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On the Coefficients of an Analytic Function Represented by Dirichlet Series

by

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1. Introduction

Consider the Dirichlet series

$$f(s)=\sum_{n=1}^{\infty} a_n \exp(s\lambda_n) \tag{1.1}$$

where $0\leqq\lambda_1<\lambda_2<\cdots<\lambda_n<\cdots$, $\lambda_n\to\infty$ as $n\to\infty$, $s<\sigma+ it$ (σ, t being real variables), $\{a_n\}_{n=1}^{\infty}$ is a sequence of complex numbers and

$$\limsup_{n\to\infty} (n/\lambda_n)=D<\infty . \tag{1.2}$$

If the series given by (1.1) converges absolutely in the half plane $\mathrm{Re}\, s<\alpha$ $(-\infty<\alpha<\infty)$ then it is known ([4, p. 166]) that the series (1.1) represents an analytic function in $\mathrm{Re}\, s<\alpha$, and since (1.2) is satisfied we have

$$-\alpha=\limsup_{n\to\infty} (\log |a_n|)/\lambda_n .$$

Let D_α denote the class of all functions $f(s)$ of the form (1.1), which are analytic in the half plane $\mathrm{Re}\, s<\alpha$ $(-\infty<\alpha<\infty)$ and satisfy (1.2). For $f\in D_\alpha$, set

$$M(\sigma)\equiv M(\sigma, f)=\max_{-\infty<t<\infty} |f(\sigma+it)| ,$$

$$m(\sigma)\equiv m(\sigma, f)=\max_{n\geqq 1} \{|a_n| \exp(\sigma\lambda_n)\}$$

and

$$N(\sigma)\equiv N(\sigma, f)=\max \{n: m(\sigma)=|a_n| \exp(\sigma\lambda_n)\} .$$

$M(\sigma)$, $m(\sigma)$ and $N(\sigma)$ are called, respectively, the maximum modulus, the maximum term and the rank of the maximum term of $f(s)$ for $\mathrm{Re}\ s=\sigma$. The elements in the range set of $N(\sigma)$ are called the principal indices of $f(s)$. It is known ([2]) that $\log M(\sigma)$ is an increasing convex function of σ for $\sigma<\alpha$.

For a function $f\in D_\alpha$, Krishna Nandan ([3]) has defined the order ρ and lower order λ $(0\leqq\lambda\leqq\rho\leqq\infty)$ of $f(s)$ as

$$\lim_{\sigma\to\alpha} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log\log M(\sigma)}{-\log(1-\exp(\sigma-\alpha))} = \begin{matrix}\rho\\ \lambda\end{matrix}.$$

The above growth parameters do not give any specific information about the growth of $f(s)$ when $\rho=0$. Recently Awasthi and Dixit ([1]) have studied the functions of zero order by comparing the growth of log log $M(\sigma)$ with that of log log $(1-\exp(\sigma-\alpha))^{-1}$. In the present paper an attempt has been made to study the growth of such functions in a more general way. Our results generalise the results of ([1]).

Let Λ be the class of all functions β satisfying the following conditions (H, i) and (H, ii):

(H, i) $\beta(x)$ is defined on $[a, \infty)$ and is positive, strictly increasing, differentiable and tends to ∞ as $x\to\infty$.

(H, ii) $$\lim_{x\to\infty} \frac{\beta(cx)}{\beta(x)}=1 \quad \text{for all } c, \quad 0<c<\infty.$$

For a function $f\in D_\alpha$, set

$$\begin{matrix}\rho(\beta, f)\\ \lambda(\beta, f)\end{matrix} = \lim_{\sigma\to\alpha} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\beta(\log M(\sigma))}{\beta(-\log(1-\exp(\sigma-\alpha)))}$$

where $\beta\in\Lambda$. Then $\rho(\beta, f)$ and $\lambda(\beta, f)$ are called, respectively, *β-order* and *lower β-order* of $f(s)$. Taking, in particular; $\beta(x)=\log x$, $\rho(\beta, f)$ and $\lambda(\beta, f)$ reduce to the growth parameters intorduced by Awasthi and Dixit ([1]). To avoid some trivial cases we shall assume throughout that $M(\sigma)\to\infty$ as $\sigma\to\alpha$. A function $f(s)\in D_\alpha$ is said to be of *β-regular growth* if $\lambda(\beta, f)=\rho(\beta, f)$ and $f(s)$ is said to be of *β-irregular growth* if $\rho(\beta, f)>\lambda(\beta, f)$.

