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Finite Element Approximation Theory for Strongly Nonlinear Problems

by

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Abstract

The application of finite element method to a strongly nonlinear Dirichlet problem, which arises in the field of oceanography, is described and new inequalities involving the error in the finite element solutions are derived. As a special case, we obtain the well-known error estimates for the corresponding mildly nonlinear and linear elliptic boundary value problems.

§1. Introduction and formulation

This paper is concerned with the approximate solution by finite element methods of a fairly large class of strongly nonlinear elliptic boundary value problems encountered in the study of very large, quasistatic deformations of isotropic hyperelastic bodies and in the field of glaciology. We develop a priori error estimates for these nonlinear problems, assuming that Galerkin approximations are made on certain subspaces endowed with standard finite element interpolation properties. The estimates given in Theorem 1 are distinctly nonlinear in character, that is there is no counterpart in the linear theory, although these reduced to the estimates for linear second order elliptic equations, when the governing equations are linearized. For instance, we find that the approximation error in the $W_p^1(\Omega)$-norm is of order $h^{1/p-1}$. Thus, when $p=2$, corresponding to linear and mildly nonlinear theory, we obtain an error of order h, which agrees with recent results, see Ciarlet [1], Strang and Fix [10], and Noor and Whiteman [7].

The mathematical model discussed in this paper arises in the field of oceanography, see [8]. The velocity u of the glacier is required to satisfy the nonlinear boundary value problem of the type:

$$(1)\qquad \begin{cases} Tu=f(u) & \text{in } \Omega \\ u=0, & \text{on } \partial\Omega, \end{cases}$$

where $Tu\equiv -\nabla(|\nabla u|^{p-2}\nabla u)$, Ω is the cross-section of the glacier and $f(u)$ is the Coulomb friction. The presence of $f(u)$ may be interpreted as a body heating term, it arises from resistivity and is termed a local Joule heating effect. In fact, some of the results and methods to be described in this paper may be extended to more complicated problems or to problems with other boundary conditions.

The problem (1) is a generalization of the nonlinear problem of finding u such that

$$Tu=f \quad \text{in} \quad \Omega$$
$$u=0 \quad \text{on} \quad \partial\Omega\,,$$

for which the error estimates, see Glowinski and Marroco [4], have been derived using the finite element approximation. *The presence of the nonlinear term $f(u)$ needs a different approach for deriving the error estimates and this is the main motivation of this paper.*

Let $\Omega\subset\mathbf{R}^n$ be a open bounded domain with smooth boundary $\partial\Omega$. We denote by $H\equiv W_0^{1,p}(\Omega)$, a reflexive Banach space with norm

$$\|v\|=\left(\int_\Omega |\nabla v|^p\, d\Omega\right)^{1/p}$$

and the dual space $H'=W^{-1,p'}(\Omega)$, $1/p+1/p'=1$. The pairing between H' and H is denoted by $\langle .,.\rangle$. It is assumed that the nonlinear function $f(u)\in L_p(\Omega)$ is antimonotone and Lipschitz continuous, see Noor [6].

Multiplying equation (1) by a test function $v\in H$ and integration by parts, we obtain

$$\langle Tu,\, v\rangle=(f(u),\, v)\,, \tag{2}$$

where

$$\langle Tu,\, v\rangle\equiv\int_\Omega |\nabla u|^{p-2}\nabla u\nabla v\, d\Omega\,, \quad \text{and} \quad (f(u),\, v)\equiv\int_\Omega f(u)v\, d\Omega\,.$$

This formulation is known as the weak formulation.

We need the following results, which are due to Glowinski and Marroco [4].

LEMMA 1. *For all $u,\, v\in H$, we have*

$$\langle Tu-Tv,\, u-v\rangle\geqq\alpha\|u-v\|^p\,, \qquad p\geqq 2\,. \tag{3}$$

$$\langle Tu-Tv,\, u-v\rangle\geqq\alpha\|u-v\|^2(\|u\|+\|v\|)^{p-2}\,, \qquad 1<p\leqq 2\,. \tag{4}$$

$$\|Tu-Tv\|\leqq\beta\|u-v\|(\|u\|+\|v\|)^{p-2}\,, \qquad p\geqq 2\,. \tag{5}$$

$$\|Tu-Tv\|\leqq\beta\|u-v\|^{p-1}\,, \qquad 1<p\leqq 2\,, \tag{6}$$

where $\alpha>0$, $\beta>0$ are constants independent of u and v.

