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Birational Geometry of Birational Pairs

by

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1. Introduction and statements of the results

In this paper, we consider complete varieties over an algebraically closed field k of characteristic zero only, and study reduced effective divisors on them from birational point of view.

Let D be a reduced effective divisor on a complete variety X. In this paper, such a pair of D and X is denoted by the symbol $(D \& X)$, which we call a *birational pair* or simply a *pair*.

Consider another birational pair $(C \& Y)$. Let $D=D_1+\cdots+D_r$ and $C=C_1+\cdots+C_r$ be the irreducible decompositions of D and C, respectively, and

$$f: X \rightarrow Y$$

a birational map. $(D \& X)$ and $(C \& Y)$ are said to be *birationally equivalent* if

$$f|_{D_i}: D_i \rightarrow C_i$$

is also birational map for each i, and we write

$$(D \& X) \sim (C \& Y).$$

We consider such a birational map as one of birational pairs, and we say f is a birational map of pairs. We write

$$f_p(D_i) := C_i, \qquad f^p(C_i) := D_i \qquad \text{for each} \quad i,$$
$$f_p(D) := C, \quad \text{and} \quad f^p(C) := D,$$

and we call $f_p(D)$ and $f^p(C)$ proper transform of D and C, respectively. By the result of Hironaka [2], for a birational pair $(D \& X)$, there exists a birational morphism

$$\mu: V \rightarrow X$$

such that V is a complete nonsingular variety and the $\mu^p(D)$ is a disjoint union of nonsingular prime divisors. We say that the pair $(\mu^p(D) \& V)$ (or $\mu: (\mu^p(D) \& V) \rightarrow (D \& X)$) is a *nonsingular model* of $(D \& X)$. Let $D=D_1+\cdots+D_r$ be the irreducible decomposition of D, and put $Z_i=\mu^p(D_i)$. Now, we fix an r-tuple of rational numbers $a_1, \cdots, a_r$ with $0 \leqq a_i \leqq 1$. For a positive integer m, we define

$$P_m\left(\sum_{i=1}^{r} a_i D_i \ \& \ X\right)$$

to be

$$\dim_k H^0\left(V, \mathcal{O}_V\left(\left[m\left(K(V)+\sum_{i=1}^{r} a_i Z_i\right)\right]\right)\right),$$

where [ ] denotes the integral part of divisor, i.e.,

$$\left[m\left(K(V)+\sum_{i=1}^{r} a_i Z_i\right)\right]=mK(V)+\sum_{i=1}^{r} [ma_i]\, Z_i\,,$$

and $K(V)$ is a canonical divisor of V. Then by a general result of Iitaka [3], there exist $\kappa \geqq 0$ (or $-\infty$) and α, $\beta>0$, $m_1>0$ such that

$$\alpha m^{\kappa} \leqq P_{mm_1}\left(\sum_{i=1}^{r} a_i D_i \ \& \ X\right) \leqq \beta m^{\kappa}$$

for sufficiently large m. We define

$$\kappa\left(\sum_{i=1}^{r} a_i D_i \ \& \ X\right)$$

to be the above κ. In particular, if

$$P_m\left(\sum_{i=1}^{r} a_i D_i \ \& \ X\right)=0$$

for each $m>0$, then

$$\kappa\left(\sum_{i=1}^{r} a_i D_i \ \& \ X\right)=-\infty\,.$$

We shall prove

$$P_m\left(\sum_{i=1}^{r} a_i D_i \ \& \ X\right) \quad \text{and} \quad \kappa\left(\sum_{i=1}^{r} a_i D_i \ \& \ X\right)$$

are independent of the choice of μ. Hence we can consider them as the birational invariants of the pair (D & X).

We have the following theorem.

THEOREM 1.1. *If (D & X) is birationally equivalent to (C & Y), and $\mu: X \to Y$ is a birational map of birational pairs, for any r-tuple of rational numbers $a_1, \cdots, a_r$ with $0 \leqq a_i \leqq 1$, and a natural number m, we have*

$$P_m(\textstyle\sum a_i D_i \ \& \ X)=P_m(\sum a_i C_i \ \& \ Y))\,,$$

$$\kappa(\textstyle\sum a_i D_i \ \& \ X)=\kappa(\sum a_i C_i \ \& \ Y)\,,$$

where $C=C_1+\cdots+C_r$ *is the irreducible decomposition of* C, *and* $D_i=\mu^p(C_i)$.

