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A Method for Obtaining Proof Figures of Valid Formulas in the First Order Predicate Calculus

by

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If the validity of a given formula A is confirmed by Herbrand's method (such as the resolution procedure [3] or [4]), it exhibits some informations for the construction of a proof figure of A in Gentzen's sequent calculus LK.

In this paper, we give a procedure for this construction. In fact, by using the informations exhibited by Herbrand's method as guide, a tree-form LK proof of the formula A can be derived upwards, without trial and error for selecting formulas to be decomposed at each stage and for searching a term or a free variable to be substituted for the bound variables indicated in the chief formula on the decomposition corresponding to an inference on $\forall$ or $\exists$.

This method can be used for some interactive proof checker to make a proof in a semi-automatic mannar employing resolution method locally.

§1. The adjoint formula of a formula A and A-formulas

(1) For an LK formula A, we define *the adjoint formula* $\tilde{A}\langle X_1, \cdots, X_n\rangle$ *of* A as follows:

Let negative $\forall$'s and positive $\exists$'s in A be $\mathcal{Q}_1 x_1, \cdots, \mathcal{Q}_n x_n$ from left to right. Then A' is the scheme which is obtained from A by replacing $\mathcal{Q}_i x_i$ with $\mathcal{Q}_i x_i(X_i)$ for each i ($i=1, \cdots, n$), where each $\mathcal{Q}_i$ is $\forall$ or $\exists$.

Let positive $\forall$'s and negative $\exists$'s in A' be $\mathcal{R}_1 y_1, \cdots, \mathcal{R}_l y_l$ from left to right, where $\mathcal{R}_j$ is $\forall$ or $\exists$. Then $\tilde{A}\langle X_1, \cdots, X_n\rangle$ is the scheme which is obtained from A' by replacing $\mathcal{R}_j y_j$ with $\mathcal{R}_j y_j[F_j]$ for each j ($j=1, \cdots, l$), where for each j, F_j is a scheme of the form $f_j(X_{i_1}, \cdots, X_{i_k})$ such that $\mathcal{Q}_{i_1} x_{i_1}(X_{i_1}), \cdots, \mathcal{Q}_{i_k} x_{i_k}(X_{i_k})$ ($i_1 < i_2 < \cdots < i_k$) are all of the form $\mathcal{Q}_i x_i(X_i)$ in A' whose scopes contain $\mathcal{R}_j y_j$ and f_j is a new function symbol which does not appear in A (Skolem function).

In the following, (X_i)'s and $[f_j(X_{i_1}, \cdots, X_{i_k})]$'s are called the guide-indeces of $\tilde{A}\langle X_1, \cdots, X_n\rangle$.

(2) For an LK formula A, let $\mathrm{H}(\tilde{A}\langle X_1, \cdots, X_n\rangle)$ be the Herbland universe, that is, the set of the terms which are obtained by using functions in $\tilde{A}\langle X_1, \cdots, X_n\rangle$

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and an additional constant symbol f^0 if $\tilde{A}\langle X_1, \cdots, X_n\rangle$ contains no constant symbols. We abbreviate $H(\tilde{A}\langle X_1, \cdots, X_n\rangle)$ as $H(\tilde{A})$.

For a fixed LK formula A, *A-formulas* are all of the schema which are obtained from $\tilde{A}\langle X_1, \cdots, X_n\rangle$ and $H(\tilde{A})$ by the following procedure:

1. If $\tau, \cdots, \tau_n \in H(\tilde{A})$, then $\tilde{A}\langle \tau_i, \cdots, \tau_n\rangle$ is an A-formula. 2. If $B \vee C$ is an A-formula, then so are B and C. 3. If $B \wedge C$ is an A-formula, then so are B and C. 4. If $B \supset C$ is an A-formula, then so are B and C. 5. If $\neg B$ is an A-formula, then so is B. 6. If $\mathcal{Q}x(\tau)B(x)$ is an A-formula, then so is $B(\tau)$. If $\mathcal{Q}y[F]C(y)$ is an A-formula, then so is $C(F)$.

(3) For each A-formula B, we denote by $sb(B)$ an LK formula which is obtained from B by replacing every $\mathcal{Q}x(\tau)$ $(\cdots x \cdots)$ in B with $(\cdots \tau \cdots)$ and every $\mathcal{Q}y[F](\cdots y \cdots)$ in B with $(\cdots F \cdots)$, that is, an LK formula obtained from B, by taking every $\mathcal{Q}x(\tau)$ in B for a substitution-operator $\begin{pmatrix} x \\ \tau \end{pmatrix}$ and by taking every $\mathcal{Q}y[F]$ in B for a substitution-operator $\begin{pmatrix} y \\ F \end{pmatrix}$.

(4) For a fixed LK formula A, *sub-A-formulas* are of the A-formulas and subschema of A-formulas which are obtained as follows:

1. Each A-formula is a sub-A-formula. 2. If $B \vee C$ is a sub-A-formula, then so are B and C. 3. If $B \wedge C$ is a sub-S-formula, then so are B and C. 4. If $B \supset C$ is a sub-A-formula, then so are B and C. 5. If $\neg B$ is a sub-A-formula, then so is B. 6. If $\mathcal{Q}x(\tau)B(x)$ is a sub-A-formula, then so is $B(x)$. 7. $\mathcal{Q}y[F]C(y)$ is a sub-A-formula, then so is $C(y)$.

(5) For each sub-A-formula B, we denote by $cl(B)$ a scheme which is obtained from B, by elasing all the guide-indeces (τ)'s and $[F]$'s at the quantifiers in B.

§2. A procedure for obtaining LK proofs of valid formulas

By using adjoint formulas, Herbrand's theorem is expressed in the following form.

THEOREM. *For an arbitrary* LK *formula A, A is* LK *provable if and only if*

(*) *"There exist* $m \geqq 1$ *and* $\tau_{ij} \in H(\tilde{A})$ $(i=1, \cdots, m;\ j=1, \cdots, n)$ *such that* $sb(\tilde{A}\langle \tau_{11}, \cdots, \tau_{1n}\rangle) \vee \cdots \vee sb(\tilde{A}\langle \tau_{m1}, \cdots, \tau_{mn}\rangle)$ *is tautology," where* $\tilde{A}\langle X_1, \cdots, X_n\rangle$ *is the adjoint formula of A.*

"If part" holds by the fact that an algorithm PAL described in the next section, derives an LK-proof of A under the above condition (*).

"Only if part" holds since we can obtain a quantifier-free LK proof of $\rightarrow sb(\tilde{A}\langle \tau_{11}, \cdots, \tau_{1n}\rangle), \cdots, sb(\tilde{A}\langle \tau_{m1}, \cdots, \tau_{mn}\rangle)$ (for some $m \geqq 1$ and $\tau_{ij} \in H(\tilde{A})$) by modifying an arbitrary LK proof of $\rightarrow A$.

We notice that the most aim of this paper is to describe effectively the above algorithm PAL, and to verify the correctness of it for "if part".

Moreover, by using the algorithm we can offer a procedure for trying to obtain an LK proof of an arbitrary formula A.

The procedure consists of two parts.

1. First part is a procedure for checking the validity of a given formula A which is explained by Fig. 1, where ⬭ may be implemented as an algorithm.