We remark that if $f(u)$ is antimonotone and Lipschitz continuous and T satisfies (3)–(6), then there exists a unique solution of (1), see Noor [5]. Furthermore, the energy functional $I[v]$ associated with (1) can be given

(7) $$I[v]=\frac{1}{p}\int_{\Omega}|\nabla v|^{p}d\Omega-2\int_{\Omega}\int_{0}^{v}f(\eta)\,d\eta\,d\Omega$$
$$\equiv J(v)-2F(v)\,.$$

In this case Noor and Whiteman [7] has shown that the function $u\in H$ minimizes $I[v]$ on H if and only if $u\in H$ satisfies

(8) $$\langle J'(u),\,v\rangle=\langle F'(u),\,v\rangle\,,\qquad \text{for all}\quad v\in H\,,$$

with

(9) $$\langle J'(u),\,v\rangle\equiv\int_{\Omega}|\nabla u|^{p-2}\nabla u\nabla v\,d\Omega\,,\qquad \text{see Sibony [9]}\,,$$

and

(10) $$\langle F'(u),\,v\rangle\equiv\int_{\Omega}f(u)v\,d\Omega\,,\qquad \text{see [7]}\,.$$

Thus it is evident from (7)–(10) that problems equivalent to (1) are;

The Weak Problem:

Find $u\in H$ such that

$$\langle Tu,\,v\rangle=(f(u),\,v)\,,\qquad \text{for all}\quad v\in H\,.$$

The Variational Problem:

Find $u\in H$ which gives the minimum value to the functional

$$I\,[v]=\frac{1}{p}\int_{\Omega}|\nabla v|^{p}\,d\Omega-2\int_{\Omega}\int_{0}^{v}f(\eta)\,d\eta\,d\Omega\,.$$

We here consider the weak formulation (2) to derive the error estimates for $u-u_h$.

§2. Approximation and error analysis

We introduce a standard finite triangulation T_h of Ω, see [1] such that

(i) $T\subset\bar{\Omega}$ for all $T\in T_h$, $\bigcup_{T\in T_h}=\bar{\Omega}$

(ii) $T,\,T'\in T_h\Rightarrow T\cap T'=\varnothing$
or
T and T' have either only one common vertex or a whole common edge;

as usual h is the length of the largest side of T_h.

Let S_h be a finite dimensional subspace of H approximated as:

$$S_h=\{v_h : v_h \in C^0(\bar{\Omega}),\ v_h|_{\Omega}=0,\ v_h|_T \in P_1,\ \text{for all } T\in T_h\},$$

where P_1 = space of polynomials of degree $\leqslant 1$.

The weak problem (2) can in practice seldom be solved, and so, approximation u_h to u from a finite dimensional subspace $S_h \subset H$ are sought. Thus the finite element approximation u_h of u is:

Find $u_h \in S_h$ such that

(11) $$\langle Tu_h, v_h\rangle = (f(u_h), v_h), \qquad \text{for all } v_h \in S_h.$$

Subtracting (11) from (2), we obtain

(12) $$\langle Tu - Tu_h, v_h\rangle = (f(u)-f(u_h), v_h), \qquad \text{for all } v_h \in S_h.$$

This shows that in the nonlinear case, the approximate solution u_h is not the projection of u on S_h as it is in the linear case, i.e., $f(u)=f$. We, therefore, introduce the concept of *pseudo-projection* in order to obtain the inequality bounding the error $u-u_h$. Similar projections have been introduced by Dailey and Pierce [3], and Noor and Whiteman [7] to derive the error bound for $u-u_h$ in the case of mildly nonlinear elliptic boundary value problems.

Define $\bar{u}_h \in S_h$ to be pseudo-projection of $u \in H$ by the orthogonality condition

(13) $$\langle Tu - T\bar{u}_h, w_h\rangle = 0, \qquad \text{for all } w_h \in S_h.$$

We remark that for the linear case $f(u)=f$, $\bar{u}_h \in S_h$ is the finite element approximation of the solution u of (2) and hence $\bar{u}_h$ is the projection of u on $S_h \subset H$.

From (13), we get the following result, whose proof is obvious.

LEMMA 2.

(14) $$\|\bar{u}_h\| \leqslant \|u\|.$$

The relation (13) and Lemma 2 will play a key role in deriving the error estimate for $u-u_h$.

