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Heegaard Diagrams of Torus Bundles Over S^1

by

Moto-o TAKAHASHI and Mitsuyuki OCHIAI

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1. Introduction

It is well known that every closed connected 3-manifold has a Heegaard splitting. A 3-manifold M is said to be of genus n, if M has a Heegaard splitting of genus n. Every 3-manifold of genus 1 is either a lens space or $S^2 \times S^1$ in the orientable case and is the twisted S^2-bundle over S^1 in the non-orientable case. Moreover, 3-manifolds of genus 1 are completely classified in [2], [4] and [5]. In this paper, we shall try to classify a certain class of 3-manifolds of genus 2. Indeed, we shall verify that torus bundles (over S^1) of genus 2 are completely classified by a new invariant (Theorem 3). Moreover, since every orientable 3-manifold of genus 2 is a 2-fold branched covering space of S^3 branched along a link, by Birman-Hilden-Viro-Takahashi [1], [10], [11], we can verify that every orientable torus bundle of genus 2 is a 2-fold branched covering space of S^3 branched along some specified link (Corollary 3.1).

In this paper, we work in the piecewise linear category. S^n, D^n denote n-sphere and n-disk, respectively. Let X be a manifold and Y be a submanifold properly embedded in X. Then $N(Y, X)$ denotes a regular neighborhood of Y in X. Closure, boundary, interior over one symbol are denoted by $\mathrm{cl}(\cdot)$, $\partial(\cdot)$, $\mathrm{int}\,(\cdot)$, respectively.

2. Surface-bundles over S^1

Let F be a closed connected surface and $\Phi: F \to F$ be a homeomorphism. Moreover let M be the 3-manifold obtained from $F \times I$ by identifying $(x, 0)$ in $F \times 0$ with $(\Phi(x), 1)$ in $F \times 1$. Then M is called a *surface-bundle over* S^1. We denote M also by $M(\Phi)$. It will be noticed that if F is orientable then M is orientable or non-orientable, according as Φ being oreitnation-preserving or orientation-reversing. Then by Neuwirth [8], we have;

PROPOSITION 1. *Let Φ_1 and Φ_2 be self-homeomorphisms of F. Then $M(\Phi_1)$ is homeomorphic to $M(\Phi_2)$, if there is a self-homeomorphism Ψ such that $\Psi\Phi_1$ is isotopic to $\Phi_2\Psi$.*

Next we consider the relationship between surface-bundles over S^1 and their Heegaard splittings. Let F be a closed connected surface and $g(F)$ be the genus of F.

That is, if F is orientable (resp. non-orientable), there exist $2\times g(F)$ (resp. $g(F)$) circles on F such that if we cut F along these circles, the resulting manifold is a 2-disk. We may assume that if F is non-orientable then all of such $g(F)$ circles are one-sided circles. Then we have;

THEOREM 1. *Let M be an F-bundle over S^1. If F is orientable (resp. non-orientable), M has a Heegaard splitting of genus $2g(F)+1$ (resp. $g(F)+1$).*

Proof. Let Φ be a self-homeomorphism of F such that $M=F\times I/\Phi$. We may assume without loss of generality that there exists a point p on F such that $\Phi(p)=p$. Next let $C_1, C_2, \cdots, C_n$ be circles on F satisfying the following conditions;

(1) $n=2g(F)$ (resp. $g(F)$), if F is orientable (resp. non-orientable),

(2) $C_i\cap C_j=p$, for all $i\neq j$,

(3) $F-\bigcup_{k=1}^{n}\operatorname{int}(N(C_k, F))$ is a 2-disk.

Let C be the circle $(p\times I)/\Phi$ in M and C'_k be the circle $C_k\times 0$ in M ($k=1, 2, \cdots, n$). Furthermore let $U=N(\bigcup_{k=1}^{n} C'_k\cup C, M)$ and $V=M-\operatorname{int}(U)$. We note that U is a non-oreintable handle if either F is orientable and Φ is orientation-reversing or F is non-orientable. (For the definition of non-orientable handles, see [9].) Let V' be $F\times I-\operatorname{int}(N(p\times I, F\times I))$ and $D_i=F\times i-\operatorname{int}(N_i)$, where $i=0$, 1 and $N_0=N(\bigcup_{k=1}^{n}(C_k\times 0), F\times 0)$, $N_1=\Phi(N_0)$. Then D_i is a 2-disk in $F\times i$ ($i=0, 1$). Now we may assume that V is obtained from V' by identifying points x in D_0 with points $\Phi(x)$ in D_1. Since V' is a handle of genus n, V is also a handle of genus $n+1$. Thus M has a Heegaard splitting of genus $n+1$. That is, $M=U\cup V$ with $U\cap V=\partial U=\partial V$ and U and V are homeomorphic handles. This completes the proof of the theorem.