In Section 2 coefficient characterization of $\rho(\beta, f)$ has been obtained. Coefficient characterization of $\lambda(\beta, f)$ has been found in Section 3. A decomposition theorem for a function of *β-irregular growth* has been obtained in Section 4. Finally, in the same section, it has been illustrated that our growth parameters are more general than those of Awasthi and Dixit.

Note. Throughout the rest of the paper we shall assume that $\beta(x)$ has been extended over $(-\infty, a)$ by the definition $\beta(x)=\beta(a)$ for $x\in(-\infty, a)$.

2. Coefficient characterization of $\rho(\beta, f)$

LEMMA 1. *Let $f(s)\in D_\alpha$ with β-order $\rho(\beta, f)$ $(0\leqq\rho(\beta, f)\leqq\infty)$. Then*

$$\rho(\beta, f)\geqq\limsup_{n\to\infty} \frac{\beta(\log|a_n|+\alpha\lambda_n)}{\beta(\log\lambda_n)}. \tag{2.1}$$

Proof. Assume that $\rho(\beta, f)<\infty$, since there is nothing to prove if $\rho(\beta, f)=\infty$. Then, given $\varepsilon>0$, there exists $\sigma_0=\sigma_0(\varepsilon)$ such that for $\sigma>\sigma_0$ we have,

$$\log M(\sigma) < \beta^{-1}(\bar{\rho}\beta(-\log(1-\exp(\sigma-\alpha))))$$

where $\bar{\rho}=\rho(\beta,f)+\varepsilon$. Using Cauchy's inequality, the above inequality gives

$$\begin{aligned}\beta^{-1}(\bar{\rho}\beta(-\log(1-\exp(\sigma-\alpha)))) &> \log M(\sigma)\\ &\geqq \log|a_n|+\sigma\lambda_n\\ &= \log|a_n|+\alpha\lambda_n+(\sigma-\alpha)\lambda_n\end{aligned}$$

for $\sigma>\sigma_0$ and all n. Taking, in particular, $\sigma=\alpha+\log(1-(1/\lambda_n))$ the above inequality gives

$$\beta^{-1}(\bar{\rho}\beta(\log\lambda_n))-\lambda_n\log(1-(1/\lambda_n))\geqq \log|a_n|+\alpha\lambda_n$$

Since $\beta\in\Lambda$, the above inequality easily gives that

$$\limsup_{n\to\infty}\frac{\beta(\log|a_n|+\alpha\lambda_n)}{\beta(\log\lambda_n)}\leqq\bar{\rho}=\rho(\beta,f)+\varepsilon.$$

Now, since $\varepsilon>0$ is arbitrary, (2.1) follows from the above inequality. This proves the lemma.

LEMMA 2. *Let $f(s)\in D_\alpha$ with β-order $\rho(\beta,f)$. Set $F(x,c)=\alpha^{-1}(c\beta(\log x))$, $1\leqq c<\infty$. Assume that*

(i) *for every c, $1\leqq c<\infty$, there exists a constant $x_0(c)$ such that for $x>x_0(c)$ we have*

$$\frac{dF(x,c)}{dx}\leqq\frac{F(x,c)}{x}.$$

and

(ii) *$F(x,c)/x\to0$ as $x\to\infty$ for every c, $1\leqq c<\infty$.*
hold. Then

$$\rho(\beta,f)\leqq\max(1,\theta)$$

where

$$\theta=\limsup_{n\to\infty}\frac{\beta(\log|a_n|+a\lambda_n)}{\beta(\log\lambda_n)}.$$

Note (i) Condition (i) of the Lemma ensures that $F(x,c)/x$ is ultimately a nonincreasing function of x for every c, $1\leqq c<\infty$.

(ii) Henceforth, a function $\beta\in\Lambda$ satisfying the conditions (i) and (ii) of the Lemma will be called an admissible function.

(iii) In particular $\log_p x$, $p\geqq1$, is an admissible function. Here $\log_1 x=\log x$ and $\log_p(x)=\log(\log_{p-1}x)$ for $p\geqq2$.