Hence these invariants are birational invariants for pairs in the above sense, which are however not proper birational invariants for $X\setminus D$. (See Section 6.)

Using these invariants, we can solve the following problem.

Problem. When is the birational pair of the form (a curve & $\boldsymbol{P}^2$) birationally equivalent to the pair of the form (a nonsingular curve & $\boldsymbol{P}^2$)?

Let C be a curve, and g be the geometric genus of it. We define the (first) *virtual degree* of C to be

$$(3+\sqrt{8g+1})/2\,.$$

and denote it by $d_1(C)$ or d_1.

Our solution is,

THEOREM 1.2. *Let* C *be an irreducible curve on* $\boldsymbol{P}^2$. *Then,* $(C\ \&\ \boldsymbol{P}^2)\sim$ *(a nonsingular curve &* $\boldsymbol{P}^2$*) if and only if either*

(i) $\kappa(C\ \&\ \boldsymbol{P}^2)=-\infty$, *or*

(ii) d_1 *is an integer with* $d_1\geqq 3$,

$$\kappa(3/d_1C\ \&\ \boldsymbol{P}^2)=P_{d_1}(2/d_1C\ \&\ \boldsymbol{P}^2)=0\,,$$

and $P_{d_1}(3/d_1C\ \&\ \boldsymbol{P}^2)=1$.

From this result, we have the following corollaries.

COROLLARY 1.3. $(C\ \&\ \boldsymbol{P}^2)\sim$*(a straight line &* $\boldsymbol{P}^2$*) if and only if* $\kappa(C\ \&\ \boldsymbol{P}^2)=-\infty$.

(A similar statement is contained in Coolidge [1, p. 398, Theorem 4]. Recently, this result is also obtained by N. Mohan Kumar and M. Pavaman Murthy independently by another motivation and different method [15].)

COROLLARY 1.4. $(C\ \&\ \boldsymbol{P}^2)\sim$*(a nonsingular cubic curve &* $\boldsymbol{P}^2$*) if and only if* $\kappa(C\ \&\ \boldsymbol{P}^2)=0$ *and* $g(C)=1$, *where* $g(C)$ *denotes the geometric genus of* C.

(A similar statement is contained in the same book p. 408, Theorem 12.) Furthermore, using the results of Kuramoto [8] and Tsunoda [13], we have the following result by Corollary 1.3.

COROLLARY 1.5. *Let* S *be a complete algebraic surface and* D *be an irreducible curve on it. Then,* $P_{12}(D\ \&\ S)=0$ *if and only if either* $(D\ \&\ S)\sim(C\ \&\ B\times\boldsymbol{P}^1)$ *for some nonsingular complete curve* B, *and a section* C *of the first projection of* $B\times\boldsymbol{P}^1$, *or there exists a birational map* $\mu\colon S\to B\times\boldsymbol{P}^1$ *such that* D *is mapped to a point on* $B\times\boldsymbol{P}^1$ *by* μ.

Using Theorem 1.1 and Corollary 1.5, we have another corollary as follows.

COROLLARY 1.6. *Let* S *be a ruled surface. Then, there exists an irreducible curve* C *with* $\kappa(C\ \&\ S)=0$ *if and only if* S *is a rational surface.*

By the proof of Theorem 1.2, we have the following result immediately.

COROLLARY 1.7. *Let C be an irreducible singular plane curve, which has degree d and multiple points P_i (possibly including infinitely near singular points) with multiplicities m_i. If $m_0 \geqq m_1 \geqq \cdots$, and either*

(i) $m_0 + m_1 + m_2 \leqq d$, *or*

(ii) $m_0 + 2m_2 \leqq d$, *and C is not an elliptic curve of degree 4 with two double points, then C cannot be transformed into a nonsingular plane curve by any birational automorphism of $\boldsymbol{P}^2$.*

Furthermore, applying the above Theorem 1.2 to the rational ruled surfaces $\sum_e := \boldsymbol{P}(\mathcal{O}_{\boldsymbol{P}^1} \oplus \mathcal{O}_{\boldsymbol{P}^1}(e))$ with $e \geqq 0$, we have another corollary as follows.

