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On Analytic Functions Related with Functions of Bounded Boundary Rotation

by

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ABSTRACT. A class of analytic functions in the unit disc is defined in which the concept of the functions of bounded boundary rotation is generalized. A necessary condition for a function f to belong to this class, coefficient results and Hankel determinant problem are solved.

Let V_k be the class of functions f with bounded boundary rotation. Paatero [7] showed that a function f, analytic in $E=\{z: |z|<1\}, f(0)=0, f'(0)=1, f'(z)\neq 0$; is in V_k if and only if

$$\int_0^{2\pi}\left|\operatorname{Re}\frac{(zf'(z)'}{f'(z)}\right|d\theta\leqq k\pi$$

It is geometrically obvious that $k\geqq 2$. By the Paatero representation theorem [7] for $f\in V_k$, we can write

$$\frac{(zf'(z))'}{f'(z)}=H(z),$$

where

$$H(z)=\left(\frac{k}{4}+\frac{1}{2}\right)H_1(z)-\left(\frac{k}{4}-\frac{1}{2}\right)H_2(z);\qquad \operatorname{Re}H_i(z)>0\qquad i=1,2.$$

We introduce a new class of analytic functions related with the class V_k.

Definition 1. Let f be analytic in E, $f(0)=0$, $f'(0)=1$ and $f'(z)\neq 0$. Then $f\in\mathscr{K}_{kk}$, if there exists a function $g\in V_k$ such that for $z\in E$

$$\int_0^{2\pi}\left|\operatorname{Re}\frac{f'(z)}{g'(z)}\right|d\theta\leqq k\pi,\qquad k\geqq 2.$$

Clearly $\mathscr{K}_{22}\equiv K$, the class of close-to-convex functions introduced by Kaplan [4].

Definition 2. Let f be analytic in $E, f(0)=0, f'(0)=1, f'(z)\neq 0$. Then $f\in\mathscr{K}_{2k}$ if there exists a functions $g\in V_k$ such that for $z\in E$,

$$\operatorname{Re}\frac{f'(z)}{g'(z)}>0.$$

The class $\mathscr{K}_{2k}$ is a subclass of $\mathscr{K}_{kk}$ and was discussed in some detail in [6]. Here we shall deal with $\mathscr{K}_{kk}$.

THEOREM 1. *Let $f \in \mathscr{K}_{kk}$. Then*

(i) $$|f'(z)| \leqq \frac{k(1+r)^{k/2}}{2(1-r)^{(k/2)+2}},$$

(ii) $$|f(z)| \leqq \frac{k}{2(k+2)}\left\{\left(\frac{1+r}{1-r}\right)^{(k/2)+1} - 1\right\}.$$

The function $f_0 \in \mathscr{K}_{kk}$ defined as

$$f_0(z) = \frac{k}{2(k+2)}\left\{\left(\frac{1+z}{1-z}\right)^{(k/2)+1} - 1\right\}, \tag{1}$$

shows that these upper bounds are sharp.

THEOREM 2. *Let $f \in \mathscr{K}_{kk}$. Then, with $z = re^{i\theta}$, and $\theta_1 < \theta_2$,*

$$\int_{\theta_1}^{\theta_2} \mathrm{Re}\left\{\frac{(zf'(z))'}{g'(z)}\right\} d\theta > -(k-1)\pi, \qquad k \geqq 2.$$

Theorem 1 and Theorem 2 follow easily from Definition 1.

Remark 1. From Theorem 2, we can interpret some geometric meaning for $\mathscr{K}_{kk}$. For simplicity let us suppose that the image domain is bounded by an analytic curve C. At a point on C, the outward drawn normal has an angle arg $\{e^{i\theta} f'(e^{i\theta})\}$. Then it follows that the angle of the outward drawn normal turns back at most $(k-1)\pi$. This is a necessary condition for a function f to belong to $\mathscr{K}_{kk}$.

Remark 2. Goodman [3] defines the class $K(\beta)$ of fucntions f as follows. Let f with

$$f(z) = z + \sum_{2}^{\infty} a_n z^n$$

be analytic in E and $f'(z) \neq 0$. Then for $\beta \geqq 0, f \in K(\beta)$ if and only if for $z = re^{i\theta}$, $\theta_1 < \theta_2$

$$\int_{\theta_1}^{\theta_2} \mathrm{Re}\,\frac{(zf'(z))'}{f'(z)} d\theta > -\beta\pi.$$

We note that $\mathscr{K}_{kk} \subset K(k-1)$, $(k-1)$, $(k \geqq 2)$ the functions in $K(k-1)$ for $k > 2$ are not necessarily univalent [3].