Proof. First, let $1\leqq\theta<\infty$. Then, given $\varepsilon>0$ there exists $n_0=n_0(\varepsilon)$ such that for $n>n_0$ we have

$$\log|a_n|+\alpha\lambda_n\leqq F(\lambda_n,\bar{\theta}),\ \bar{\theta}=\theta+\varepsilon. \tag{2.2}$$

Now, since (1.2) holds, we have $n < D'\lambda_n$ for $n > n' = n'(D')$, where $D' > D$.

Let λ_{N_0} be a fixed λ_n greater than λ_{n_0} and $x_0(\bar{\theta})$, where $x_0(\bar{\theta})$ is as defined in condition (i) of the lemma. Then, using (2.2), we have

$$(2.3)\qquad M(\sigma) \leqq \sum_{n=1}^{\infty} |a_n| \exp(\sigma\lambda_n)$$

$$\leqq P(N_0) + \sum_{n=N_0+1}^{\infty} \exp(F(\lambda_n, \bar{\theta}) + (\sigma-\alpha)\lambda_n)$$

where $P(N_0)$, the sum of first N_0 terms, is bounded.

For each σ define a natural number $n(\sigma)$ as

$$\lambda_{n(\sigma)} \leqq \frac{2}{(\alpha-\sigma)} F\left(\frac{2}{(\alpha-\sigma)}, 2\bar{\theta}\right) < \lambda_{n(\sigma)+1}\,.$$

Then (2.3) gives that

$$(2.4)\qquad M(\sigma) \leqq P(N_0) + \left(\sum_{n=N_0+1}^{n(\sigma)} + \sum_{n=n(\sigma)+1}^{\infty}\right)(\exp(F(\lambda_n, \bar{\theta}) + (\sigma-\alpha)\lambda_n))$$

$$\leqq P(N_0) + n(\sigma) \exp F(\lambda_{n(\sigma)}, \bar{\theta})$$

$$+ \sum_{n=n(\sigma)+1}^{\infty} \exp(F(\lambda_n, \bar{\theta}) + (\sigma-\alpha)\lambda_n)\,.$$

Now, for σ sufficiently near to α and for all $n > n(\sigma)$, we have

$$\frac{F(\lambda_n, \bar{\theta})}{\lambda_n} \leqq \frac{F(\lambda_{n(\sigma)+1}, \bar{\theta})}{\lambda_{n(\sigma)+1}} \leqq \frac{F\left(\frac{2}{\alpha-\sigma}F\left(\frac{2}{\alpha-\sigma}, 2\bar{\theta}\right), \bar{\theta}\right)}{\frac{2}{\alpha-\sigma}F\left(\frac{2}{\alpha-\sigma}, 2\bar{\theta}\right)}$$

$$\leqq \frac{(\alpha-\sigma)}{2}$$

since $\beta \in \Lambda$. Thus, for σ sufficiently near to α, we have

$$\sum_{n=n(\sigma)+1}^{\infty} \exp(F(\lambda_n, \bar{\theta}) + (\sigma-\alpha)\lambda_n) \leqq \sum_{n=n(\sigma)+1}^{\infty} \exp\left(\frac{(\sigma-\alpha)}{2}\lambda_n\right)$$

$$\leqq \sum_{n(\sigma)+1}^{\infty} \exp\left(\frac{(\sigma-\alpha)n}{2D'}\right) \leqq \sum_{n=0}^{\infty} \exp\left(\frac{(\sigma-\alpha)n}{2D'}\right)$$

$$= 1/(1-\exp((\sigma-\alpha)/2D'))$$

$$\sim \frac{2D'}{(\alpha-\sigma)}\,.$$

Now, using (2.4) and the definition of $n(\sigma)$, we have

$$M(\sigma) \leqq P(N_0) + n(\sigma) \exp (F(\lambda_{n(\sigma)}, \bar{\theta}))(4D'/(\alpha-\sigma))$$

or

$$\begin{aligned}\log M(\sigma) &\leqq \log n(\sigma) + F(\lambda_{n(\sigma)}, \bar{\theta}) + \log (2/(\alpha-\sigma)) + O(1) \\ &\leqq \log (2/(\alpha-\sigma)) + \log F(2/(\alpha-\sigma), 2\bar{\theta}) + \log (2/(\alpha-\sigma)) \\ &\quad + F\left(\frac{2}{(\alpha-\sigma)} F\left(\frac{2}{(\alpha-\sigma)}, 2\bar{\theta}\right), \bar{\theta}\right) + O(1) . \\ &\leqq 4 \log (2/(\alpha-\sigma)) + F((2/(\alpha-\sigma))^2, \bar{\theta}) \\ &\leqq 5F((2/(\alpha-\sigma))^2, \bar{\theta})\end{aligned}$$