From now on, we shall consider surface-bundles over S^1 with Heegaard splittings of rather small genus. Let F be a closed surface with positive genus $g(F)$ and M be an F-bundle over S^1. It is easily verified that M has no Heegaard splittings of genus one. Thus we are interested in the existence of surface-bundles over S^1 with Heegaard splittings of genus two. As the first observation, we have;

THEOREM 2. *For an arbitrary positive integer n, there exists an orientable F-bundle over S^1 such that $g(F)=n$ and M has a Heegaard splitting of genus two.*

Proof. Let K be a torus knot of type (p, q) in S^3 with $n=(p-1)(q-1)/2$. Then the knot exterior $E(K)=S^3-\operatorname{int}(N(K, S^3))$ of K is an F_1-bundle over S^1 such that $\partial F_1\subseteq\partial E(K)$, $g(F_1)=n$, and $\partial E(K)=S^1\times S^1$. Since K is a torus knot, we may assume that K lies on the boundary of an unknotted solid torus H in S^3. Let α be a simple arc in ∂H joining distinct points of K with the interior of it disjoint from K such that it is not homotopic on ∂H to any arcs in K joining points $K\cap\alpha$. Then $N(\alpha\cup K, S^3)=V$ is a handle of genus two. Furthermore, $U=S^3-\operatorname{int}(V)$ is also a handle of genus two, since $H-\operatorname{int}(V)$ and $(S^3-\operatorname{int}(H))-\operatorname{int}(V)$ are both solid tori and their intersection is a 2-disk $\partial H-\operatorname{int}(V)$. Let M be a closed 3-manifold obtained by attaching a 2-handle $D^2\times I$ to $E(K)$ along ∂F_1. Then M is an F-bundle over S^1 such that F is a closed surface with $g(F)=n$ and that M has a Heegaard splitting of genus two. This completes the proof of the theorem.

It will be noticed that by Moser [6] all the 3-manifolds given by Theorem 2 are Seifert fibered spaces.

3. Torus-bundles over S^1

In this section, we consider only torus-bundles over S^1. Let G be the group of 2×2 matrices over Z with determinant plus or minus one. Moreover, let T be a torus and $\Lambda(T)$ be the homeotopy group of T. Then $\Lambda(T)$ is isomorphic to G. Let Φ be a homeomorphism of T onto itself. Then Φ is given by a matrix $\begin{pmatrix} p & q \\ r & s \end{pmatrix}$ in G. Let $M(\Phi)$ be the torus bundle over S^1 determined by Φ. A presentation of $\pi_1(M(\Phi))$ is given by

$$\pi_1(M(\Phi))=\langle x, y, t \mid [x, y]=1,\ txt^{-1}=x^p y^q,\ tyt^{-1}=x^r y^s\rangle ,$$

where x, y correspond to generators of $\pi_1(T)$.

PROPOSITION 2. *Let Φ_1 and Φ_2 be self-homeomorphisms of T, whose matrices are A_1 and A_2, respectively. Moreover let M_1 and M_2 be the torus-bundles over S^1 determined by Φ_1 and Φ_2, respectively. Then M_1 is homeomorphic to M_2 if and only if A_1 is a conjugate of A_2 or A_2^{-1} in G.*