We now prove the following.

THEOREM 3. *Let $f \in \mathscr{K}_{kk}$ and be given by*

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n.$$

Then

$$a_n = O(1)\cdot n^{k/2}, \qquad \textit{where} \quad O(1)$$

is a constant and depends upon k only.

The function f_0 defined by (1) shows that the exponent $k/2$ is best possible.

THEOREM 4. *Let $f \in \mathscr{K}_{kk}$ and be given by*

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n .$$

Then

$$\big|\, |a_n| - |a_{n+1}| \,\big| \leqq A(k) n^{(k/2)-1}, \qquad (k \geqq 2),$$

where $A(k)$ is a constant depending upon k only.

The function f_0 defined by (1) shows that the exponent $((k/2)-1)$ is best possible.

Let f be analytic and be given by

$$f(z) = z + \sum_{2}^{\infty} a_n z^n .$$

Suppose that the qth Hankel determinant of f is defined for $q \geqq 1$, $n \geqq 1$ by

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} \cdots\cdots & a_{n+q-1} \\ a_{n+1} \cdots\cdots & \cdots\cdots & \cdots, \\ a_{n+q-1} \cdots\cdots & \cdots\cdots & a_{n+2q-2} \end{vmatrix} \tag{2}$$

In [6], it was shown that if $f \in V_k$,

$$H_q(n) = O(1) \begin{cases} n^{(k/2)-1}, & q = 1 \\ n^{(kq/2)-q^2}, & k \geqq 8q-10, \quad q \geqq 2 \end{cases}$$

The exponent $(kq/2) - q^2$ is best possible in some sense. see [5]. Here we estimate the rate of growth of the Hankel determinant for $f \in \mathscr{K}_{kk}$.

THEOREM 5. *Let $f \in \mathscr{K}_{kk}$ and let the qth Hankel determinant of f for $q \geqq 1$, $n \geqq 1$ be defined by* (2). *Then*

$$H_q(n) = O(1) \begin{cases} n^{k/2}, & q = 1 \\ n^{(kq/2)-q^2+q}, & q \geqq 2, \quad k \geqq 8q-10 . \end{cases}$$

The O(1) is a constant depending upon k, q and f.

To prove Theorem 5, we need the following known lemmas.

LEMMA 1. *Let H be analytic in E, $|H(0)| \leqq 1$ and be defined as*

$$H(z) = \left(\frac{k}{4} + \frac{1}{2}\right) H_1(z) - \left(\frac{k}{4} - \frac{1}{2}\right) H_2(z); \quad \mathrm{Re} H_i(z) > 0, \qquad i = 1, 2 .$$

Then

$$\frac{1}{2\pi}\int_0^{2\pi}|H(z)|^2 d\theta \leqq \frac{1+(k^2-1)r^2}{1-r^2} \qquad (z=re^{i\theta})$$

and

$$\frac{1}{2\pi}\int_0^{2\pi}|H'(z)|d\theta \leqq \frac{k}{1-r^2}$$

Lemma 1 is an easy generalization of one in [8].

LEMMA 2. *Let f be analytic in E and*

$$f(z)=z+\sum_{n=2}^{\infty} a_n z^n .$$

Let the qth Hankel determinant of f for $q\geqq 1$, $n\geqq 1$ be defined by (2) *Then writing $\Delta_j(n)=\Delta_j(n, z_1, f)$, we have*

$$H_q(n)=\begin{vmatrix} \Delta_{2q-2}(n) & \Delta_{2q-3}(n+1) & \cdots & \Delta_{q-1}(n+q-1) \\ \Delta_{2q-3}(n+1) & \Delta_{2q-4}(n+2) & \cdots & \Delta_{q-2}(n+q) \\ \vdots & & & \\ \Delta_{q-1}(n+q-1) & \Delta_{q-2}(n+q) & \cdots & \Delta_0(n+2q-2) \end{vmatrix};$$

where, with $\Delta_0(n, z_1, f)=a_n$, we define for $j\geqq 1$,

$$\Delta_j(n, z_1, f)=\Delta_{j-1}(n, z_1, f)-z_1\Delta_{j-1}(n+1, z_1, f) \tag{3}$$

LEMMA 3. *With $x=(n/(n+1)y)$, $v\geqq 0$ any integer*

$$\Delta_j(n+v, x, zf'(z))=\sum_{k=0}^{j}\binom{j}{k}\frac{y^k(v-(k-1)n)}{(n+1)^k}\Delta_{j-k}(n+v+k, y, f)$$

Lemmas 2 and 3 are due to Noonan and Thomas [5].