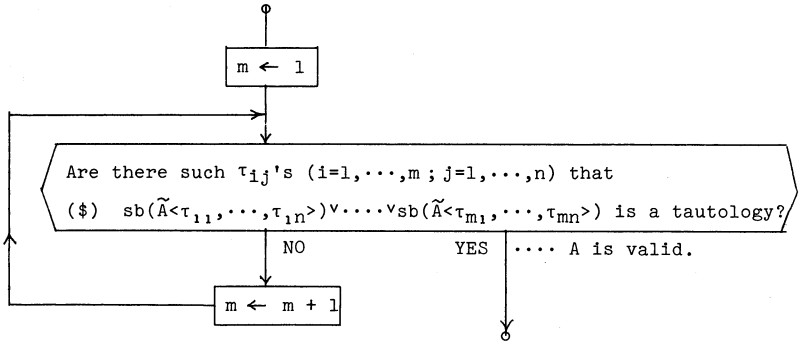

Fig. 1.

2. When the first part terminates (that is, A is valid), we get informations ($m \geqq 1$ and τ_{ij}) which satisfies (*). Then, by using help of the informations, Second part (algorithm PAL) derives deterministically a cut-free LK proof of $\rightarrow A$, as follows:

PAL consists of three phases.

Phase 1 is an algorithm *US* which draws uniquely upwards a tree form proof scheme $\boldsymbol{US}[\mathfrak{S}_0]$ with the end sequent $\mathfrak{S}_0 = \rightarrow \tilde{A}\langle\tau_{11}, \cdots, \tau_{1n}\rangle, \cdots, \tilde{A}\langle\tau_{m1}, \cdots, \tau_{mn}\rangle$, by successive applications of *US*-operation to $\mathfrak{S}_0$. Then domain of *US*-operation is a set of certain sequents of A-formulas which are called q-A-sequents and are precisely defined in the next section. Each application of *US*-operation to a q-A-sequent $\mathfrak{S}$ which has at least one logical symbol, derives uniquely one (or two) upper q-A-sequent(s) $US(\mathfrak{S}) = \mathfrak{S}_1$ (or $\mathfrak{S}_1$; $\mathfrak{S}_2$) of A-formulas, and induces an inference scheme $\dfrac{\mathfrak{S}_1}{\mathfrak{S}} \left(\text{or } \dfrac{\mathfrak{S}_1;\ \mathfrak{S}_2}{\mathfrak{S}}\right)$ which is similar to an LK inference.

The definition of *US*-operation is also described in the next section.

Phase 2 elases all the guide-indeces (τ)'s and [F]'s in $\boldsymbol{US}[\mathfrak{S}_0]$ by *cl*-operation. So, "$cl(\boldsymbol{US}[\rightarrow \tilde{A}\langle\tau_{11}, \cdots, \tau_{1n}\rangle, \cdots, \tilde{A}\langle\tau_{m1}, \cdots, \tau_{mn}\rangle])$" is a cut-free LK proof with the end sequent "$cl(\mathfrak{S}_0) = \rightarrow \underbrace{A, \cdots, A}_{m}$", except that ∀-right and ∃-left inferences have terms of the form $f_j(\tau_1, \cdots, \tau_k)$ for their eigen variables.

Phase 3 (eigen variable adjustment) replaces every maximal Skolem term $F_j = f_j(\tau_1, \cdots, \tau_k)$ in $cl(\boldsymbol{US}[\mathfrak{S}_0])$, with a new free variable α_{F_j}. ··· (α-operation)

Then we obtain finally a cut-free LK proof $\alpha(cl(\boldsymbol{US}[\mathfrak{S}_0]))$ of $\rightarrow \underbrace{A, \cdots, A}_{m}$, consequently $\rightarrow A$.

In the above, $f_j(\tau_1, \cdots, \tau_k)$ in a formula B is called a maximal Skolem term, if B has no Skolem functions which are operated later than the indicated $f_j(\tau_1, \cdots, \tau_k)$ in B.

The above descriptions will be verified in the following sections.

§3. *q-A*-sequents and *US*-operation

The objects of *US*-operation are *q-A*-sequents which are defined as follows:

A sequent $B_1, \cdots, B_p \to C_1, \cdots, C_q$ of A-formulas is called a *q-A-sequent* if (i) quantifiers of the form $\exists x(\tau)$ or $\forall y[F]$ in $\neg B_1, \cdots \neg B_p, C_1, \cdots, C_q$ are positive, and (ii) quantifiers of the form $\exists y[F]$ or $\forall x(\tau)$ in $\neg B_1, \cdots, \neg B_p, C_1, \cdots, C_q$ are negative.

We notice that $\to \tilde{A}\langle \tau_{11}, \cdots, \tau_{1n}\rangle, \cdots, \tilde{A}\langle \tau_{m1}, \cdots, \tau_{mn}\rangle$ is a *q-A*-sequent, and if $\mathfrak{S}$ is a *q-A*-sequent, then each sequent of $US(\mathfrak{S})$ is also a *q-A*-sequent, according to the definition of *US*-operation below.

(def.) *Degrees of terms in* $\mathrm{H}(\tilde{A})$ *and degrees of q-A-sequents*

(i) For each Skolem function f_j and $\tau_i \in \mathrm{H}(\tilde{A})$,

$$deg(f_j(\tau_1, \cdots, \tau_k)) = \omega \cdot lg(f_j(\tau_1, \cdots, \tau_k)) + j\,,$$

(ii) For each function g in A which is not a Skolem function, and $\tau_i \in \mathrm{H}(\tilde{A})$,

$$deg(g(\tau_1, \cdots, \tau_k)) = \omega \cdot lg(g(\tau_1, \cdots, \tau_k))\,.$$

In (i) and (ii), $lg(h(\tau_1, \cdots, \tau_k)) = lg(\tau_1) + \cdots + lg(\tau_k) + 1$ and $lg(c) = 1$, if c is a letter.

(iii) For a *q-A*-sequent $\mathfrak{S}$,

$$deg(\mathfrak{S}) = \begin{cases} min\,\{deg(F) \mid \mathcal{Q}y[F] \text{ in } \mathfrak{S}\}, & \\ & \text{if } \mathfrak{S} \text{ has at least one } \mathcal{Q}y[F] \text{ form;} \\ 0, & \text{if } \mathfrak{S} \text{ has no } \mathcal{Q}y[F] \text{ forms.} \end{cases}$$

Definition of US-operation

For each *q-A*-sequent $\Gamma \to \Delta$ which contains at least one logical symbol, we define the upper *q-A*-sequent(s) $US(\Gamma \to \Delta)$ as follows:

Case 0. Either of Γ or Δ has two formulas* of the same form.

0.1. Γ has such two formulas.

$$\begin{aligned} &US(\Gamma \to \Delta) = \Gamma_0, D, \Gamma_1, \cdots, \Gamma_k \longrightarrow \Delta\,, \\ &\text{if } \;\Gamma \to \Delta\; = \Gamma_0, D, \Gamma_1, D, \cdots, D, \Gamma_k \longrightarrow \Delta\,, \end{aligned}$$

where $\Gamma_0, \Gamma_1, \cdots, \Gamma_k$ have not D (and formulas of Γ_0 are different to each other).

0.2. Γ has no two formulas of the same form, but Δ has.

$$\begin{aligned} &US(\Gamma \to \Delta) = \Gamma \longrightarrow \Delta_0, C, \Delta_1, \cdots, \Delta_k\,, \\ &\text{if } \;\Gamma \to \Delta\; = \Gamma \longrightarrow \Delta_0, C, \Delta_1, C, \Delta_2, \cdots, C, \Delta_k\,, \end{aligned}$$

where $\Gamma, \Delta_0, \cdots, \Delta_k$ does not have C (and formulas of Δ_0 are different to each other).

