COMMENTARII MATHEMATICI
UNIVERSITATIS SANCTI PAULI
Vol. 31, No. 1, 1982

ed. RIKKYO UNIV/MATH
IKEBUKURO TOKYO
171 JAPAN

Strict Boolean-valued Models

by

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(Received January 16, 1981)

§ 1. Introduction

The object of the category BVM(T) of Boolean-valued models for the theory T is a map of

$$\mu : \mathrm{Alg}_F(Tm, M) \longrightarrow \mathrm{Bool}\,(L(T), A)$$

in **Set**, satisfying the conditions of Definition 2.1 [1]. This map is called an A-valued model with the domain M, where A is a Boolean algebra and M is an F-type algebra.

If we substitute any Boolean algebra B under the conditions $A \subseteqq B$ for A in μ above, then the A-valued model μ becomes a B-valued model at the same time. Clearly, the elements of $B-A$ can not be used under the given interpretation. Here, we call these elements "dummy values." The purpose of this paper is to construct a model with a few dummy values as possible or a model without any dummy values, and to pursue its behavior.

§ 2. Definition

In this section, we will introduce the notion of strict Boolean-valued models. This notion will achieve the purpose of §1.

We need at least the set of values

$$\bigcup_{\sigma : Tm \to M} \mu(\sigma)''L(T) \qquad (\subseteqq A)$$

for the interpretation of all formulas. So, the subalgebra of A generated by $\bigcup_{\sigma : Tm \to M} \mu(\sigma)''L(T)$ must comply with our definition.

However,

LEMMA 2.1. *$\bigcup_{\sigma : Tm \to M}\mu(\sigma)''L(T)$ is a Boolean algebra.*

Proof. First of all, $\mathbf{0}, \mathbf{1} \in \mu(\sigma)''L(T)$ for some σ. Let $\mu(\sigma_1)[\varphi_1]$, $\mu(\sigma_2)[\varphi_2] \in \bigcup_{\sigma : Tm - M} \mu(\sigma)''L(T)$, and let $\binom{x_1, \cdots, x_n}{a_1, \cdots, a_n} : Tm \to M$ be one of the arrows in Alg_F, such that

$$\binom{x_1, \cdots, x_n}{a_1, \cdots, a_n}(t) = \begin{cases} a_i & \text{if} \quad t = x_i \\ \text{arbitrary in } M & \text{otherwise}. \end{cases}$$

Then

$$\begin{aligned}
\mu(\sigma_1)[\varphi_1] \wedge \mu(\sigma_2)[\varphi_2] &= \mu\binom{x_1, \cdots, x_n}{\sigma_1(x_1), \cdots, \sigma_1(x_n)}[\varphi_1] \wedge \mu\binom{y_1, \cdots, y_m}{\sigma_2(y_1), \cdots, \sigma_2(y_m)}[\varphi_2] \\
&= \mu\binom{x_1, \cdots, x_n, \quad z_1, \cdots, z_m}{\sigma_1(x_1), \cdots, \sigma_1(x_n), \sigma_2(y_1), \cdots, \sigma_2(y_m)}[\varphi_1] \\
&\qquad \wedge \mu\binom{x_1, \cdots, x_n, \quad z_1, \cdots, z_m}{\sigma_1(x_1), \cdots, \sigma_1(x_n), \sigma_2(y_1), \cdots, \sigma_2(y_m)}[\varphi_2] \\
&= \mu\binom{x_1, \cdots, x_n, \quad z_1, \cdots, z_m}{\sigma_1(x_1), \cdots, \sigma_1(x_n), \sigma_2(y_1), \cdots, \sigma_2(y_m)}[\varphi_1 \wedge \varphi_2] \\
&\in \bigcup_{\sigma : Tm \to M} \mu(\sigma)''L(T)
\end{aligned}$$

where the free variables of φ_1 are among $x_1, \cdots, x_n$, these φ_2 are among $y_1, \cdots, y_m$ and

$$z_j = \begin{cases} x_i & \text{there exisrs } i \text{ for which } \sigma_1(x_i) = \sigma_2(y_j) \\ y_j & \text{otherwise} . \end{cases}$$

This is similar to the cases of $\vee$, $\neg$.