for all σ sufficiently near to α. The above inequality easily gives that

$$\rho(\beta, f) \leqq \bar{\theta} = \theta + \varepsilon .$$

Since $\varepsilon > 0$ is arbitrary, we have

$$\rho(\beta, f) \leqq \theta .$$

Next, let $\theta < 1$. Then the above analysis, with $\bar{\theta} = 1$, gives that the *β-order* $\rho(\beta, f)$ of $f(s)$ is atmost one. The lemma is thus proved.

Lemma 1 and Lemma 2 lead to

THEOREM 1. *Let $f(s) \in D_\alpha$ with β-order $\rho(\beta, f)$. Assume that $\rho(\beta, f) \geqq 1$ and that β is an admissible function. Then*

$$\rho(\beta, f) = \max (1, \theta)$$

where θ is as defined in Lemma 2.

Remark. With $\beta(x) = \log x$, the above theorem generalises a result in [1].

3. Coefficient characterization of $\lambda(\beta, f)$

We need the following lemmas. Lemmas 3 and 4 are due to Krishna Nandan ([3]).

LEMMA 3. *Let $f(s) \in D_\alpha$. Then*

$$\log m(\sigma) = \log m(\sigma_0) + \int_{\sigma_0}^{\sigma} \lambda_{N(x)} dx, \quad -\infty < \sigma_0 < \sigma < \alpha .$$

LEMMA 4. *If*

$$f(s) = \sum_{n=1}^{\infty} a_n \exp (s\lambda_n)$$

belongs to D_α and satisfies

(3.1)
$$\liminf_{n\to\infty} (\lambda_n-\lambda_{n-1})=\delta>0$$

then for every $\delta'<\delta$ and for all σ sufficiently near to α, we have

$$M(\sigma)\leqq m(\sigma)\left[1+\frac{1+\delta'}{\delta'}N\left(\sigma+\frac{1-\exp(\sigma-\alpha)}{N(\sigma)}\right)\right](1-\exp(\sigma-\alpha))^{-1}$$

Note. In Lemmas 3 and 4 we assume that $m(\sigma)$ and $N(\sigma)$ are unbounded functions of σ.

LEMMA 5. *Let $f(s)\in D_\alpha$ with β-order $\rho(\beta,f)$ and lower β-order $\lambda(\beta,f)$. Assume that β is an admissible function and that* (3.1) *holds. Set*

$$\left.\begin{matrix}\varphi_1\\ \varphi_2\end{matrix}\right\}=\lim_{\sigma\to\alpha}\begin{matrix}\sup\\ \inf\end{matrix}\frac{\beta(\log m(\sigma))}{\beta(-\log(1-\exp(\sigma-\alpha)))}.$$

Then, if $\rho(\beta,f)>1$, we have $\rho(\beta,f)=\varphi_1$. Further if $\lambda(\beta,f)>1$, then $\lambda(\beta,f)=\varphi_2$ holds.

Proof. Let $D'>D$ be a fixed number. Since (1.2) is satisfied, we have, by Lemma 3, for all σ sufficiently close to α.

(3.2)
$$\begin{aligned}N(\sigma)(\alpha-\sigma)&<D'\lambda_{N(\sigma)}(\alpha-\sigma)\\ &\leqq 2D'\int_\sigma^{\sigma+1/2(\alpha-\sigma)}\lambda_{N(x)}dx\\ &\leqq 2D'\log m\left(\sigma+\frac{1}{2}(\alpha-\sigma)\right).\end{aligned}$$

Using (3.2), for σ sufficiently close to α, we have

$$\begin{aligned}N\left(\sigma+\frac{1}{2}(1-\exp(\sigma-\alpha))\right)&\leqq 2D'\frac{\log m\left(\sigma+\frac{1}{2}(\alpha-\sigma)+\frac{1}{4}(1-\exp(\sigma-\alpha))\right)}{(\alpha-\sigma)-\frac{1}{2}(1-\exp(\sigma-\alpha))}\\ &\leqq 4D'\frac{\log m\left(\sigma+\frac{3}{4}(\alpha-\sigma)\right)}{(\alpha-\sigma)}.\end{aligned}$$