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* In this paragraph, a word "formula" is used for "A-formula."

Case 1. Both Γ and Δ have no two formulas of the same form and $\Gamma\to\Delta$ has either of the form $B\vee C$, $B\wedge C$, $B\supset C$ or $\neg B$. Let D be the leftmost formula of those.

1.*l*.1. $\Gamma\ni D=B\vee C$:

$$US(\Gamma\to\Delta)=\Gamma_1, B, \Gamma_2 \longrightarrow \Delta \ ; \ \Gamma_1, C, \Gamma_2 \longrightarrow \Delta ,$$
$$\text{if } \Gamma\to\Delta = \Gamma_1, B\vee C, \Gamma_2 \longrightarrow \Delta .$$

1.*l*.2. $\Gamma\ni D=B\wedge C$:

$$US(\Gamma\to\Delta)=\Gamma_1, B, C, \Gamma_2 \longrightarrow \Delta ,$$
$$\text{if } \Gamma\to\Delta = \Gamma_1, B\wedge C, \Gamma_2 \longrightarrow \Delta .$$

1.*l*.3. $\Gamma\ni D=B\supset C$:

$$US(\Gamma\to\Delta)=\Gamma_1, \Gamma_2\to B, \Delta \ ; \ \Gamma_1, C, \Gamma_2 \longrightarrow \Delta ,$$
$$\text{if } \Gamma\to\Delta = \Gamma_1, B\supset C, \Gamma_2 \longrightarrow \Delta .$$

1.*l*.4. $\Gamma\ni D=\neg B$:

$$US(\Gamma\to\Delta)=\Gamma_1, \Gamma_2 \longrightarrow B, \Delta ,$$
$$\text{if } \Gamma\to\Delta = \Gamma_1, \neg B, \Gamma_2 \longrightarrow \Delta .$$

1.*r*.1. $\Delta\ni D=B\vee C$:

$$US(\Gamma\to\Delta)=\Gamma \longrightarrow \Delta_1, B, C, \Delta_2 ,$$
$$\text{if } \Gamma\to\Delta = \Gamma \longrightarrow \Delta_1, B\vee C, \Delta_2 .$$

1.*r*.2. $\Delta\ni D=B\wedge C$:

$$US(\Gamma\to\Delta)=\Gamma \longrightarrow \Delta_1, B, \Delta_2 \ ; \ \Gamma \longrightarrow \Delta_1, C, \Delta_2 ,$$
$$\text{if } \Gamma\to\Delta = \Gamma \longrightarrow \Delta_1, B\wedge C, \Delta_2 .$$

1.*r*.3. $\Delta\ni D=B\supset C$:

$$US(\Gamma\to\Delta)=B, \Gamma \longrightarrow \Delta_1, C, \Delta_2 ,$$
$$\text{if } \Gamma\to\Delta = \Gamma \longrightarrow \Delta, B\supset C, \Delta_2 .$$

1.*r*.4. $\Delta\ni D=\neg B$:

$$US(\Gamma\to\Delta)=B, \Gamma \longrightarrow \Delta_1, \Delta_2 ,$$
$$\text{if } \Gamma\to\Delta=\Gamma \longrightarrow \Delta_1, B\supset C, \Delta_2 .$$

Case 2. Both Γ and Δ have no two formulas of the same form and $\Gamma\to\Delta$ has no formulas of the forms $B\vee C$, $B\wedge C$, $B\supset C$ and $\neg B$.

2.1. $deg(\Gamma\to\Delta)>0$: Here $\Gamma\to\Delta$ contains at least one $\mathscr{Q}y[F]$ form.

2.1.1. *$\Gamma\to\Delta$ has $\mathscr{Q}x(\tau)B(x)$ such that $deg(\tau)<deg(\Gamma\to\Delta)$.*
Let D be the leftmost formula of those $\mathscr{Q}x(\tau)B(x)$'s.

2.1.1.*l*. $D\in\Gamma$: Here $D=\forall x(\tau)B(x)$, since the pointed $\mathscr{Q}x(\tau)$ in $\neg D=\neg\mathscr{Q}x(\tau)B(x)$ is negative and $\Gamma\to\Delta$ is a q-A-sequent. Then define that

$$US(\Gamma \to \Delta) = \Gamma_1,\ B(\tau),\ \Gamma_2 \ \longrightarrow\ \Delta\ ,$$
$$\text{if}\quad \Gamma \to \Delta\ = \Gamma_1,\ \forall x(\tau)\ B(x),\ \Gamma_2 \ \longrightarrow\ \Delta\ .$$

2.1.1.*r*. $D \in \Delta$: Here $D = \exists x(\tau)B(x)$, by a similar reasoning to the case (2.1.1.*l*.). Then we define that

$$US(\Gamma \to \Delta) = \Gamma \ \longrightarrow\ \Delta_1,\ B(\tau),\ \Delta_2\ ,$$
$$\text{if}\quad \Gamma \to \Delta\ = \Gamma \ \longrightarrow\ \Delta_1,\ \exists x(\tau)B(x),\ \Delta_2\ .$$

2.1.2. *No* $\mathcal{Q}x(\tau)B(x)$ *such that* $deg(\tau) < deg(\Gamma \to \Delta)$, *appears in* $\Gamma \to \Delta$: In this case, $\Gamma \to \Delta$ has at least one $\mathcal{Q}y[F]B(y)$ such that $deg(F) = deg(\Gamma \to \Delta)$. Let $D = \mathcal{Q}y[F]B_1(y)$ be the leftmost one of those.

2.1.2.*l*. $D \in \Gamma$: Here $D = \exists y[F]B_1(y)$, since $\Gamma \to \Delta$ is a *q*-*A*-sequent.
Then we define that

$$US(\Gamma \to \Delta) = \Gamma_0,\quad B_1(F),\ \Gamma_1,\quad B_2(F),\ \cdots,\quad B_k(F),\ \Gamma_k \ \longrightarrow\ \Delta\ ,$$
$$\text{if}\ \Gamma \to \Delta\ = \Gamma_0,\ \exists y[F]B_1(y),\ \Gamma_1,\ \exists y[F]B_2(y),\ \cdots,\ \exists y[F]B_k(y),\ \Gamma_k \ \longrightarrow\ \Delta$$

by indicating all the formulas with the same $\exists y[F]$ at front.

(In this case, we denote that $cl(B_1(F)) = \cdots = cl(B_k(F))$.)

2.1.2.*r*. $D \in \Delta$: Here $D = \forall y[F]B_1(y)$. Then we define that

$$US(\Gamma \to \Delta) = \Gamma \to \Delta_0,\quad B_1(F),\ \Delta_1,\quad B_2(F),\ \cdots,\quad B_k(F),\ \Delta_k\ ,$$
$$\text{if}\quad \Gamma \to \Delta\ = \Gamma \to \Delta_0,\ \forall y[F]B_1(y),\ \Delta_1,\ \forall y[F]B_2(y),\ \cdots,\ \forall y[F]B_k(y),\ \Delta_k$$

by indicating all the formulas with the same $\forall y[F]$ at front.

(In this case, also $cl(B_1(F)) = \cdots = cl(B_k(F))$.)