By Lemma 2.1 the step of generation falls into disuse. Therefore, the next definition comes into effect.

The Boolean-valued model

$$\mu : \mathrm{Alg}_F(Tm, M) \longrightarrow \mathrm{Bool}\,(L(T), A)$$

is said to be strict if and only if

$$A = \bigcup_{\sigma : Tm \to M} \mu(\sigma)''L(T)$$

The full subcategory of BVM(T) determined by the strict Boolean-valued models will be denoted by SBVM(T).

Clearly, these well-known two-valued models are strict.

COROLLARY 2.2. *For any Boolean-valued model* $\mu_{(M, A)}$, *there is exactly one strict Boolean-valued model such as following*:

$$s(\mu) : \mathrm{Alg}_F(Tm, M) \longrightarrow \mathrm{Bool}\,(L(T), s\,(A)),$$

where $s(A) = \bigcup_{\sigma : Tm \to M} \mu(\sigma)''L(T)$, *and for any* $\sigma : Tm \to M$ *and* $\varphi \in wff$

$$s(\mu)(\sigma)[\varphi] = \mu(\sigma)[\varphi] .$$

This $s(\mu)$ *will be called the cut down model of* μ.

§3. Limits in SBVM(T)

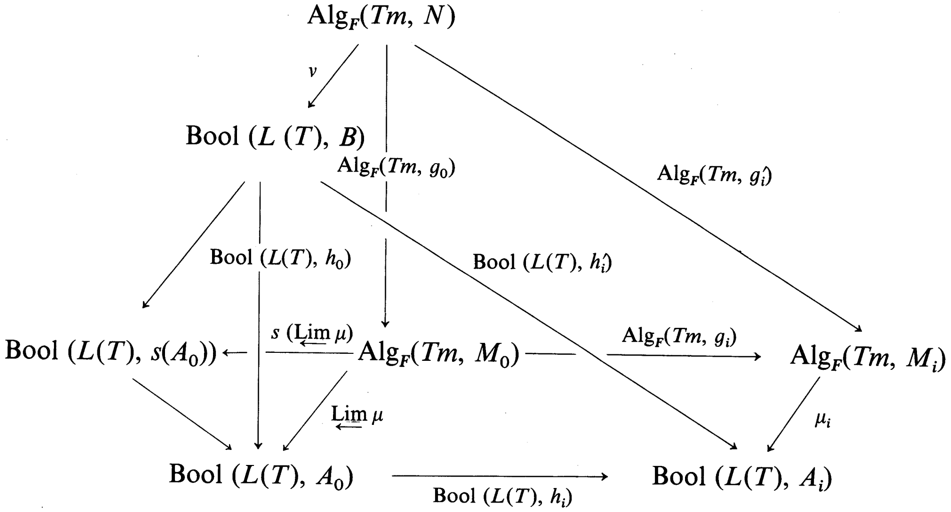
This section examines the construction of limits in SBVM(T). Note especially that the terminal object

$$t : \mathrm{Alg}_F(Tm, *) \longrightarrow \mathrm{Bool}\,(L(T), \mathbf{1})$$

in BVM(T) is strict, and it is also terminal in SBVM(T).

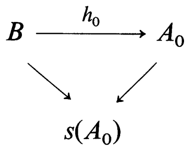
It is notable that every product $\mu \times \nu$ of two strict Boolean-valued models is not always strict. By Corollary 2.2, the cut down model of this product in BVM(T) is strict, and we can easily prove that $s(\mu \times \nu)$ becomes the product in SBVM(T). This argument can be substantiated not only in the case of products but also in the case of limits.