Proof. One direction comes from Proposition 1. Furthermore, if the Betti number $b(M(\Phi_1))=1$, then the converse follows from Theorem 1 in [7]. Suppose that $M(\Phi_1)$ is homeomorphic to $M(\Phi_2)$ and $b(M(\Phi_i))\geqq 2$ $(i=1, 2)$. Thus we have that $H_1(M(\Phi_i), Z)=Z+Z+Z_k$. Let E be the unit matrix and $B_i=A_i-E$ $(i=1, 2)$. It is easily seen that the determinant of B_i is zero. Let $B_i=\begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}$ $(i=1, 2)$. Then there are integers v_i and w_i such that $(a_i, b_i)=v_i(\alpha_i, \beta_i)$ and $(c_i, d_i)=w_i(\alpha_i, \beta_i)$, where $i=1, 2$ and α_i and β_i are relatively prime integers. Thus there are integers γ_i and δ_i such that $\det\begin{pmatrix} \alpha_i & \beta_i \\ \gamma_i & \delta_i \end{pmatrix}=1$ $(i=1, 2)$. Then we have that $\begin{pmatrix} \alpha_i & \beta_i \\ \gamma_i & \delta_i \end{pmatrix}\begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}\begin{pmatrix} \delta_i & -\beta_i \\ -\gamma_i & \alpha_i \end{pmatrix}=\begin{pmatrix} a_i+d_i & 0 \\ u_i & 0 \end{pmatrix}$, where $u_i=\delta_i(\gamma_i a_i+\delta_i c_i)-\gamma_i(\gamma_i b_i+\delta_i d_i)$ $(i=1, 2)$. Thus the matrix A_i is conjugate to $\begin{pmatrix} a_i+b_i+1 & 0 \\ u_i & 1 \end{pmatrix}$ $(i=1, 2)$. Let $z_i=a_i+b_i+1$. Since $\det(A_i)=\pm 1$, we have that $|z_i|=1$. Then two cases happen;

Case (1): $M(\Phi_i)$ is orientable. In this case, we have that $z_i=1$. Since $H_1(M(\Phi_i), Z)=Z+Z+Z_k$, we have that $k=|u_i|$. Thus A_1 is conjugate to A_2, since $\begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}=\begin{pmatrix} 1 & 0 \\ -u & 1 \end{pmatrix}$.

Case (2): $M(\Phi_i)$ is non-orientable. In this case, we have that $z_i=-1$. By Hempel [4], A_1 is also conjugate to A_2, since $\begin{pmatrix} -1 & 0 \\ u & 1 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ u & 1 \end{pmatrix}=E$.

This completes the proof.

By the above argument, if M is a torus-bundle with $H_1(M, Z)=Z+Z+Z_k$, then the corresponding matrix A is conjugate to one of $\begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix}$, $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}$.

From now on, we are interested in torus-bundles with Heegaard splittings of genus two. By Theorem 1, every torus-bundle has always a Heegaard splitting of genus three. But some of them have also Heegaard splittings of genus two.

PROPOSITION 3. *Let $M(\Phi)$ be a torus-bundle over S^1 and $\varepsilon=\pm 1$. If the matrix of Φ is $\begin{pmatrix} m & \varepsilon \\ 1 & 0 \end{pmatrix}$, then $M(\Phi)$ has a Heegaard splitting of genus two.*

Proof. By Theorem 1, $M(\Phi)$ has a Heegaard splitting of genus three and the Heegaard splitting $(U, V; F)$ is associated with the presentation of $\pi_1(M(\Phi))$, $\langle x, y, t \mid [x, y]=1,\ txt^{-1}=x^m y^\varepsilon,\ tyt^{-1}=x\rangle$. Let $u=u_1\cup u_2\cup u_3$ (resp. $v=v_1\cup v_2\cup v_3$) be a complete system of meridian-disks of U (resp. V). That is, u (resp. v) is a collection of mutually disjoint disks properly embedded in U (resp. V) such that $\mathrm{cl}(U-N(u, U))$ (resp. $\mathrm{cl}(V-N(v, V))$) is a 3-disk. Let x, y, and t be the canonical generators of the free group $\pi_1(V)$ $(=Z*Z*Z)$. Then we can easily find a homeomorphism f from ∂U onto ∂V such that the induced homomorphism $f_*: \pi_1(\partial U)\to\pi_1(V)$ satisfies $f_*(\partial u_1)=xyx^{-1}y^{-1}$, $f_*(\partial u_2)=x^m y^\varepsilon tx^{-1}t^{-1}$, and $f_*(\partial u_3)=xty^{-1}t^{-1}$. It will be noticed that $f(\partial u_1)$ bounds a torus with one hole in V. We can assume that $f(\partial u_3)$ meets ∂v_2 transversely at only one point. Then if $M(\Phi)$ is orientable, by Waldhausen [13] the intersection of ∂v_2 and $f(\partial u_1)$ or $f(\partial u_2)$ are eliminated. Next suppose that $M(\Phi)$ is non-orientable. Then we may assume that the generators x and y (resp. t) are induced by orientable circles (resp. a non-orientable circle) in V. Thus all the circles $f(\partial u_1)$, $f(\partial u_2)$, and $f(\partial u_3)$ are orientable in ∂V. Hence the elimination method of the orientable case can also apply to the non-orientable case. Let u_1' and u_2' be the resulting circles on the boundary of $V'=V-\mathrm{int}(N(v_2, V))$. Then $(V'; \partial v_1\cup\partial v_3, u_1' u_2')$ gives a Heegaard diagram of genus two. Thus $M(\Phi)$ has a Heegaard splitting of genus two. This completes the proof.