2.2. $deg(\Gamma \to \Delta) = 0$.

2.2.1. $\Gamma \to \Delta$ has a formula of the form $\mathcal{Q}x(\tau)B(x)$. let D be the leftmost one of those.

2.2.1.*l*. $D \in \Gamma$: Here $D = \forall x(\tau)B(x)$. Then define that

$$US(\Gamma \to \Delta) = \Gamma_1,\ B(\tau),\ \Gamma_1 \ \longrightarrow\ \Delta\ ,$$
$$\text{if}\quad \Gamma \to \Delta\ = \Gamma_1,\ \forall x(\tau)B(x),\ \Gamma_2 \ \longrightarrow\ \Delta\ .$$

2.2.1.*r*. $D \in \Delta$: Here $D = \exists x(\tau)B(x)$. Then define that

$$US(\Gamma \to \Delta) = \Gamma \ \longrightarrow\ \Delta_1,\quad B(\tau),\ \Delta_2\ ,$$
$$\text{if}\quad \Gamma \to \Delta\ = \Gamma \ \longrightarrow\ \Delta_1,\ \exists x(\tau)B(x),\ \Delta_2\ .$$

2.2.2. $\Gamma \to \Delta$ has no formulas of the $\mathcal{Q}x(\tau)B(x)$. Here $\Gamma \to \Delta$ consists of only primitive formulas. In this case, $US(\Gamma \to \Delta)$ is undefined.

§4. Propositions on *US*-operation and the correctness of PAL algorithm

In this section, we list up several propositions on *US*-operation.
Then, by using them, we prove the correctness of PAL algorithm for deriving

effectively LK proof figures of valid formulas by using the informations characterizing the validities of these formulas.

But, the proofs of these propositions will be left in the next section.

PROPOSITION 1. *If* $\boldsymbol{\Gamma}\to\boldsymbol{\Delta}$ *is a q-A-sequent and satisfies the following condition* (#), *then so does each sequent of* $US(\boldsymbol{\Gamma}\to\boldsymbol{\Delta})$.

(#) *If* $\mathcal{Q}y[F]$ *appears in a q-A-sequent* $\mathfrak{S}$, *then* F *does not appear in* $cl(\mathfrak{S})$.

PROPOSITION 2. *If* $\boldsymbol{\Gamma}\to\boldsymbol{\Delta}$ *is a q-A-sequent, then*

$$(c)\qquad \frac{sb(US(\boldsymbol{\Gamma}\to\boldsymbol{\Delta}))}{sb(\boldsymbol{\Gamma}\to\boldsymbol{\Delta})}$$

is a quantifier-free LK *deduction. If the lower sequent of* (*c*) *is a tautology, then so is each of the upper sequents of* (*c*).

We notice that a quantifier-free LK sequent $B_1, \cdots, B_p\to C_1, \cdots, C_q$ ($p\geqq 1$ or $q\geqq 1$) is called a tautology, if $\neg B_1\vee\cdots\vee B_p\vee C_1\vee\cdots\vee C_q$ is a tautology.

PROPOSITION 3. *If* $\boldsymbol{\Gamma}\to\boldsymbol{\Delta}$ *is a q-A-sequent and satisfies the condition* (#), *then*

$$\frac{cl(US(\boldsymbol{\Gamma}\to\boldsymbol{\Delta}))}{cl(\boldsymbol{\Gamma}\to\boldsymbol{\Delta})}$$

is an LK* *deduction, by inserting some* LK *inferences.*

In the above, LK* is defined as a deduction system which is obtained from LK, by adding Skolem functions f_j's in $\tilde{A}\langle X_1, \cdots, X_n\rangle$ to the language given in the biginning and by replacing inference-rules on quantifiers with the following:

$$\forall^*\text{ left:}\quad \frac{C(\tau),\ \Gamma\to\Delta}{\forall xC(x),\ \Gamma\to\Delta}$$

$$\exists^*\text{ right:}\quad \frac{\Gamma\to\Delta,\ C(\tau)}{\Gamma\to\Delta,\ \exists xC(x)}$$

where $C(\tau)$ contains no terms of the form $f_d(\sigma_1, \cdots, \sigma_p)$ which contains the indicated τ in proper part.

$$\exists^*\text{ left:}\quad \frac{D(f_j(\tau_1, \cdots, \tau_k)),\ \Gamma\to\Delta}{\exists yD(y),\ \Gamma\to\Delta}$$

$$\forall^*\text{ right:}\quad \frac{\Gamma\to\Delta,\ D(f_j(\tau_1, \cdots, \tau_k))}{\Gamma\to\Delta,\ \forall yD(y)}$$

where

(i) $F=f_j(\tau_1, \cdots, \tau_k)$ does not appear in the lower sequent,

(ii) $D(f_j(\tau_1, \cdots, \tau_k))$ contains no terms of the form $f_d(\sigma_1, \cdots, \sigma_p)$ which contains the indicated $F=f_j(\tau_1, \cdots, \tau_k)$ in proper part.

(def.) For each formula D in LK*, $\alpha(D)$ is a formula which is obtained from D, by replacing every maximal Skolem term F in D with a free variable α_F with subscript F.

PROPOSITION 4. *If $\mathscr{P}$ is an* LK* *proof, then $\alpha(\mathscr{P})$ is an* LK *proof, where $\alpha(\mathscr{P})$ is a proof scheme which is obtained from $\mathscr{P}$, by replacing every formula D in $\mathscr{P}$ with $\alpha(D)$.*

(deg.) For a q-A-sequent $\mathfrak{S}_0$, $\boldsymbol{US}[\mathfrak{S}_0]$ is a tree form proof scheme which is derived by successive applications of US-operation, as follows:

Let $\mathscr{P}_0=\mathfrak{S}_0$ and let $\mathscr{P}_i$ be a tree form which is obtained at the i-th stage. If $\mathscr{P}_i$ has uppermost sequents $\mathfrak{S}_{i_1},\cdots,\mathfrak{S}_{i_p}$ $(p\geqq 1)$ which are not of primitive formulas, then $\mathscr{P}_{i+1}$ is the scheme which is obtained from $\mathscr{P}_i$, by drawing $US(\mathfrak{S}_{i_j})$ over $\mathfrak{S}_{i_j}$ $(j=1,\cdots,p)$. If every uppermost sequent of $\mathscr{P}_i$ is of primitive formulas, then $\boldsymbol{US}[\mathfrak{S}_0]=\mathscr{P}_i$.

THEOREM. *Let $\tilde{A}\langle X_1,\cdots,X_n\rangle$ be the adjoint formula of A and let $\mathfrak{S}_0=\rightarrow\tilde{A}\langle\tau_{11},\cdots,\tau_{1n}\rangle,\cdots,\tilde{A}\langle\tau_{m1},\cdots,\tau_{mn}\rangle$.*

If $sb(\mathfrak{S}_0)=\rightarrow sb(\tilde{A}\langle\tau_{11},\cdots,\tau_{1n}),\cdots,sb(\tilde{A}\langle\tau_{m1},\cdots,\tau_{mn}\rangle)$ is a tautology, then $\alpha(cl(\boldsymbol{US}[\mathfrak{S}_0]))$ is a cut-free LK proof of $\rightarrow\underbrace{A,\cdots,A}_{m}$.