COROLLARY 1.8. *Let $\sum_e$ be Hirzebruch's surface with $e \neq 1$, and let C and C' be irreducible nonsingular curves on $\sum_e$ and $\boldsymbol{P}^2$, respectively. Then, $(C \,\&\, \sum_e) \sim (C' \,\&\, \boldsymbol{P}^2)$ if and only if*

(i) *C is a section of $\sum_e$,*

(ii) *C is a fiber of $\sum_e$, or*

(iii) *C is an elliptic curve, $e \leqq 2$, $p|_C \colon C \to \boldsymbol{P}^1$ is a double covering, and $C \cdot a = 2 - e$, where $p \colon \sum_e \to \boldsymbol{P}^1$ is the canonical projection and a is the canonical section of $\sum_e$ with $a^2 = -e$.*

2. The proof of Theorem 1.1.

Here, in general, we let $h^0(\Delta)$ denote $\dim_k H^0(V, \mathcal{O}_V(\Delta))$ for a divisor Δ on V. First we prove the following fact.

PROPOSITION 2.1. *Let $D = D_1 + \cdots + D_r$ and $C = C_1 + \cdots + C_r$ be disjoint unions of nonsingular prime divisors on complete nonsingular varieties V and W, respectively. Let $f \colon V \to W$ be a birational morphism such that $D_i = f^p(C_i)$ for each i. Then,*

$$h^0([m(K(V) + \sum a_i D_i)]) = h^0([m(K(W) + \sum a_i C_i)])$$

for a positive integer m, and rational numbers $a_1, \cdots, a_r$ with $0 \leqq a_i \leqq 1$.

Proof.

Step 1. First we prove.

$$h^0([m(K(V) + \sum a_i D_i)]) \leqq h^0([m(K(W) + \sum a_i C_i)]) .$$

Since $f^*C_i \geqq f^p(C_i) = D_i$, and $K(V) = f^*K(W) + R_f$ by ramification formula, where R_f is the ramification divisor for f,

$$\begin{aligned}[m(K(V) + \sum a_i D_i)] &= mK(V) + \sum n_i D_i \\ &\leqq mK(V) + \sum n_i f^* C_i \\ &= f^*[m(K(W) + \sum a_i C_i)] + mR_f ,\end{aligned}$$

where $n_i=[ma_i]$. Since R_f is an effective divisor and codim $(f(\text{supp }(R_f)))\geqq 2$, we have

$$h^0(f^*[m(K(W)+\sum a_iC_i)]+mR_f)=h^0([m(K(W)+\sum a_iC_i)])\ ,$$

and

$$h^0([m(K(V)+\sum a_iD_i)])\leqq h^0(f^*[m(K(W)+\sum a_iC_i)]+mR_f)$$
$$=h^0([m(K(W)+\sum a_iC_i)])\ .$$

Step 2. If f is a blowing up with nonsingular center Z of codimension r, then $R_f=(r-1)E$, where $E=f^{-1}(Z)$, and

$$f^*C_i=\begin{cases} D_i+E & (\text{if } \ Z\subseteqq C_i) \\ D_i & (\text{if } \ Z\nsubseteqq C_i)\ . \end{cases}$$

Since $0\leqq a_i\leqq 1$, $m\geqq n_i$ and $r\geqq 2$,

$$[m(K(V)+\sum a_iD_i)]-f^*[m(K(W)+\sum a_iC_i)]=mR_f+\sum n_i(D_i-f^*C_i)$$
$$\geqq m(r-1)E-n_iE\geqq m(r-2)E\ .$$

Hence,

$$h^0([m(K(V)+\sum a_iD_i)])\geqq h^0([m(K(W)+\sum a_iC_i)])\ .$$

Consequently, we have by Step 1,

$$h^0([m(K(V)+\sum a_iD_i)])=h^0([m(K(W)+\sum a_iC_i)])\ .$$

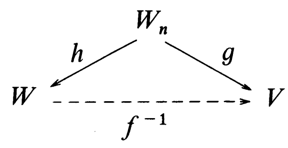
Step 3. In general case, we have a finite number of blowings up

$$h_j\colon W_j\to W_{j-1}$$

with nonsingular centers for $1\leqq j\leqq n$, with $W_0=W$, and a birational morphism

$$g\colon W_n\to V$$

such that $h=f\circ g$ by Hironaka [2], where $h=h_1\circ\cdots\circ h_n$. W_n is said to be nonsingular reduction model of f^{-1} (see [3]).



