha+\beta-\gamma:\sigma_1,\cdots,\sigma_n], & [(a):\theta',\cdots,\theta^{(n)}]: \\ [(c):\, \psi',\cdots,\psi^{(n)}],\, [1-\eta+\alpha-\gamma:\sigma_1,\cdots,\sigma_n], & [1-\eta+\beta-\gamma:\sigma_1,\cdots,\sigma_n]: \end{matrix}\right.$$

$$\left.\begin{matrix} [(b'):\, \phi'];\cdots;[(b^{(n)}):\, \phi^{(n)}]; \\ [(d'):\, \delta'];\cdots;[(d^{(n)}):\, \delta^{(n)}]; \end{matrix} z_1,\cdots,z_n\right),$$

provided that $\sigma_1, \cdots, \sigma_n > 0$, $\mathrm{Re}(\gamma) > 0$, $\mathrm{Re}(\gamma-\alpha-\beta) > 0$, and

$$\text{(1.11)} \qquad \mathrm{Re}\left(\eta + \sum_{i=1}^{n} \sigma_i \alpha_i\right) > 0,$$

$\alpha_1, \cdots, \alpha_n$ being given by (1.7).

Remark. In our main results (1.8) and (1.10), we assume that the conditions corresponding appropriately to (1.3), (1.4) and (1.5) hold true so that each member of a given result exists.

2. Evaluation of (1.8) and (1.10)

We first replace the multivariable H-function in the integrand of (1.8) by its Mellin-Barnes contour integral [8, p. 271, Eq. (4.1)], and interchange the order of integration which is permissible under the conditions already stated. We then evaluate the innermost t-integral by applying the known integral [2, p. 172, Eq. (24)]

$$\text{(2.1)} \qquad \int_0^1 t^{\xi-1}(1-t^2)^{-s/2} P_k^s(t)\, dt$$

$$= \frac{\sqrt{\pi}\, s^{s-\xi}\Gamma(\xi)}{\Gamma[(\xi-s-k+1)/2]\Gamma[(\xi-s+k+2)/2]},$$

where $\mathrm{Re}(\xi) > 0$, $\mathrm{Re}(s) < 1$, and $P_\nu^\mu(z)$ denotes the associated Legendre function. Finally, we interpret the resulting multiple contour integral as an H-function of several variables, and (1.8) follows at once.

In a similar manner, the integral (1.10) can be evaluated by appealing to the known formula [1, p. 399, Eq. (4)], *viz*

$$\text{(2.2)} \qquad \int_0^1 t^{\eta-1}(1-t)^{\gamma-1}{}_2F_1(\alpha,\beta;\gamma;1-t)\, dt$$

$$= \frac{\Gamma(\gamma)\Gamma(\eta)\Gamma(\gamma+\eta-\alpha-\beta)}{\Gamma(\gamma+\eta-\alpha)\Gamma(\gamma+\eta-\beta)},$$

where $\mathrm{Re}(\gamma)>0$, $\mathrm{Re}(\eta)>0$, $\mathrm{Re}(\gamma+\eta-\alpha-\beta)>0$, and we omit the details.

3. Particular cases

In view of the known relationship [8, p. 272, Eq. (4.7)], both of our results (1.8) and (1.10) can be specialized in terms of the generalized Lauricella function

$$F^{A:\ B';\cdots;\ B^{(n)}}_{C:\ D';\cdots;\ D^{(n)}}\begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix}$$

which was introduced a decade ago (see [3], p. 454 *et seq.*), and we shall thus arrive at our earlier integrals [4, p. 117, Eq. (2.6); p. 118, Eq. (2.8)]; these integrals will, in turn, yield many known results which were discussed systematically in our earlier paper [4].

On the other hand, when $\lambda=A=C=0$, the multivariable H-functions occurring on the left-hand sides of (1.8) and (1.10) would immediately reduce to the product of n different H functions of a single variable; our results will thus yield formulas for the corresponding definite integrals involving various products of H functions.

Finally, if each of the positive coefficients listed in (1.2) is equated to 1, and if the positive parameters $\sigma_1, \cdots, \sigma_n$ are chosen suitably, our integral formulas (1.8) and (1.10) will lead to the corresponding results for the relatively more familiar G functions of several variables.

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Department of Mathematics
University of Victoria
Victoria, British Columbia, Canada
V8W 2Y2