LEMMA 3. *If $f(u)$ is antimonotone and Lipschitz continuous, then for $p \geqslant 2$,*

(15) $$\|\bar{u}_h - u_h\| \leqslant \left(\frac{\gamma}{\alpha}\right)^{1/p-1} \|u-\bar{u}_h\|^{1/p-1},$$

where u_h, $\bar{u}_h$ are defined by the relations (11) *and* (13) *respectively.*

Proof. It follows from (3) and the antimonotonicity of $f(u)$ that

$$\begin{aligned}
\alpha\|\bar{u}_h - u_h\|^p &\leqslant \langle T\bar{u}_h - Tu_h, \bar{u}_h - u_h\rangle \\
&\leqslant \langle T\bar{u}_h - Tu_h, \bar{u}_h - u_h\rangle - (f(\bar{u}_h)-f(u_h), \bar{u}_h - u_h) \\
&= \langle T\bar{u}_h - Tu, \bar{u}_h - u_h\rangle + \langle Tu - Tu_h, \bar{u}_h - u_h\rangle - (f(\bar{u}_h)-f(u), \bar{u}_h - u_h) \\
&\qquad - (f(u)-f(u_h), \bar{u}_h - u_h).
\end{aligned}$$

From (13), we have that $\langle Tu-T\bar{u}_h, w_h\rangle=0$, for all $w_h\in S_h$,
Hence,

$$\alpha\|\bar{u}_h-u_h\|^p\leqslant\langle Tu-Tu_h, \bar{u}_h-u_h\rangle-(f(\bar{u}_h)-f(u), \bar{u}_h-u_h)-(f(u)-f(u_h), \bar{u}_h-u_h)\,.$$

Using (2) and (11), we have

$$\alpha\|\bar{u}_h-u_h\|^p\leqslant(f(u)-f(\bar{u}_h), \bar{u}_h-u_h)\,.$$

Applying the Cauchy-Schwartz inequality, we obtain

$$\begin{aligned}\alpha\|\bar{u}_h-u_h\|^p&\leqslant\|f(u)-f(\bar{u}_h)\|\ \|\bar{u}_h-u_h\|\\&\leqslant\gamma\|u-\bar{u}_h\|\ \|\bar{u}_h-u_h\|\,,\end{aligned}$$

after use has been made of the Lipschitz continuity of $f(u)$ and the Sobolev imbedding theorem, see Ciarlet [1]. Thus we have

$$\|\bar{u}_h-u_h\|\leqslant\left(\frac{\gamma}{\alpha}\right)^{1/p-1}\|u-\bar{u}_h\|^{1/p-1}\,,$$

which is (15), the required result.

LEMMA 4. *If u and $\bar{u}_h$ satisfy* (2) *and* (13), *then for* $p\geqslant 2$,

$$\|u-\bar{u}_h\|\leqslant Ch^{1/p-1}\|u\|^{p-2/p-1}\|u\|_2^{1/p-1}\,,\tag{16}$$

where C is a constant independent of u and h.

Proof. From (3), it follows that

$$\begin{aligned}\alpha\|u-\bar{u}_h\|^p&\leqslant\langle Tu-T\bar{u}_h, u-\bar{u}_h\rangle\\&=\langle Tu-T\bar{u}_h, u-v_h\rangle+\langle Tu-T\bar{u}_h, v_h-\bar{u}_h\rangle\,.\end{aligned}$$

But from (13), $\langle Tu-T\bar{u}_h, v_h-\bar{u}_h\rangle=0$, for all $v_h\in S_h$, and so,

$$\alpha\|u-u_h\|^p\leqslant\langle Tu-T\bar{u}_h, u-v_h\rangle\,.$$

If $u\in H\cap W^{2,p}(\Omega)$, then we can take v_h as I_hu, the linear interpolation of u on S_h, i.e.,

$$\begin{aligned}&I_hu\in S_h\\&I_hu(P)=u(P),\qquad\text{for all}\quad P\quad\text{vertex of}\quad T_h\,.\end{aligned}\tag{17}$$

Thus it follows from Ciarlet [1] and Strang and Fix [10] that

$$\|I_hu-u\|\leqslant C_1h\|u\|_2\,,\tag{18}$$

where C_1 is a constant independent of u and h.