Therefore, letting $B_i=g^p(D_i)$, and $B=B_1+\cdots+B_r$, we have by Step 1,

$$h^0([m(K(W_n)+\sum a_iB_i)])\leqq h^0([m(K(V)+\sum a_iD_i)])\ ,\quad\text{and}$$
$$h^0([m(K(V)+\sum a_iD_i)])\leqq h^0([m(K(W)+\sum a_iC_i)])\ .$$

Furtheremore, we have by Step 2,

$$h^0([m(K(W)+\sum a_iC_i)])=h^0([m(K(W_n)+\sum a_iB_i)]) .$$

Therefore

$$h^0([m(K(W)+\sum a_iC_i)])=h^0([m(K(V)+\sum a_iD_i)]) . \quad \blacksquare$$

Proof of Theorem 1.1. Take two nonsingular models

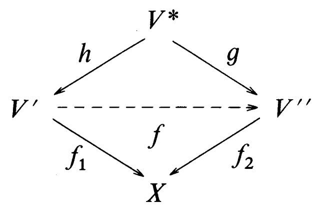
$$f_1: (D' \ \& \ V')\to(D \ \& \ X)$$

$$f_2: (D'' \ \& \ V'')\to(D \ \& \ X) .$$

We choose a nonsingular reduction model of $f=f_2^{-1}\circ f_1$

$$g: (D^* \ \& \ V^*)\to(D'' \ \& \ V'')$$

such that $h=f^{-1}\circ g$ is a birational morphism.



Then, $D'=f_1^p(D)$, $D''=f_2^p(D)$, $D^*=h^p(D')$ by definition of nonsingular model. Applying Proposition 2.1 to h and g, we have

$$h^0([m(K(V')+\sum a_iD_i')])=h^0([m(K(V^*)+\sum a_iD_i^*)])$$
$$=h^0([m(K(V'')+\sum a_iD_i'')]) .$$

Thus, $P_m(\sum a_iD_i \ \& \ X)$ is independent of the choice of f_1 and f_2, and is consequently a well defined birational invariant. Theorem 1.1 follows immediately from this fact. $\blacksquare$

3. The simplification algorithm for plane curves

To prove Theorem 1.2, we need the technique used in Shafarevich [10, pp. 100–105].

Consider a plane curve C of degree d which has singular points $P_0, \cdots, P_k$ with multiplicities $m_0, \cdots, m_k$, respectively. Here, singular points mean also infinitely near singular points. We assume $m_0 \geqq m_1 \geqq m_2 \geqq \cdots \geqq m_k$.

Let $j=(d-m_0)/2$, and define the integer h by the condition

$$m_h > j \geqq m_{h+1} .$$

We say that (j, h) is the parameters of C. For another curve C' with the parameters (j', h'), we say that C' is *simpler* than C if

$$(j', h') < (j, h)$$

with respect to the lexicographical order.

We prove the following proposition.

PROPOSITION 3.1. *If a plane curve C satisfies the following conditions*:
(a) $h \geqq 2$, *and*

(b) $d - 3j \leqq \sum_{i=1}^{h} (m_i - j)$,

then there exists a Cremona transformation f such that $f_p(C)$ *is simpler than* C.

Sketch of the proof.
Step 1. If P_0, P and Q are all distinct points of $\boldsymbol{P}^2$ such that P_0, P, and Q are not collinear and if $C' = c_p(C)$, where $c = c(P_0, P, Q)$, i.e., c is the standard Cremona transformation with fundamental points P_0, P and Q, then $j \geqq j'$. Furthermore, $j = j'$ if and only if the point corresponding P_0 with respect to c has highest multiplicity. For a proof, see Shafarevich [10, p. 103].

Step 2. Assume that there exist two distinct points P_r and P_s, with $1 \leqq r, s \leqq h$, lying on the plane $\boldsymbol{P}^2$. Then by the definition of h, P_0, P_r and P_s are not collinear, and $c_p(C)$ is simpler than C, where $c = c(P_0, P_r, P_s)$. Saying more precisely, if $j = j'$, then $h' = h - 2$. For a proof, see Shafarevich [10, p. 103, Lemma 17].