Proof. Tree-form proof scheme $\boldsymbol{US}[\mathfrak{S}_0]$ has the end sequent $\mathfrak{S}_0=\rightarrow\tilde{A}\langle\tau_{11},\cdots,\tau_{1n}\rangle,\cdots,\tilde{A}\langle\tau_{m1},\cdots,\tau_{mn}\rangle$ and has the topmost sequents of only the form $P_1,\cdots,P_p\rightarrow Q_1,\cdots,Q_q$ where P_i and Q_j are primitive formulas.

So, $sb(\boldsymbol{US}[\mathfrak{S}_0])$ has also the topmost sequents of only the form $P_1,\cdots,P_p\rightarrow Q_1,\cdots,Q_q$ and has the end sequent $\rightarrow sb(\tilde{A}\langle\tau_{11},\cdots,\tau_{1n}\rangle),\cdots,sb(\tilde{A}\langle\tau_{m1},\cdots,\tau_{mn}\rangle)$ which is a tautology, by the assumption. Then, by Proposition 2, every topmost sequents $P_1,\cdots,P_p\rightarrow Q_1,\cdots,Q_q$ is a tautology, that is, $\neg P_1\vee\cdots\vee\neg P_p\vee Q_1\vee\cdots\vee Q_q$ is a tautology. So, $P_i=Q_j$ for some i and j.

On the other hand, the tree form proof scheme $cl(\boldsymbol{US}[\mathfrak{S}_0])$ has the end sequent $cl(\mathfrak{S}_0)=\rightarrow\underbrace{A,\cdots,A}_{m}$ and the topmost sequents of the form $P_1,\cdots,P_p\rightarrow Q_1,\cdots,Q_q$ such that $P_i=Q_j$ for some i and j, that is, $P_1,\cdots,P_p\rightarrow Q_1,\cdots,Q_q$ is deduced from a beginning sequent $P_i\rightarrow Q_j$.

Since $\mathfrak{S}_0$ satisfies the condition (#) by Proposition 1, every q-A-sequent of $\boldsymbol{US}[\mathfrak{S}_0]$ also satisfies the condition (#).

Then, by these facts and Proposition 3, $cl(\boldsymbol{US}[\mathfrak{S}_0])$ induces a cut-free LK* proof of $\rightarrow\underbrace{A,\cdots,A}_{m}$, by inserting some inferences.

Therefore, by Proposition 4, $\alpha(cl(\boldsymbol{US}[\mathfrak{S}_0]))$ is a cut-free LK proof of $\rightarrow\underbrace{A,\cdots,A}_{m}$, and derives consequently $\rightarrow A$.

§ 5. Proofs of propositions

In order to prove the propositions in the preceding sections, we use preliminary lemmas on A-formulas.

We have immediately the next lemma, according to the definition of A-formulas.

LEMMA 1. *For an arbitrary A-formula B,* (*) *every scheme of the form $f_d(T_1, \cdots, T_k)$ in B never contains variables y_j and x_i where f_d is a Skolem function.*

In the following, $\boldsymbol{S}$, $\boldsymbol{c}$ and $\#$ denote quantifier-symbols.

LEMMA 2. *For an arbitrary* LK *formula A, if $\boldsymbol{S}y_i[F]B(y_i)$ and $\boldsymbol{c}y_j[F]C(y_j)$ are A-formulas, then $i=j$ and $\boldsymbol{S}=\boldsymbol{c}$.*

Proof. In general, by the definition of A-formulas, we can easily see that if $\boldsymbol{S}y_i[F]B(y_i)$ is an A-formula, then F must be of the form $f_i(\tau_1, \cdots, \tau_p)$ and $\boldsymbol{S}y_i$ appears only as the i-th quantifier with [ ]-guide-index in $\tilde{A}\langle X_1, \cdots, X_n\rangle$ from the left side.

Therefore, by the assumption, F is of the form $f_i(\tau_1, \cdots, \tau_p)$ and F is also of the form $f_j(\sigma_1, \cdots, \sigma_q)$. So, $i=j$. Thus $\boldsymbol{S}y_i$ and $\boldsymbol{c}y_j(=\boldsymbol{c}y_i)$ both are the i-th quantifier with [ ]-guide-index in $\tilde{A}\langle X_1, \cdots, X_n\rangle$.

Therefore, $\boldsymbol{S}=\boldsymbol{c}$.

LEMMA 3. *If $\boldsymbol{S}y_j[F]B(y_j)$ and $\boldsymbol{S}y_j[F]C(y_j)$ both are A-formulas, then $cl(B(F))=cl(C(F))$.*

Proof. Let $\tilde{A}\langle\tau_1, \cdots, \tau_n\rangle$ and $\tilde{A}\langle\sigma_1, \cdots, \sigma_n\rangle$ be A-formulas, from which $\boldsymbol{S}y_j[F]B(y_j)$ and $\boldsymbol{S}y_j[F]C(y_j)$ are obtained respectively.

Let $\mathcal{Q}_{i_1}x_{i_1}(X_{i_1}), \cdots, \mathcal{Q}_{i_r}x_{i_r}(X_{i_r})$ $(i_1<\cdots<i_r)$, $\mathcal{R}_{j_1}y_{j_1}[f_{j_1}(X_{i_1}, \cdots, X_{i_{p1}})], \cdots, \mathcal{R}_{j_s}y_{j_s}[f_{j_s}(X_{i_1}, \cdots, X_{i_{ps}})]$ $(j_1<\cdots<j_s<j: p_1\leqq\cdots\leqq p_s\leqq r)$ be all of the quantifiers in $\tilde{A}\langle X_1, \cdots, X_n\rangle$, whose scopes contains $\boldsymbol{S}y_j$.

Then, the adjoint formula $\tilde{A}\langle X_1, \cdots, X_n\rangle$ has a subscheme $H=\boldsymbol{S}y_j[f_j(X_{i_1}, \cdots, X_{i_r})]D(x_{i_1}, \cdots, x_{i_r}, y_{j_1}, \cdots, y_{j_s}, y_j; \langle X_{i_1}, \cdots, X_{i_r}, \cdots, X_{i_t}\rangle)$ where (1) $x_{i_1}, \cdots, x_{i_r}, y_{j_1}, \cdots, y_{j_s}$ are not bounded in H and (2) $\langle X_{i_1}, \cdots, X_{i_r}, \cdots, X_{i_t}\rangle$ in H $(t\geqq r)$ means that $X_{i_1}, \cdots, X_{i_r}, \cdots, X_{i_t}$ may appear only in the guide-indeces (τ)'s and $[F]$'s at some quantifiers in $D(x_{i_1}, \cdots, x_{i_r}, y_{j_1}, \cdots, y_{j_s}, y_j; \langle X_{i_1}, \cdots, X_{i_r}, \cdots, X_{i_t}\rangle)$.