It will be noticed that if $\varepsilon=-1$ and $m=2$ (resp. $\varepsilon=+1$ and $m=3$), $M(\Phi)$ has an orientable (resp. non-orientable) Heegaard diagram of genus two, illustrated in Figure 1.1 (resp. Figure 1.2).

Next we shall verify that the torus-bundles of genus two given by Proposition 3 cover all torus-bundles of genus two.

LEMMA 1. *Let A be a matrix in G and M be a torus-bundle determined by A. If $\pi_1(M)$ is generated by two generators, then A is conjugate to a matrix $\begin{pmatrix} p' & q' \\ r' & s' \end{pmatrix}$ with $q'=1$ or $r'=1$.*

Proof. To avoid complexity, we will verify only the case when M is orientable, and the proof in the case when M is non-orientable is similar. Let $\prod=\pi_1(M)$ and $A=\begin{pmatrix} p & q \\ r & s \end{pmatrix}$. Suppose that $\prod=\langle a, b\rangle$, that is, two elements a and b in $\prod$ generate $\prod$. By $txt^{-1}=x^p y^q$ and $tyt^{-1}=x^r y^s$, we have $t^{-1}xt=x^s y^{-q}$ and $t^{-1}yt=x^{-r}y^p$, since $ps-qr=1$. Thus we have that $tx=x^p y^q t$, $ty=x^r y^s t$, $t^{-1}x=x^s y^{-q} t^{-1}$, and $t^{-1}y=x^{-r}y^p t^{-1}$. Let z be an arbitrary element in $\prod$. By the above four equations and $xy=yx$, there are three integers α, β, γ, such that $z=x^\alpha y^\beta t^\gamma$. Furthermore such expression of z is unique. For, if $x^\alpha y^\beta t^\gamma=1$, then the equation $\alpha x+\beta y+\gamma t=0$ holds in $H_1(M, Z)$. Since $H_1(M, Z)=Z+Z_k$, x and y generate Z_k, and t generates Z, we have that $\gamma=0$. Hence $x^\alpha y^\beta=1$ in $\pi_1(M)$. Here x, y are contained in $\pi_1(T)$. Let i_*: $\pi_1(T)\to\pi_1(M)$ be the inclusion induced homomorphism. Since i_* is monic, $x^\alpha y^\beta=$

1 in $\pi_1(T)$. But T is a torus, and so $\alpha=\beta=0$.

Now suppose that $a=x^{\alpha_1}y^{\beta_1}t^{\gamma_1}$ and $b=x^{\alpha_2}y^{\beta_2}t^{\gamma_2}$. We may assume that $0\leqq\gamma_1\leqq\gamma_2$. Then $b=x^{\alpha_1}y^{\beta_1}t^{\gamma_1}x^{\alpha'}y^{\beta'}t^{\gamma_2-\gamma_1}=ax^{\alpha'}t^{\beta'}t^{\gamma_2-\gamma_1}$ for some integer α', β'. Thus we may assume that $\prod=\langle a, b\rangle$ with $a=x^{\alpha_1}y^{\beta_1}t^{\gamma_1}$ and $b=x^{\alpha_2}y^{\beta_2}$. Next we can assume without loss of generality that α_2 and β_2 are relatively prime. Then the element b can be thought of as a simple loop in T, which is not homotopic in T to zero. And there is a simple loop c in T which meets b transversely at only one point. Let $c=c^{\alpha_3}y^{\beta_3}$ with $\det\begin{pmatrix}\alpha_2 & \beta_2\\ \alpha_3 & \beta_3\end{pmatrix}=1$. Consequently a new presentation of $\prod$, $\langle b, c, t \mid [b, c]=1, tbt^{-1}=b^{p_1}c^{q_1}, tct^{-1}=b^{r_1}c^{s_1}\rangle$ is obtained and $\prod=\langle a, b\rangle$ with $a=b^{\alpha}c^{\beta}t^{\gamma}$. And so $\prod=\langle a_1, b\rangle$ with $a_1=c^{\beta}t^{\gamma}$. Since a_1 and b generate t, we have that $\gamma=1$. Thus $\prod=\langle a_1, b\rangle$ with $a_1=c^{\beta}t$. Since $t=c^{-\beta}a_1$, the following presentation of $\prod$ follows;

$$\prod=\langle b, c, a_1 \mid [b, c]=1,\ a_1ba_1^{-1}=b^{p_1}c^{q_1},\ a_1ca_1^{-1}=b^{r_1}c^{s_1}\rangle\,.$$