Now, using Lemma 4 and the above inequality, we have

(3.3)
$$\begin{aligned}\log M(\sigma)&\leqq\log m(\sigma)+\log\log m\left(\sigma+\frac{3}{4}(\alpha-\sigma)\right)-\log(\alpha-\sigma)\\ &\quad+\log(1-\exp(\sigma-\alpha))^{-1}+O(1)\\ &\leqq 2\log m\left(\sigma+\frac{3}{4}(\alpha-\sigma)\right)\\ &\quad+2\log(1-\exp(\sigma-\alpha))^{-1}+O(1).\end{aligned}$$

From (3.3), it easily follows that if $\varphi_1<1$ then $\rho(\beta,f)\leqq 1$ and if $\varphi_1\geqq 1$ then $\varphi_1\geqq\rho(\beta,f)$. We also have $\varphi_1\leqq\rho(\beta,f)$ in view of the inequality $m(\sigma)\leqq M(\sigma)$. Thus $\rho(\beta,f)=\varphi_1$ if $\rho(\beta,f)>1$.

The remaining part of the lemma also follows similarly by using (3.3). The lemma is thus proved.

Remark. Lemma 5 generalises a result in [1].

THEOREM 2. *Let $f(s)\in D_\alpha$ with lower β-order $\lambda(\beta,f)$. Then*

$$\lambda(\beta,\, f)\geqq\liminf_{k\to\infty}\frac{\beta(\log|a_{n_k}|+\alpha\lambda_{n_k})}{\beta(\log\lambda_{n_k+1})} \tag{3.4}$$

for any increasing sequence $\{n_k\}$ of positive integers.

Proof. Let the limit inferior on the right hand side of (3.4) be denoted by S. Clearly $0\leqq S\leqq\infty$. First, let $0<S<\infty$. Then, given $\varepsilon>0$, $S>\varepsilon$, there exists $k_0=k_0(\varepsilon)$ such that for all $k>k_0$ we have

$$\log|a_{n_k}|>F(\lambda_{n_k+1},\,\bar{S})-\alpha\lambda_{n_k}$$

where $\bar{S}=S-\varepsilon$. Choose $\sigma_k=\alpha-(1/\lambda_{n_k})$. Then, for $\sigma_k\leqq\sigma\leqq\sigma_{k+1}$, using Cauchy's inequality and the above inequality, we have

$$\begin{aligned}\log M(\sigma)&\geqq\log|a_{n_k}|+\sigma\lambda_{n_k}\geqq\log|a_{n_k}|+\sigma_k\lambda_{n_k}\\&\geqq F(\lambda_{n_k+1},\,\bar{S})+(\sigma_k-\alpha)\lambda_{n_k}\\&=F((1/(\alpha-\sigma_{k+1})),\,\bar{S})-1\\&\geqq F((1/(\alpha-\sigma)),\,\bar{S})-1\end{aligned}$$

Since $\beta\in\Lambda$, the above inequality easily gives

$$\lambda(\beta,f)\geqq\bar{S}=S-\varepsilon$$

As ε is arbitrary this in turn implies that

$$\lambda(\beta,f)\geqq S \tag{3.5}$$

Obviously, (3.5) holds for $S=0$. For $S=\infty$, the above arguments give that $\lambda(\beta,f)=\infty$. This proves the theorem.

COROLLARY Let $f(s)\in D_\alpha$ *with β-order $\rho(\beta,f)$ and lower β-order $\lambda(\beta,f)$. Assume that β is an admissible function. Further, let* (i) $\beta(\log\lambda_n)\sim\beta(\log\lambda_{n+1})$ *as $n\to\infty$ and* (ii) $\lim_{n\to\infty}\dfrac{\beta(\log|a_n|+\alpha\lambda_n)}{\beta(\log\lambda_n)}=S_0$ *exist with $1<S_0<\infty$. Then $f(s)$ is of β-regular growth with $\rho(\beta,f)=\lambda(\beta,f)=S_0$.*

The corollary is immediate in view of Theorem 1.

Remark. Taking $\beta(x)=\log x$, Theorem 2 gives a result of Awasthi and Dixit [1].