Step 3. Assume that it is impossible to find any points P_r and P_s satisfying the condition of Step 2. We choose two points A and B on $\boldsymbol{P}^2$ such that none of the points P_i for $1 \leqq i \leqq k$, lie on the lines P_0A and P_0B, and the direction of P_0A and P_0B do not correspond to any of the points lying over P_0. Set $c = c(P_0, A, B)$ and $C' = c_p(C)$. Then,

a) we have $j = j'$ and $h' = h + 2$, and

b) no other singular points of C' lies over the point of highest multiplicity of C'.

Remark. Furthermore, (b) is preserved by this transformation. For a proof, see Shafarevich [10, p. 104, Lemma 18].

Step 4. Assume that none of the singular points of C lies over P_0, but for some $r, s \leqq h$ the point P_r lies over P_s, of order one and P_s lies on $\boldsymbol{P}^2$. We choose a point R on the plane such that no singular point lies on the lines P_0R and P_sR, and such that the direction of P_sR does not correspond to the point P_r. Set $c = c(P_0, P_s, R)$. Then,

a) no singular point of C of multiplicities greater than j lies on the line P_0P_s, and the direction of this line does not correspond to any point P_r, and

b) either $j' < j$, or $j' = j$, $h' = h$ and the number of singular points of $c_p(C)$ lying on the plane and having multiplicity $> j$ is greater than the analogus number for C.

Remark. If $j = j'$, then (b) is preserved by this transformation. For a proof, see

Shafarevich [10, p. 104, Lemma 19].

Step 5. We shall now describe the simplification algorithm for C. If C satisfies the condition of Step 2, then it can be simplified by the transformation described in Step 2. Considering the Step 1, we may assume that j is preserved by every transformation of each steps. If C does not satisfy the condition of Step 2, we apply to C the transformation of Step 3, and applying Step 4 successively, we arrive at a curve with parameters $(j, h+2)$ for which all the singular points $Q_1, \cdots, Q_{h+2}$ of multiplicities $>j$ lie on a plane. Since $h+2\geqq 4$, by the inequality (b), we have

$$d<\sum_{i=1}^{h+2} m_i .$$

Hence not all of the points $Q_1, \cdots, Q_{h+2}$ lie on the same line. Similarly by inequalities

$$m_0+m_r+m_s\geqq m_0+2m_s>d ,$$

it does not happen that all of the points Q_0, Q_r, Q_s with $r, s\leqq h+2$ lie on the same line. Then it is possible to find points Q_r, Q_s with $r, s\leqq h+2$, such that there exist two more points Q_u, Q_v with $u, v\leqq h+2$, not lying on any of the lines Q_0Q_r, $Q_r Q_s$, Q_0Q_s. (If all but one, say Q_t, of the points Q lie on the line Q_rQ_s, then we choose the pair (Q_s, Q_t) instead of the pair (Q_r, Q_s). Since $h\geqq 2$, the points $Q_0, Q_r, Q_s, Q_t, Q_u, \cdots$ must lie on the plane, that is, we have at least five points on $\boldsymbol{P}^2$.) After this we apply successively $c(Q_0, Q_r, Q_s)$ and $c(Q_0, Q_u, Q_v)$. By Step 2 we arrive at a curve with parameters $(j, h-2)$. Thus we obtain the desired simplification. ■

4. The proof of Theorem 1.2.

We use the following lemmas.

LEMMA 4.1. *Assume that* $P_0, P_1, \cdots, P_h$ *are on* $\boldsymbol{P}^2$, *and* $m_1, \cdots, m_h$ *and* m *are non-negative integers, and* $E_0, E_1, \cdots, E_h$ *are the exceptional curves of the blowing up with center* $\{P_0, P_1, \cdots, P_h\}$. *If*

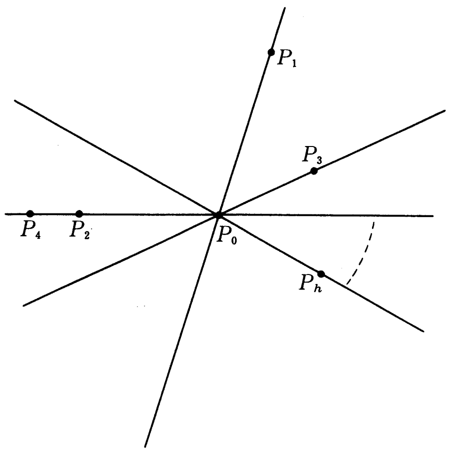
$$m\geqq\sum_{i=1}^{h} m_i ,$$

then

$$\left|m(L-E_0)-\sum_{i=1}^{h} m_iE_i\right|\neq\varnothing ,$$

where L *denotes a general line on* $\boldsymbol{P}^2$.