Let $a_1=g$. For every integer m, we have the following,

$$(1)\quad gb^mg^{-1}=(b^{p_1}c^{q_1})^m \qquad (2)\quad g^{-1}b^mg=(b^{s_1}c^{-q_1})^m$$

$$(3)\quad gc^mg^{-1}=(b^{r_1}c^{s_1})^m \qquad (4)\quad g^{-1}c^mg=(b^{-r_1}c^{p_1})^m$$

Since $\prod=\langle g, b\rangle$, we have that $c=g^{\nu_1}b^{\nu_2}g^{\nu_3}\cdots b^{\nu}k$ for some integer $\nu_1, \nu_2, \cdots, \nu_k$. Then we will verify that c has an expression $b^{\alpha}c^{\beta}g^{\gamma}$ such that q_1 divides β. Since both b and c are contained in $\pi_1(T)$, we may assume without loss of generality that all of the three integers ν_1, ν_2, ν_3 are non-zero. It is sufficient to verify that an element $g^{\tau}b^{\lambda}$, with non-zero integers τ and λ, in $\prod$ has an expression $b^{\alpha_1}c^{\beta_1}g^{\gamma_1}$ with q_1 divides β_1. To avoid complexity, we assume that τ and λ are both positive. Then by the equations (1) and (2), we have the following,

$$g^{\tau}b\lambda=b^{p_1^{\tau}\lambda}c^{p_1^{\tau-1}q_1\lambda}gc^{p_1^{\tau-2}q_1\lambda}g\cdots c^{p_1q_1\lambda}gc^{q_1\lambda}g\,.$$

Furthermore, by equation (3) we have that for any integer m, $gc^m=(b^{r_1}c^{s_1})^mg=b^{r_1m}c^{s_1m}g$. Thus, at the final step we can obtain the expression of $g^{\tau}b^{\lambda}$, $b^{\alpha_1}c^{\beta_1}g^{\gamma_1}$, such that q_1 divides β_1. Consequently, $c=b^{\alpha}c^{\beta}g^{\gamma}$ for some integers α, β, γ and q_1 divides β. But by the uniqueness of the expression of c, we have that $\beta=1$. Hence $q=\pm 1$. Here $\begin{pmatrix}p & -1\\ r & s\end{pmatrix}$ is conjugate to $\begin{pmatrix}p & 1\\ -r & s\end{pmatrix}$. Thus we conclude that $q=1$. This completes the proof of the lemma.

LEMMA 2. *Let* $A=\begin{pmatrix}p & q\\ r & s\end{pmatrix}$ *be a matrix in* G. *If* $(q-1)(r-1)=0$, *then* A *is conjugate to a matrix* $\begin{pmatrix}m & \varepsilon\\ 1 & 0\end{pmatrix}$ *in* G *with* $\varepsilon=\pm 1$.

Proof. Suppose that $q=1$. In this case, if $\det(A)=1$, then $A=\begin{pmatrix}p & 1\\ ps-1 & s\end{pmatrix}$. If $\det(A)=-1$, then $A=\begin{pmatrix}p & 1\\ ps+1 & s\end{pmatrix}$. Then the following hold;

$$\begin{pmatrix}p & 1\\ ps-1 & s\end{pmatrix}\begin{pmatrix}1 & 0\\ s & -1\end{pmatrix}=\begin{pmatrix}1 & 0\\ s & -1\end{pmatrix}\begin{pmatrix}p+s & -1\\ 1 & 0\end{pmatrix}, \qquad \begin{pmatrix}p & 1\\ ps+1 & s\end{pmatrix}\begin{pmatrix}-1 & 0\\ -s & -1\end{pmatrix}=\begin{pmatrix}-1 & 0\\ -s & -1\end{pmatrix}\begin{pmatrix}p+s & 1\\ 1 & 0\end{pmatrix}.$$

Thus we set $m=p+s$. If $r=1$, then the same result is obtained.