We now prove Theorem 5.

Proof. We shall prove this result by using the differences (3). Since $f\in\mathscr{K}_{kk}$ there exists $g\in V_k$ such that

$$f'(z)=g'(z)H(z),$$

where H is defined as in Lemma 1.

Set $$F(z)=(zf'(z))'=g'(z)\{H(z)h(z)+zH'(z)\},$$

where

$$(zg'(z))'=g'(z)h(z)$$

Now for $j\geqq 0$, z_1 any non-zero complex number, consider

$$\Delta_j(n, z_1, F)=\frac{1}{2\pi r^{n+j}}\left|\int_0^{2\pi}(z-z_1)^j(zf'(z))'e^{-i(n+j)\theta}d\theta\right|$$
$$\leqq\frac{1}{2\pi r^{n+j}}\int_0^{2\pi}|z-z_1|^j|g'(z)|\,|H(z)h(z)+zH'(z)|\,d\theta \tag{4}$$

It is known [1] that for $g\in V_k$,

$$g'(z)=\frac{\left(\frac{S_1(z)}{z}\right)^{(k/4)+(1/2)}}{\left(\frac{S_2(z)}{z}\right)^{(k/4)-(1/2)}}, \tag{5}$$

where S_1 and S_2 are starlike.

Also it is known [2, p.162] that we can choose a $z_1=z_1(r)$ with $|z_1|=r$ such that for any univalent function S

$$\max_{|z|=r}|(z-z_1)S(z)|\leqq\frac{2r^2}{1-r^2} \tag{6}$$

Thus, for $k\geqq(4j-2)$

$$|\Delta_j(n, z_1, F)|\leqq\frac{1}{2\pi r^{n+j+1}}\int_0^{2\pi}|z-z_1|^j\frac{|S_1(z)|^{(k/4)+(1/2)}}{|S_2(z)|^{(k/4)-(1/2)}}|H(z)h(z),\ +zH'(z)|\,d\theta$$
$$\leqq r^{1/(n+j+1)}\left(\frac{2r^2}{1-r^2}\right)^j\left(\frac{r}{(1-r)^2}\right)^{(k/4)-j+(1/2)}\left(\frac{4}{5}\right)^{(k/4)-(1/2)}$$
$$\frac{1}{2\pi}\int_0^{2\pi}|H(z)h(z)+zH'(z)|d\theta,$$

where we have used (5), (6) and the fact that for any univalent functions S,

$$\frac{r}{(1+r)^2}\leqq|S(z)|\leqq\frac{r}{(1-r)^2}$$

Now using Lemma 1, we have for $k\geqq(4j-2)$

$$|\Delta_j(n, z_1, F)|\leqq\frac{A(k, j)}{(1-r)^{(k/2)-j+2}},$$

where $A(k, j)$ is a constant depending upon k and j only. Choosing $r=1-(1/n)$, we have for $k\geqq 4j-2$,

$$\Delta_j(n, z_1, F)=O(1)\cdot n^{(k/2)-j+2}\qquad(n\to\infty)$$

Applying Lemma 3, twice, we obtain for $k\geqq 4j-2$,

$$\Delta_j(n, e^{i\theta}n, f)=O(1)\cdot n^{(k/2)-j},\qquad(n\to\infty) \tag{7}$$

where

$$z_1 = \left(\frac{n}{n+1}\right)^2 e^{i\theta_n}$$

Theorems 3 and 4 follow by putting $j=0$, 1 in (4) and proceeding in the same way.

Using (7) and Lemma 2, along with the similar argument due to Noonan and Thomas [5], we have

$$H_q(n) = O(1) \cdot n^{(kq/2) - q^2 + q}, \qquad k \geqq 8q - 10 \qquad q \geqq 2 .$$

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