The proof is easy.



LEMMA 4.2. *If* $D=D_1+\cdots+D_r$ *is an irreducible decomposition of reduced effective divisor of* $\boldsymbol{P}^n$, *then*

$$P_1(D \,\&\, \boldsymbol{P}^n)=\sum_{i=1}^{r} g(D_i)\,,$$

where $g(D_i)$ *denotes the geometric genus of* D_i.

Proof. Take a nonsingular model

$$f:(W \,\&\, V)\to(D \,\&\, \boldsymbol{P}^n)\,,$$

and put $f^p(D_i)=W_i$, and consider the following short exact sequence.

$$0\to\mathcal{O}_V(K(V))\to\mathcal{O}_V(K(V)+W)\to\mathcal{O}_W(K(W))\to 0\,.$$

Since

$$\dim_k H^0(V,\,\mathcal{O}_V(K(V)))=0\,,\quad \text{and}$$
$$\dim_k H^1(V,\,\mathcal{O}_V(K(V)))=\dim_k H^{n-1}(V,\,\mathcal{O}_V)=\dim_k H^{n-1}(\boldsymbol{P}^n,\,\mathcal{O}_{\boldsymbol{P}^n})=0$$

by Serre duality and birational invariance of $\dim_k H^i(V,\,\mathcal{O}_V)$, we have the desired equality by the long exact sequence of the above short exact sequence. ∎

Assume $(C \,\&\, \boldsymbol{P}^2)\sim$(a nonsingular curve & $\boldsymbol{P}^2$). If $g=0$, then $(C \,\&\, \boldsymbol{P}^2)$ satisfies (i) of Theorem 1.2, and if $g\geqq 1$, then it satisfies (ii) of the same theorem by the birational invariance of $\kappa(\sum a_iD_i \,\&\, \boldsymbol{P}^2)$ in Theorem 1.1, where g denotes the geometric genus of C.

Let's prove the converse.

Step 1. If $(C \,\&\, \boldsymbol{P}^2)$ satisfies (ii) of Theorem 1.2, then for any rational number ε with $3/d_1>\varepsilon\geqq 0$, it holds that

$$\kappa(\varepsilon C \,\&\, \boldsymbol{P}^2)=-\infty\,.$$

Indeed, let μ: $(D \,\&\, S)\to(C \,\&\, \boldsymbol{P}^2)$ be a nonsingular model of $(C \,\&\, \boldsymbol{P}^2)$, and $F=d_1K(S)+3D$. Then there exists an effective divisor $A\in|F|$ by the hypothesis. Suppose there exists an effective divisor $E\in|mF-D|$ for some $m>0$. Since $mF\sim E+D\sim mA$ and dim $|mF|=0$, we have

$$E+D=mA\ , \quad \text{and} \quad E=mA-D\geqq 0\ .$$

Therefore we have $A\supset D$, because D is irreducible. Hence,

$$d_1K(S)+2D\sim A-D\geqq 0\ .$$

We have $P_{d_1}(2/d_1C \,\&\, \boldsymbol{P}^2)>0$, this is a contradiction. Thus we have for any $m>0$

$$|mF-D|=|md_1K(S)+(3m-1)D|=\varnothing\ .$$

Hence, for any rational number ε with $3/d_1>\varepsilon\geqq 0$, we have for sufficient large m

$$K(S)+\varepsilon D<K(S)+(3m-1)/md_1D\ .$$

Therefore,

$$\kappa(\varepsilon C \,\&\, \boldsymbol{P}^2)\leqq\kappa((3m-1)/md_1C \,\&\, \boldsymbol{P}^2)=-\infty\ .$$

Hereafter, we consider the simplest counter example $(C \,\&\, \boldsymbol{P}^2)$ with respect to the simplification algorithm, that is, we assume C is a singular plane curve with minimum parameters (j, h) such that

$$h<2\ , \tag{1}$$

or

$$d-3j>\sum_{i=1}^{h}(m_i-j)\ . \tag{2}$$

Let μ: $(D \,\&\, S)\to(C \,\&\, \boldsymbol{P}^2)$ be a nonsingular model of $(C \,\&\, \boldsymbol{P}^2)$ such that μ^{-1} is the composition of successive blowings up with center P_i.