Let $M(m, \varepsilon)$ be a 3-manifold determined by a matrix $\begin{pmatrix}m+2 & \varepsilon\\ 1 & 0\end{pmatrix}$ with $\varepsilon=\pm 1$. Then by Lemma 1 and Lemma 2, and Proposition 2, we have;

THEOREM 3. *Every torus-bundle over S^1 with a Heegaard splitting of genus two is homeomorphic to $M(m, \varepsilon)$ for some integer m, and if it is orientable (resp. non-orientable) then $\varepsilon=-1$ (resp. $\varepsilon=1$). In particular, $M(m, \varepsilon)= M(m', \varepsilon)$ if and only if $m=m'$.*

Birman-Hilden-Viro-Takahashi [1], [10], and [11] proved that every orientable

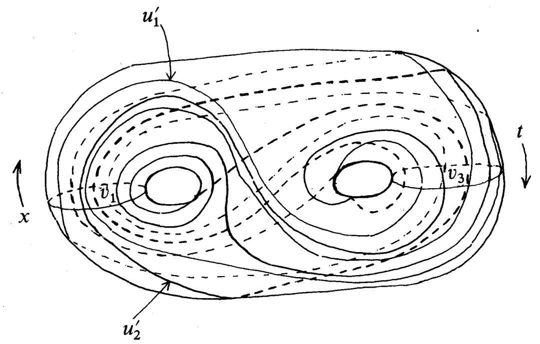

Figure 1.1. A Heegaard diagram in the orientable case of $m=2$.

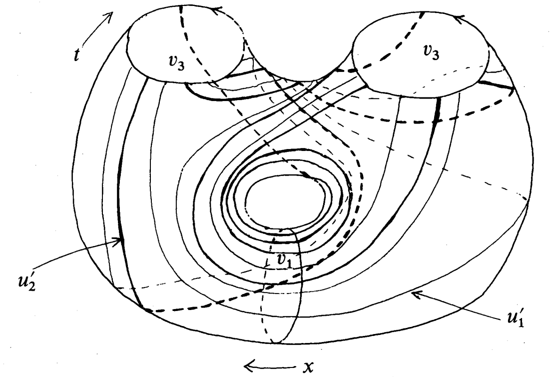

Figure 1.2. A Heegaard diagram in the non-orientable case of $m=3$.

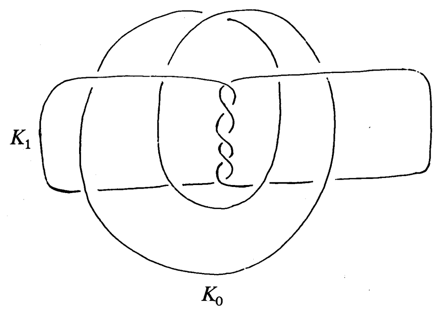

Figure 2. A link $K(m+4)=K_0 \cup K_1(\cup K_2)$.

closed 3-manifold with Heegaard splittings of genus two is a 2-fold branched covering space of S^3 branched along a link. As illustrated in preceding remark, the manifold $M(2, -1)$ has a Heegaard diagram of genus two given by Figure 1.1. Thus we can determine one type of branched sets of torus-bundles of genus two. Let $K(m+4)$ be the link illustrated in Figure 2. It has two components K_0 and K_1 (resp. three components K_0, K_1, and K_2) if m is odd (resp. even). We note that the component K_0 is unknotted and that $m+4$ is the number of double points in $K_1 \cup K_2$ (resp. K_1), when m is even (resp. odd). Then we have;

COROLLARY 3.1. *Every orientable torus-bundle of form $M(m, -1)$ is a 2-fold branched covering space of S^3 branched along $K(m+4)$.*

By the way, there are infinitely many torus-bundles of genus three but not two. It is an interesting problem to decide whether such torus-bundles are 2-fold branched covering spaces of S^3 or not. Fox had proved in [3] that $S^1 \times S^1 \times S^1$ is not a 2-fold branched covering space of S^3. Thus we will set up the following problem;

PROBLEM 1. *Which torus-bundles are 2-fold branched covering spaces of S^3?*

In view of Lemma 1, we raise the following;

PROBLEM 2. *Are link types of branched sets of every torus-bundle of genus two unique?*

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Institute of Mathematics
The University of Tsukuba
Ibaraki, Japan

and

Department of Mathematics
Osaka University
Osaka, Japan