THEOREM 3. *Let $f(s)\in D_\alpha$ with lower β-order $\lambda(\beta,f)$. Let $\lambda(\beta,f)>1$. Assume that β is an admissible function and that*

$$x^2\frac{dF(x,c)}{dx}\to\infty \quad as \quad x\to\infty \tag{3.6}$$

for all c, $1<c<\infty$. Further, let (3.1) *hold and let*

$$\psi(n)=(\log|a_n/a_{n+1}|)/(\lambda_{n+1}-\lambda_n) \tag{3.7}$$

be a nondecreasing function of n for $n>n_0$. Then

$$\lambda(\beta,f)\leqq\liminf_{n\to\infty}\frac{\beta(\log|a_n|+\alpha\lambda_n)}{\beta(\log\lambda_n)}.$$

Note. In particular, $\log_p x$, $p\geqq 1$, is an admissible function which satisfies (3.6).

Proof. As $\psi(n)$ forms a nondecreasing function for $n>n_0$, it follows that $\psi(n)>\psi(n-1)$ for infinitely many values of n, since otherwise $\rho(\beta,f)\leqq 1$. Clearly $\psi(n)\to\alpha$ as $n\to\infty$. When $\psi(n)>\psi(n-1)$, the term $a_n\exp(s\lambda_n)$ becomes the maximum term and we have

$$m(\sigma)=|a_n|\exp(\sigma\lambda_n) \qquad \text{for} \quad \psi(n-1)\leqq\sigma<\psi(n).$$

Now, since (3.1) is satisfied, we have by Lemma 5,

$$\lambda(\beta,f)=\liminf_{\sigma\to\alpha}\frac{\beta(\log m(\sigma))}{\beta(-\log(1-\exp(\sigma-\alpha)))}.$$

Suppose first that $1<\lambda(\beta,f)<\infty$. Then, given $\varepsilon>0$ there exists $\sigma_1=\sigma_1(\varepsilon)$ such that for $\sigma>\sigma_1$ we have

$$\log m(\sigma)\geqq F((1-\exp(\sigma-\alpha))^{-1},\bar{\lambda})$$

where $\bar{\lambda}=\lambda(\beta,f)-\varepsilon$. Let $a_{n_1}\exp(s\lambda_{n_1})$ and $a_{n_2}\exp(s\lambda_{n_2})$ $(n_1>n_0,\ \psi(n_1-1)>\sigma_1)$ be two consecutive maximum terms so that $n_1\leqq n_2-1$. Then

$$\log|a_{n_2}|+\lambda_{n_2}\sigma\geqq F((1-\exp(\sigma-\alpha))^{-1},\bar{\lambda})$$

for all σ satisfying $\psi(n_2-1)\leqq\sigma<\psi(n_2)$. Let $n_1\leqq n\leqq n_2-1$. It is easily seen that $\psi(n_1)=\psi(n_1+1)=\cdots=\psi(n)=\cdots=\psi(n_2-1)$ and that

$$|a_n|\exp(\sigma\lambda_n)=|a_{n_2}|\exp(\sigma\lambda_{n_2}) \qquad \text{for} \quad \sigma=\psi(n).$$

Hence

$$\log|a_n|+\lambda_n\psi(n)\geqq F((1-\exp(\psi(n)-\alpha))^{-1},\bar{\lambda})$$

or

$$\begin{aligned}\log|a_n|+\alpha\lambda_n&\geqq\lambda_n(\alpha-\psi(n))+F((1-\exp(\psi(n)-\alpha))^{-1},\bar{\lambda})\\&\geqq\lambda_n(1-\exp(\psi(n)-\alpha))+F((1-\exp(\psi(n)-\alpha))^{-1},\bar{\lambda})\end{aligned} \tag{3.8}$$