THEOREM 3.1. *If a functor* $\mu : J \to \mathrm{SBVM}(T)$ *has a limit* $\varprojlim \mu$ *in* BVM(T), *then the cut down model* $s(\varprojlim \mu)$ *of it is a limit in* SBVM(T).



Proof. Let $\mu : J \to \mathrm{SBVM}(T)$ be a functor with a limiting cone (g_i, h_i): $\varprojlim \mu_{(M_0, A_0)} \to \mu_{i(M_i, A_i)}$ in BVM(T), and let $\nu_{(N, B)}$ be any strict Boolean-valued model with a cone $(g_i', h_i') : \nu_{(N, B)} \to \mu_i$. Then, from the universal property of $\varprojlim \mu$, there is a unique arrow (g_0, h_0): $\nu \dashrightarrow \varprojlim \mu$ such that $(g_i', h_i') = (g_i, h_i)(g_0, h_0)$ for all $i \in J$. The Boolean homomorphism $h_0 : B \to A_0$ can be factored as $B \to s(A_0)$ followed by an inclusion $s(A_0) \rightarrowtail A_0$. Since $\nu_{(N, B)}$ is strict, any element b of B can be written in the form $\nu(\sigma)[\varphi]$ and we have the following equation:

$$h_0(b) = h_0 \nu(\sigma)[\varphi] = \varprojlim \mu(g_0 \sigma)[\varphi] \in s(A_0)\,.$$



$\S 4.$ Adjunction $\mathrm{SBVM}(T) \underset{\xi}{\overset{U_s}{\rightleftarrows}} \mathrm{Alg}_F$

In [2] Theorem 3.1, we introduce the adjunction

$$\varepsilon : \mathrm{BVM}(T) \underset{\xi}{\overset{U}{\rightleftarrows}} \mathrm{Alg}_F$$

where U is the forgetful functor, ξ is its left-adjoint and ε is the counit of this adjunction. In this section we will examine more closely the construction of ξ.

Among other things, we have

LEMMA 4.1. *For any F-algebra M, $\xi(M)$ is strict.*

Proof. Clearly $\bigcup_{\sigma: Tm\to M} \xi(M)(\sigma)''L(T) \subseteqq L_M$. The converse inclusion also holds. Since any element of L_M can be written in the form

$$[\varphi(\begin{smallmatrix} x_1, & \cdots, & x_n \\ a_1, & \cdots, & a_n \end{smallmatrix})]$$

there is a $\sigma : Tm \to M$ and $\varphi \in wff$ such that

$$[\varphi(\begin{smallmatrix} x_1, & \cdots, & x_n \\ a_1, & \cdots, & a_n \end{smallmatrix})] = \xi(M)(\sigma)[\varphi] .$$

This means $[\varphi(\begin{smallmatrix} x_1, & \cdots, & x_n \\ a_1, & \cdots, & a_n \end{smallmatrix})] \in \bigcup_{\sigma: Tm\to M} \xi(M)(\sigma)''L(T)$.

By this Lemma, the full subcategory SBVM(T) contains all the objects $\xi(M)$ for $M \in \mathrm{Alg}_F$, and it leads to another adjunction

$$\mathrm{SBVM}(T) \underset{\xi_s}{\overset{U_s}{\rightleftarrows}} \mathrm{Alg}_F$$

where the functor ξ_s is just ξ with its codomain restricted from BVM(T) to SBVM(T), U_s is U with a domain restricted to SBVM(T).

Putting together the information in the above lemma, we have the theorem:

THEOREM 4.2. The forgetful functor

$$U_s : \mathrm{SBVM}(T) \longrightarrow \mathrm{Alg}_F$$

has a left adjoint ξ_s.

COROLLARY 4.3. *U_s preserves limits.*

Acknowledgments. The author wishes to thank Professor T. Simauti for his encouragement suggestions, and criticism in connection with the writting this paper.

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