For $p\geqslant 2$, it follows from (17), (18) and (5) that

$$\alpha\|u-\bar{u}_h\|^p \leqslant \|Tu-T\bar{u}_h\| \, \|I_hu-u\|$$
$$\leqslant \beta C_0 h(\|u\|+\|\bar{u}_h\|)^{p-2}\|u-\bar{u}_h\| \, \|u\|_2 ,$$
$$\leqslant \beta C_0 2^{p-2} h\|u\|^{p-2}\|u\|_2\|u-\bar{u}_h\| , \qquad \text{by Lemma 2},$$

from which, we get

$$\|u-\bar{u}_h\| \leqslant \left(\frac{\beta C_0}{\alpha}\right)^{1/p-2} 2^{p-2/p-1} h^{1/p-1}\|u\|^{p-2/p-1}\|u\|_2^{1/p-1}$$
$$= Ch^{1/p-1}\|u\|^{p-2/p-1}\|u\|_2^{1/p-1} ,$$

which is (16).

Our main aim is to derive the error estimate for $u-u_h$, which is the object of the next theorem.

THEOREM 1. *If u and u_h are the solutions of* (2) *and* (11) *respectively, then for $p \geqslant 2$, we have*

$$\|u-u_h\| \leqslant Ch^{1/p-1}\left\{1+\left(\frac{\gamma}{\alpha}\right)^{1/p-1}(C_1 h\|u\|^{p-2}\|u\|_2)^{2-p/(p-1)^2}\right\} \times \|u\|^{p-2/p-1}\|u\|_2^{1/p-1} , \tag{19}$$

where C and C_1 are constants independent of h and u.

Proof. Using the triangle inequality,

$$\|u-u_h\| \leqslant \|u-\bar{u}_h\|+\|\bar{u}_h-u_h\| , \qquad \text{for all} \quad \bar{u}_h \in S_h ,$$

and Lemma 3, we obtain

$$\|u-u_h\| \leqslant \left\{1+\left(\frac{\gamma}{\alpha}\right)^{1/p-1}\|u-\bar{u}_h\|^{2-p/p-1}\right\}\|u-\bar{u}_h\| .$$

Now combining the above inequality with (16), we obtain (19), the required result.

Similarly using the arguements of Lemma 3 and Lemma 4, we have the following error estimate for $u-u_h$, when $1<p\leqslant 2$.

THEOREM 2. *If u and u_h are the solutions of* (2) *and* (11) *respectively, then for $1<p\leqslant 2$, we have*

$$\|u-u_h\| \leqslant Ch^{1/3-p}\left\{1+\left(\frac{\gamma}{\alpha}\right)2^{2-p}\|u\|^{2-p}\right\}\|u\|^{2-p/3-p}\|u\|_2^{1/3-p} , \tag{20}$$

where C is a constant independent of h and u.

Remarks. We note that when $f(u)=f$, then the results obtained by us are exactly those of Glowinski and Marroco [4], since in this case the Lipschitz constant γ is zero. This shows that our results include the results obtained by Glowinski and Marroco as a special case.

If $p=2$, then the error estimates (19) and (20) reduce to the well known results, see Noor and Whiteman [7] and Ciarlet, Schultz and Varga [2].

References

[1] Ciarlet, P. G.; *The Finite Method of Eilliptic Problems*, North-Holland, 1978.

[2] Ciarlet, P. G., Schultz, M. and Varga, R. S.; Numerical methods of high-order accuracy for nonlinear boundary value problems, IV, *Num. Math.*, **13** (1969), 51–77.

[3] Dailey, J. W. and Pierce, J. G.; Error bounds for the Galerkin method applied to singular and nonsingular boundary value problems, *Num. Math.*, **19** (1972), 266–282.

[4] Glowinski, R. and Marroco, A.; Sur l'approximation par elements finis d'ordrè une, et la resolution, par penalisation-dualite, d'une classe de problemes de Dirichlet non lineaires, RAIRO, 1975.

[5] Noor, M. Aslam ; variational inequalities and approximation, *Math. Jour. Punjb. Univ.*, **8** (1975), 25–40.

[6] ———; On Variational Inequalities, Ph. D. Thesis, Brunel University, U. K., 1975.

[7] Noor, M. A. and Whiteman, J. R.; Error bounds for finite element solutions of mildly nonlinear elliptic boundary value problens, *Num. Math.*, **26** (1976), 107–116. MR 55 #11649.

[8] Pelissier, M. C.; Sur Quelques Problemes Nonlineairesen Glaciologie, T/R 110–75–24, Universite Paris XI, U.E.R. Math., Orsay, France, 1975. MR 55 #11916.

[9] Sibony, M.; Sur l'approximation d'equations et inequations aux derivees partielles nonlineaires de type monotone, *J. Math. Anal. Appl.*, **36** (1971), 502–564.

[10] Strang, G. and Fix, G.; *An Analysis of the Finite Element Method*, Prentice-Hall, New Jersey, 1973.

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