Therefore, $\boldsymbol{S}y_j[F]B(y_j)=\boldsymbol{S}y_j[f_j(\tau_{i_1}, \cdots, \tau_{i_r})]D(\tau_{i_r}, \cdots, \tau_{i_r}, f_{j_1}(\tau_{i_1}, \cdots, \tau_{i_{p_1}}), \cdots, f_{j_s}(\tau_{i_1}, \cdots, \tau_{i_{ps}}), y_j; \langle\tau_{i_1}, \cdots, \tau_{i_r}, \cdots, \tau_{i_t}\rangle)$, since all of the quantifiers in $\tilde{A}\langle\tau_1, \cdots, \tau_n\rangle$ whose scopes contain $\boldsymbol{S}y_j$, are $\mathcal{Q}_{i_1}x_{i_1}(\tau_{i_1}), \cdots, \mathcal{Q}_{i_r}x_{i_r}(\tau_{i_r})$, $\mathcal{R}_{j_1}y_{j_1}[f_{j_1}(\tau_{i_1}, \cdots, \tau_{i_{p_1}})], \cdots, \mathcal{R}_{j_s}y_{j_s}[f_{j_s}(\tau_{i_1}, \cdots, \tau_{i_{ps}})]$ and $\tilde{A}\langle\tau_1, \cdots, \tau_n\rangle$ has a subscheme $\boldsymbol{S}_{y_j}[f_j(\tau_{i_1}, \cdots, \tau_{i_r})]D(x_{i_1}, \cdots, x_r, y_{j_1}, \cdots, y_{j_s}, y_j; \langle\tau_{i_1}, \cdots, \tau_{i_r}, \cdots, \tau_{i_t}\rangle)$.

Similarly, $\boldsymbol{S}y_j[F]C(y_j)=\boldsymbol{S}y_j[f_j(\sigma_{i_1}, \cdots, \sigma_{i_r})]D(\sigma_{i_1}, \cdots, \sigma_{i_r}, f_{j_1}(\sigma_{i_1}, \cdots, \sigma_{i_{p_1}}), \cdots, f_{j_s}(\sigma_{i_1}, \cdots, \sigma_{i_{ps}}), y_j; \langle\sigma_{i_1}, \cdots, \sigma_{i_r}, \cdots, \sigma_{i_t}\rangle)$.

Since $f_j(\tau_{i_1}, \cdots, \tau_{i_r})=F=f_j(\sigma_{i_1}, \cdots, \sigma_{i_r})$, $\tau_{i_1}=\sigma_{i_1}, \cdots, \tau_{i_r}=\sigma_{i_r}$.

Therefore, $B(F)=D(\tau_{i_1}, \cdots, \tau_{i_r}, f_{j_1}(\tau_{i_1}, \cdots, \tau_{i_{p_1}}), \cdots, f_{j_s}(\tau_{i_1}, \cdots, \tau_{i_{ps}}), f_j(\tau_{i_1}, \cdots, \tau_{i_r}); \langle\tau_{i_1}, \cdots, \tau_{i_r}, \tau_{i_{r+1}}, \cdots, \tau_{i_t}\rangle)$ and $C(F)=D(\tau_{i_1}, \cdots, \tau_{i_r}, f_{j_1}(\tau_{i_1}, \cdots, \tau_{i_{p_1}}), \cdots, f_{j_s}(\tau_{i_1}, \cdots, \tau_{i_{p_s}}), f_j(\tau_{i_1}, \cdots, \tau_{i_r}); \langle\tau_{i_1}, \cdots, \tau_{i_r}, \sigma_{i_{r+1}}, \cdots, \sigma_{i_r}\rangle)$, since $p_1\leqq\cdots\leqq p_s\leqq r$.

So, $B(F)$ and $C(F)$ coincide with each other except differences of terms in the guide-indeces (τ)'s and $[F]$'s at the quantifiers in those A-formulas.

Thus, $cl(B(F))=cl(C(F))$.

LEMMA 4.

(1) *If* $B = \boldsymbol{S}y_i[F]C(y_i, \boldsymbol{c}y_j[G]D(y_i, y_j))$ *is an A-formula, then* $deg(F) < deg(G)$.

(2) *If* $B' = \#x_i(\tau)H(x_i, \boldsymbol{c}y_j[G]K(x_i, y_j))$ *is an A-formula, then* $deg(\tau) < deg(G)$.

Proof. Let $\tilde{A}\langle\tau_1, \cdots, \tau_n\rangle$ be an A-formula from which B is obtained.

Then, $\boldsymbol{S}y_i[F]$ appears in $\tilde{A}\langle\tau_1, \cdots, \tau_n\rangle$, and $\boldsymbol{c}y_j[G]$ also appears in the scope of $\boldsymbol{S}y_i[F]$. So $i<j$. $\cdots\cdots$(1)

Let $\mathcal{Q}_{i_1}x_{i_1}(\tau_{i_1}), \cdots, \mathcal{Q}_{i_r}x_{i_r}(\tau_{i_r})$ be all the quantifiers of the form $\mathcal{Q}x(\tau)$ in $\tilde{A}\langle\tau_1, \cdots, \tau_n\rangle$ whose scopes contain $\boldsymbol{S}y_i[F]$.

Then $F = f_i(\tau_{i_1}, \cdots, \tau_{i_r})$. And then, all the quantifiers of the form $\mathcal{Q}x(\tau)$ whose scopes contain $\boldsymbol{c}y_j[G]$, are $\mathcal{Q}_{i_1}x_{i_1}(\tau_{i_1}), \cdots, \mathcal{Q}_{i_r}x_{i_r}(\tau_{i_r}), \cdots, \mathcal{Q}_{i_s}x_{i_s}(\tau_{i_s})$ $(r \leqq s)$. Thus $G = f_j(\tau_{i_1}, \cdots, \tau_{i_r}, \cdots, \tau_{i_s})$.

So, $lg(F) \leqq lg(G)$. $\cdots\cdots$(2)

Therefore, by (1) and (2), $deg(F) < deg(G)$. / /

Let D be a sub-A-formula, let τ be a term in $\mathrm{H}(\tilde{A}\langle X_1, \cdots, X_n\rangle)$ and let z denotes either y_j or x_i. Then, in the same way as for LK formulas, we define $D\begin{pmatrix} z \\ \tau \end{pmatrix}$ as a scheme which is obtained from D, by substituting each occurrence of z in D to τ. Then we have the following lemma.

LEMMA 5. *If* $B(z)$ *is a sub A-formula with full indications of free occurrences of* z, *and* $\tau \in \mathrm{H}(\tilde{A}\langle X_1, \cdots, X_n\rangle)$, *then* $cl(B(\tau)) = (cl(B(z)))\begin{pmatrix} z \\ \tau \end{pmatrix}$.

Proof. We can easily verify this lemma by the induction on the number of logical symbols in $B(z)$. / /

Proof of Proposition 1.

First, we notice that the condition (#) is equivalent to the following (#′).

(#′) If $\mathcal{Q}y[F]$ appears in $\Gamma \to \Delta$, then every F in $\Gamma \to \Delta$ appears only in the guide-indeces in $\Gamma \to \Delta$.

(0) In the cases except four cases when US-operation removes some quantifiers in $\Gamma \to \Delta$, the treatments are as follows:

Suppose that $\Gamma \to \Delta$ has the condition (#′). An appearance of $\mathcal{Q}y[F]$ in $US(\Gamma \to \Delta)$ implies that of $\mathcal{Q}y[F]$ in $\Gamma \to \Delta$, since in those cases, "all the quantifiers with guide-indeces in $\Gamma \to \Delta$ coincides those of $US(\Gamma \to \Delta)$." $\cdots\cdots$(%)

Then, by the assumption, F occurs only in some guide-indeces in $\Gamma \to \Delta$.

Therefore, by the above (%), F also occurs only in some guide-indeces in $US(\Gamma \to \Delta)$.