Now, consider the function

$$H(x)=\frac{\lambda_n}{x}+F(x, \bar{\lambda}).$$

Differentiating $H(x)$, we get

$$H'(x)=-\frac{\lambda_n}{x^2}+\frac{dF(x, \bar{\lambda})}{dx}.$$

Now, since β is an admissible function we have $F(x, \bar{\lambda})<x$ for $x \geqq x^0=x^0(\bar{\lambda})$ and

$$\frac{dF(x, \bar{\lambda})}{dx} \leqq \frac{F(x, \bar{\lambda})}{x}$$

for $x \geqq x_0=x_0(\bar{\lambda})$. Set $x'=\max(x_0, x^0)$. Now

$$H'(x')=-\frac{\lambda_n}{(x')^2}+\left.\frac{dF(x, \bar{\lambda})}{dx}\right|_{x=x'}$$

$$\leqq -\frac{\lambda_n}{(x')^2}+\frac{F(x', \bar{\lambda})}{x'}<0$$

for $n>n'=n'(x')$. On the other hand, since (3.6) is satisfied we have $H'(x)>0$ for all sufficiently large values of x. Thus, if $x^*(n)$ is the point such that

$$\min_{x'<x<\infty} H(x)=H(x^*(n)) \qquad (n>n').$$

Then

$$\frac{\lambda_n}{(x^*(n))^2}=\left.\frac{dF(x, \bar{\lambda})}{dx}\right|_{x=x^*(n)}$$

or

(3.9) $$\lambda_n \leqq x^*(n)F(x^*(n), \bar{\lambda}) \leqq (x^*(n))^2.$$

Using (3.8), (3.9) and the definition of $x^*(n)$, we have

$$\log|a_n|+\alpha\lambda_n \geqq F(\sqrt{\lambda_n}, \bar{\lambda})$$

Since $\beta \in \Lambda$, this easily gives that

(3.10) $$\lambda(\beta, f) \leqq \liminf_{n\to\infty} \frac{\beta(\log|a_n|+\alpha\lambda_n)}{\beta(\log \lambda_n)}$$

If $\lambda(\beta, f)=\infty$, the above arguments with an arbitrarily large number in place of $\bar{\lambda}$ give that the limit inferior on the right hand side of (3.10) is also infinite. This proves the theorem.

Combining Theorems 2 and 3 we obtain a coefficient characterization of *lower β-order* for a subclass of functions of D_α given as Theorem 4 below.

THEOREM 4. *Let $f(s)\in D_\alpha$ with lower β-order $\lambda(\beta,f)$, $\lambda(\beta,f)>1$. Assume that β is an admissible function and satisfies* (3.6). *Let* (3.1) *hold and let $\psi(n)$, as given by* (3.7), *be ultimately a nondecreasing function of n. Further, if $\beta(\log \lambda_n)\sim\beta(\log \lambda_{n+1})$ as $n\to\infty$, then*

$$\lambda(\beta,f)=\liminf_{n\to\infty}\frac{\beta(\log|a_n|+\alpha\lambda_n)}{\beta(\log\lambda_{n+1})}.$$

Our next theorem gives a coefficient characterization of *lower β-order* which holds for a wider subclass of functions of the class D_α. Thus we have

THEOREM 5. *Let $f(s)\in D_\alpha$ with lower β-order $\lambda(\beta,f)$, $\lambda(\beta,f)\geqq 1$. Assume that β is an admissible function and satisfies* (3.6). *Let* (3.1) *hold. Further, if $\beta(\log\lambda_{n_m})\sim\beta(\log\lambda_{n_m+1})$ as $m\to\infty$, where $\{n_m\}$ is the sequence of the principal indices of $f(s)$, then*

$$\lambda(\beta,f)=\max(1,\theta_0) \tag{3.11}$$

where

$$\theta_0=\max_{\{n_k\}}\left\{\liminf_{k\to\infty}\frac{\beta(\log|a_{n_k}|+\alpha\lambda_{n_k})}{\beta(\log\lambda_{n_k+1})}\right\}. \tag{3.12}$$

Here maximum in (3.12) *is taken over all increasing sequences $\{n_k\}$ of positive integers.*

Proof. First, let $\lambda(\beta,f)=1$. Then, by Theorem 2, θ_0, as given by (3.12), is atmost one. Thus (3.11) holds in this case.