(1) In the case corresponding to (2.1.1.*l.*) in the definition US-operation,

$$(*) \quad US(\Gamma \to \Delta) = \Gamma_1, \quad B(\tau), \Gamma_2 \to \Delta$$
$$\text{and} \quad \Gamma \to \Delta = \Gamma_1, \forall x(\tau)B(x), \Gamma_2 \to \Delta, \quad \text{where} \quad deg(\tau) < deg(\Gamma \to \Delta).$$

Suppose that $\Gamma \to \Delta$ satisfies the condition (#′) but $US(\Gamma \to \Delta)$ does not. Then there is such a term $G = f_d(\tau_1, \cdots, \tau_s)$ that $US(\Gamma \to \Delta)$ contains some $\mathscr{Q}y[G]$ and one of G's appears at the outside of every guide-index in $US(\Gamma \to \Delta)$. ·····①

So, $\Gamma \to \Delta$ also contains $\mathscr{Q}y[G]$, because of the form of the inference scheme (*). Therefore, by the assumption, every G does not appear at the outside of any guide-indeces in $\Gamma \to \Delta$. ·····②

Thus, by ① and ②, G at the outside of every guide-index in $US(\Gamma \to \Delta)$, must appear in $B(\tau)$.

Let $\underline{G}$ be one of such G's in $B(\tau)$. Then the following two cases arises:

(i) Case when the indicated $\underline{G}$ in $B(\tau)$ is separated from the indicated τ's.

(ii) the other case.

In Case (i), $\forall x(\tau)B(x)$ in $\Gamma \to \Delta$ contains previously G at the outside of every guide-index in $\Gamma \to \Delta$. This is a contradiction.

In Case 2, the following three cases arise:

(ii-1) $\underline{G}$(in $B(\tau)$) is a proper part of an indicated τ.

(ii-2) $\underline{G}$(in $B(\tau)$) coincides to an indicated τ.

(ii-3) $\underline{G}$(in $B(\tau)$) contains some indicated τ's in a proper part of G.

In Cases (ii-1) and (ii-2), $deg(G) \leqq deg(\tau) < deg(\Gamma \to \Delta) \leqq deg(G)$.

This is a contradiction.

In Case (ii-3), since $\mathscr{Q}y[G]$ in $US(\Gamma \to \Delta)$ appears in an A-formula in $\Gamma \to \Delta$, and G is of the form $f_d(\tau_1, \cdots, \tau_s)$ where f_d is a Skolem function, a contradiction is leaded as follows:

Since G contains some of the indicated τ's, $G = f_d(\tau_1'(\tau), \cdots, \tau_s'(\tau))$ where at least one $\tau_j'(\tau)$ properly contains τ.

Thus $B(\tau) = B'(f_d(\tau_1'(\tau), \cdots, \tau_s'(\tau)), \tau)$. So an A-formula $\forall x(\tau)B(x)$ is of the form $\forall x(\tau)B'(f_d(\tau_1'(x), \cdots, \tau_s'(x))), x)$. This contradicts to Lemma 1.

(2) In the case corresponding to Case (2.1.1.r), the treatment is dual to the case (1).

(3) In the case corresponding to Case (2.1.2.l),

$$(*) \qquad \begin{array}{ll} US(\Gamma \to \Delta) = \Gamma_0, & B_1(F), \Gamma_1, \quad B_2(F), \cdots, \quad B_k(F), \Gamma_k \to \Delta \\ \text{and} \quad \Gamma \to \Delta = \Gamma_0, & \exists y[F]B_1(y), \Gamma_1, \exists y[F]B_2(y), \cdots, \exists y[F]B_k(y), \Gamma_k \to \Delta \end{array}$$

where there are no formulas of the form $\exists y[F]C(y)$ in $US(\Gamma \to \Delta)$ and $deg(F) = deg(\Gamma \to \Delta) \leqq deg(T)$ for every $\mathscr{R}y'[T]$ in $\Gamma \to \Delta$.

Suppose that $\Gamma \to \Delta$ has the condition (#′), but $US(\Gamma \to \Delta)$ does not. Then this leads to a contradiction as follows:

By the assumption, there is a term $G = f_d(\tau_1, \cdots, \tau_s)$ such that $US(\Gamma \to \Delta)$ contains $\mathscr{Q}y'[G]$ and one of G's appears at the outside of every guide-index in $US(\Gamma \to \Delta)$. By an argument similar to that in the case (1), we can see that the above G's appears in $B_{i_0}(F)$ for some i_0. Let $\underline{G}$ be one of such G's in $B_{i_0}(F)$. Then two cases arise:

(i) The case when the indicated $\underline{G}$ in $B_{i_0}(F)$ is separated from the indicated F's.

(ii) The other case.

In Cace (i), by an argument similar to the case (1), we can lead that $\mathscr{Q}y'[G]$ is contained in $\Gamma \to \Delta$ and $\exists y[F]B_{i_0}(y)$ in $\Gamma \to \Delta$ contains $\underline{G}$ at the outside of every guide-index in $\Gamma \to \Delta$. This is a contradiction.

In Case (ii), three cases arise.

(ii-1) $\underline{G}(\text{in} B_{i_0}(F))$ is a proper part of one of the indicated F's.

(ii-2) $\underline{G}(\text{in } B_{i_0}(F))$ coincides with one of the indicated F's.

(ii-3) $\underline{G}(\text{in } B_{i_0}(F))$ contains some of the indicated F's.

In Case (ii-1), $deg(G) < deg(F) \leqq deg(\Gamma \to \Delta) \leqq deg(G)$. This is a contradiction. Case (ii-3), a contradiction is leaded by the same treatment as in Case (1).

In Case (ii-2), $G = F$. On the other hand, $\mathcal{Q}y'[G]$ is in $US(\Gamma \to \Delta)$.

(ii-2-1) When such a $\mathcal{Q}y'[G](=\mathcal{Q}y'[F])$ is in $B_i(F)$ for some i, $\exists y[F]B_i(y)$ is of the form $\exists y[F]B'(y, \mathcal{Q}y'[F]C(y, y'))$.

By Lemma 4, $deg(F) < deg(F)$. This is a contradiction.

(ii-2-2) When such a $\mathcal{Q}y'[G](=\mathcal{Q}y'[F])$ is in some B of Γ_j, Δ, the treatment is as follows: Since B is not primitive and the outermost logical symbol of B is neither $\vee$, $\wedge$, $\supset$ nor $\neg$, B has a quantifier at front. Three cases arise.

(α) $B = \mathcal{Q}y'[F]D(y')$.

(β) $B = \mathcal{Q}'y''[F'']H(y'', \mathcal{Q}y'[F]K(y'', y'))$.

(γ) $B = \mathcal{Q}''x(\tau)\mathrm{E}(x, \mathcal{Q}y'[F]\mathrm{L}(x, y'))$.

In Case (α), by Lemma 2, $\mathcal{Q} = \exists$ and $y' = y$. So, $\exists y[F]D(y)$ remains in Γ_j, Δ.

This cannot occur in the case (3), corresponding to (2.1.2.*l*).

In Case (β), by Lemma 4, $deg(F'') < deg(F) = deg(\Gamma \to \Delta) \leqq deg(F'')$.

This is a contradiction.

In Case (γ), by Lemma 4, $deg(\tau) < deg(F) = deg(\Gamma \to \Delta)$. This cannot occur in the case (3).

(4) In the case corresponding to the case (2.1.2.*r*), the treatment is dual to the case (2.1.2.*l*).