Next, let $\lambda(\beta,f)>1$. Now, consider the function

$$g(s)=\sum_{m=1}^{\infty}a_{n_m}\exp(s\lambda_{n_m}),$$

where $\{n_m\}_{m=1}^{\infty}$ is the sequence of the principal indices of $f(s)$. It is easily seen that $g(s)\in D_\alpha$ and that $g(s)$ also satisfies (3.1). Further, for any s, $f(s)$ and $g(s)$ have the same maximum term and so, by Lemma 5, *β-order* and *lower β-order* of $g(s)$ are the same as those of $f(s)$. Thus *lower β-order* of $g(s)$ is $\lambda(\beta,f)$. Also, $\psi(n_m)=(\log|a_{n_m}/a_{n_m+1}|)/(\lambda_{n_m+1}-\lambda_{n_m})$ is a strictly increasing function of m. Since $g(s)$ satisfies the hypothesis of Theorem 4, we have

$$\lambda(\beta,f)=\liminf_{m\to\infty}\frac{\beta(\log|a_{n_m}|+\alpha\lambda_{n_m})}{\beta(\log\lambda_{n_m+1})}. \tag{3.13}$$

But, from Theorem 2, we have

$$\lambda(\beta,f)\geqq\max_{\{n_k\}}\left\{\liminf_{k\to\infty}\frac{\beta(\log|a_{n_k}|+\alpha\lambda_{n_k})}{\beta(\log\lambda_{n_k+1})}\right\}. \tag{3.14}$$

From (3.13) and (3.14) we get (3.11) in this case also.

The theorem is thus proved.

4. A decomposition theorem

We now prove a theorem for functions of *β-irregular growth.*

THEOREM 6. *Let $f(s)\in D_\alpha$ be of β-irregular growth with $\rho(\beta,f)>1$. Assume that β is an admissible function. Let $\lambda(\beta,f)<u<\rho(\beta,f)$. Then $f(s)$ is of the form $h_u(s)+g_u(s)$, where β-order of $g_u(s)$ is not greater than* max $(1, u)$ *and*

$$h_u(s)=\sum_{p=1}^{\infty} a_{n_p} \exp (s\lambda_{n_p})$$

satisfies

$$\lambda(\beta, f)\geqq u \liminf_{p\to\infty} \frac{\beta(\log \lambda_{n_p})}{\beta(\log \lambda_{n_p+1})}.$$

Proof. Let $g_u(s)=\Sigma' a_n \exp (s\lambda_n)$, where Σ' denotes the summation over n for which

$$\log |a_n|\leqq F(\lambda_n, u)-\alpha\lambda_n .$$

Then $g_u(s)$ is in D_α and, by Lemma 2, is of *β-order* atmost max $(1, u)$. Let

$$h_u(s)=f(s)-g_u(s)=\sum_{p=1}^{\infty} a_{n_p} \exp (s\lambda_{n_p}) .$$

Then

$$\log |a_{n_p}|>F(\lambda_{n_p}, u)-\alpha\lambda_{n_p} .$$

Let $\sigma_p=\alpha-(1/\lambda_{n_p})$. Then, for $\sigma_p\leqq\sigma\leqq\sigma_{p+1}$, using Cauchy's inequality, we have

$$\begin{aligned}\log M(\sigma)&\geqq\log |a_{n_p}|+\sigma\lambda_{n_p}\geqq F(\lambda_{n_p}, u)-(\alpha-\sigma_p)\lambda_{n_p}\\ &\geqq F(\lambda_{n_p}, u)-1\geqq\frac{1}{2}F(\lambda_{n_p}, u)\end{aligned}$$

for all sufficiently large values of p. Thus

$$\frac{\beta(2 \log M(\sigma))}{\beta(-\log (1-\exp (\sigma-\alpha)))}\geqq\frac{u\beta(\log \lambda_{n_p})}{\beta(-\log (1-\exp (\sigma_{p+1}-\alpha)))}$$

for all σ sufficiently near to α. Now, since $(1-\exp (\sigma_{p+1}-\alpha))^{-1}\sim\lambda_{n_p+1}$ as $p\to\infty$, the theorem follows from the above inequality.

We now give a simple example which shows that the growth parameters introduced here are more general than those of Awasthi and Dixit.

Example. Consider the function

$$G(s)=\sum_{n=3}^{\infty} \exp (\exp (\log_2 n)^K+sn), \qquad 1<K<\infty .$$

Then $G(s)\in D_0$ and the order ρ of G is zero. From Theorem 2 we see that

$\lambda(\log, G)=\rho(\log, G)=\infty$. Thus the growth parameters of Awasthi and Dixit fail to give any specific information about the growth of $G(s)$. But, from the corollary of Theorem 2, it is easily seen that $\rho(\log_2, G)=\lambda(\log_2, G)=K$. Thus, the function G has nonzero finite growth parameters in our sense.

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