Proof of Proposition 2.

We can easily verify that the proposition holds for each type of applications of *US*-operation in Case 0 and Case 1. For each type of applications of *US*-operation in Case 2, $sb(US(\Gamma \to \Delta))$ coincides with $sb(\Gamma \to \Delta)$. This, leads to an LK deduction of a dummy step. Then, in this case, the proposition also holds.

Proof of Proposition 3.

In order to prove the Proposition, it suffices to see that

$$\frac{cl(US(\Gamma \to \Delta))}{cl(\Gamma \to \Delta)}$$

induces an LK* deduction, for each type of applications of *US*-operation. Essential types are in only the cases when *US*-operation decomposes *A*-formulas with quantifiers at prenex.

We treat only Case 2.1.1.*l*. and Case 2.1.2.*r*., since Case 2.1.1.*r*. and Case 2.1.2.*l*. are dual to the above cases.

"Case 2.1.1.*l*.": In this case, $\boldsymbol{\Gamma}\rightarrow\boldsymbol{\Delta}=\boldsymbol{\Gamma}_1, \forall x(\tau)B(x), \boldsymbol{\Gamma}_2\rightarrow\boldsymbol{\Delta}$. Therefore,

$$\frac{cl(US(\boldsymbol{\Gamma}\rightarrow\boldsymbol{\Delta}))}{cl(\boldsymbol{\Gamma}\rightarrow\boldsymbol{\Delta})}=\frac{cl(\boldsymbol{\Gamma}_1),\ cl(B(\tau)),\ cl(\boldsymbol{\Gamma}_2)\rightarrow cl(\boldsymbol{\Delta})}{cl(\boldsymbol{\Gamma}_1),\ \forall x cl(B(x)),\ cl(\boldsymbol{\Gamma}_2)\rightarrow cl(\boldsymbol{\Delta})}$$

induces an LK* deduction, since $cl(B(\tau))=(cl(B(x)))\begin{pmatrix} x \\ \tau \end{pmatrix}$ by Lemma 5 and $cl(B(\tau))=B'(\tau)$ cannot contain any term $f_d(\sigma_1, \cdots, \sigma_p)$ which contains some of the indicated τ.

[Assume that $B'(\tau)=B''(f_d(\sigma_1'(\tau), \cdots, \sigma_p'(\tau)), \tau)$ where one of $\sigma_i'(\tau)$'s contains τ properly. Since $B(\tau)$ is reconstructed by adding certain indeces to the quantifiers of $B'(\tau)$, $B(\tau)$ is of the form $B'''(f_d(\sigma_1'(\tau), \cdots, \sigma_p'(\tau)), \tau)$. Then an A-formula $\forall x(\tau)B(x)=\forall x(\tau)B'''(f_d(\sigma_1'(x), \cdots, \sigma_p'(x)), x)$ has a bound variable x at the inner part of a Skolem function f_d in an A-formula. This contradicts to Lemma 1.]

"Case 2.1.2.*r*.": In this case, we verify that

$$\frac{cl(US(\boldsymbol{\Gamma}\rightarrow\boldsymbol{\Delta}))}{cl(\boldsymbol{\Gamma}\rightarrow\boldsymbol{\Delta})}$$

induces an LK* deduction explained in Fig. 2, by inserting some inferences.

By the assumption of the proposition, the lower sequent $\boldsymbol{\Gamma}\rightarrow\boldsymbol{\Delta}=\boldsymbol{\Gamma}\rightarrow\boldsymbol{\Delta}_0, \forall y[F]B_1(y), \boldsymbol{\Delta}_1, \forall y[F]B_2(y), \cdots, \forall y[F]B_k(y), \boldsymbol{\Delta}_k$ satisfies (#).

Then the sequent $cl(\boldsymbol{\Gamma}\rightarrow\boldsymbol{\Delta})=$

$$cl(\boldsymbol{\Gamma}) \longrightarrow cl(\boldsymbol{\Delta}_0),\ \forall y\ cl(B_1(y)),\ cl(\boldsymbol{\Delta}_1),\ \forall y\ cl(B_2(y)),\ \cdots,\ \forall y\ cl(B_k(y)),\ cl(\boldsymbol{\Delta}_k)$$

has no terms of the form F. So, the position (@) in Fig. 2 is a correct $\forall$*-right inferences of LK*. Because, the condition (i) is satisfied by the above fact, the condition (ii) is satisfied by an argument similar to Case 2.1.1.*l*. and $cl(B_1(F))=(cl(B_1(y)))\begin{pmatrix} y \\ F \end{pmatrix}$, by Lemma 5. Moreover, by Lemma 3, the position (%) can be taken for $k-1$ contructions. Then the assertion holds in this case.

$$cl(US(\Gamma \rightarrow \Delta)) =$$

$$\frac{cl(\Gamma) \rightarrow cl(\Delta_0), cl(B_1(F)), cl(\Delta_1), \cdots, cl(B_k(F)), cl(\Delta_k)}{cl(\Gamma) \rightarrow cl(\Delta_0), \cdots, cl(\Delta_k), cl(B_1(F)), \cdots, cl(B_k(F))}\ \text{Exchanges}$$

$$\frac{cl(\Gamma) \rightarrow cl(\Delta_0), \cdots, cl(\Delta_k), cl(B_1(F)), \cdots, cl(B_k(F))}{cl(\Gamma) \rightarrow cl(\Delta_0), \cdots, cl(\Delta_k), cl(B_1(F))}\ (\%)\ \text{Contractions}$$

$$\frac{cl(\Gamma) \rightarrow cl(\Delta_0), \cdots, cl(\Delta_k), cl(B_1(F))}{cl(\Gamma) \rightarrow cl(\Delta_0), \cdots, cl(\Delta_k), \forall y\, cl(B_1(y))}\ (@)\ \forall\text{*-right}$$

$$\frac{cl(\Gamma) \rightarrow cl(\Delta_0), \cdots, cl(\Delta_k), \forall y\, cl(B_1(y))}{cl(\Gamma) \rightarrow cl(\Delta_0), \cdots, cl(\Delta_k), \forall y\, cl(B_1(y)), \cdots, \forall y\, cl(B_k(y))}\ \text{Weakenings}$$

$$\frac{cl(\Gamma) \rightarrow cl(\Delta_0), \cdots, cl(\Delta_k), \forall y\, cl(B_1(y)), \cdots, \forall y\, cl(B_k(y))}{cl(\Gamma) \rightarrow cl(\Delta_0), cl(\forall y[F]B_1(y)), cl(\Delta_1), \cdots, cl(\forall y[F]B_k(y)), cl(\Delta_k)}\ \text{Exchanges}$$

$$= cl(\Gamma \rightarrow \Delta).$$

Fig. 2.

Proof of Proposition 4.

It suffices to prove that by the operation α, inference-rules of LK* turn into those of LK. Only the cases corresponding to $\exists$*-left and $\forall$*-right are essential. Since the condition (ii) in $\exists$*-left inference implies that $\alpha(D(f_j(\tau_1, \cdots, \tau_k)) = D'(\alpha_{f_j(\tau_1, \cdots, \tau_k)})$ and $\alpha(\exists y D(y)) = \exists y D'(y)$, and (i) means the condition on the eigenvariable, each inference on $\exists$*-left turns into an inference on $\exists$-left of LK. The treatment on $\forall$*-right inference is the same as that on $\exists$*-left.

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