Step 2. If $h\geqq 2$, then we have

$$d-3j\leqq\sum_{i=1}^{h}(m_i-j)$$

which is a contradiction.

Indeed, if $h\geqq 2$, we may apply Steps 3 and 4 of Proposition 3.1 to this pair successively. Therefore we can assume that $P_0, P_1, \cdots, P_h$ are on $\boldsymbol{P}^2$, $(C \,\&\, \boldsymbol{P}^2)$ is smallest with respect to j, and $(C \,\&\, \boldsymbol{P}^2)$ satisfies (2).

Case (i). Assume $3/d_1>2/(d-m_0)$.

If $(C \,\&\, \boldsymbol{P}^2)$ satisfies (i) Theorem 1.2, C is a rational curve by Lemma 4.2. Furthermore, if C has only one singular point, i.e., $k=0$, then it is easy to check that C can be transformed into a straight line by a Cremona transformation. Therefore we may assume $k\geqq 1$, i.e., $d-m_0\geqq 2$. Hence, $(C \,\&\, \boldsymbol{P}^2)$ satisfies the condition $3/d_1>2/(d-m_0)$.

Let $R_\mu = E_0 + \cdots + E_k$ be the ramification divisor of a nonsingular model μ. Then,

$$\begin{aligned}(d-m_0)K(S)+2D &= -3(d-m_0)L+(d-m_0)R+2dL-\sum 2m_iE_i \\ &= (3m_0-d)L-\sum(2m_i+m_0-d)E_i \\ &\geqq (3m_0-d)(L-E_0)-\sum_{i=1}^{h}(2m_i+m_0-d)E_i\,,\end{aligned}$$

hence by Lemma 4.1 and (2),

$$\dim|(d-m_0)K(S)+2D| = \dim\left|(3m_0-d)(L-E_0)-\sum_{i=1}^{h}(2m_i+m_0-d)E_i\right|>0\,.$$

Therefore we have

$$P_{d-m_0}(2/(d-m_0)C\ \&\ \boldsymbol{P}^2)>0\,.$$

This contradicts Step 1, or (i) of Theorem 1.2.

Case (ii). Assume $3/d_1 \leqq 2/(d-m_0)$.

Since $\dim|d_1K(S)+3D|=0$, and

$$d_1K(S)+3D=3(d-d_1)L-\sum_{i=0}^{h}(3m_i-d_1)E_i\,,$$

we have

$$3(d-d_1)\geqq 3m_0-d_1\,.$$

Thus

$$3/d_1 \geqq 2/(d-m_0)\,.$$

Therefore

$$3/d_1 = 2/(d-m_0)\,,\quad \text{and we have}\quad 2d_1=3(d-m_0)\,.$$

Hence,

$$\begin{aligned}2(d_1K(S)+3D) &= 2d_1K(S)+6D = 3(d-m_0)K(S)+6D \\ &= 3((d-m_0)K(S)+2D) \\ &= 3((3m_0-d)(L-E_0)-\sum_{i=1}^{h}(2m_i+m_0-d)E_i)\,.\end{aligned}$$

Therefore, we have by Lemma 4.1 and (2)

$$\dim|2(d_1K(S)+3D)|\geqq 3\,.$$

On the other hand, by the assumption $P_{d_1}(3/d_1\ C\ \&\ \boldsymbol{P}^2)=1$ and $\kappa(3/d_1\ C\ \&\ \boldsymbol{P}^2)=0$, we have

$$\dim|2(d_1K(S)+3D)|=0,$$

which is a contradiction.

Therefore we may assume

$$h<2.$$

If $(C \,\&\, \boldsymbol{P}^2)$ satisfies (i) of Theorem 1.2, then

$$\begin{aligned} m_2K(S)+D &= (d-3m_2)L+\sum_{i\geqq 3}(m_2-m_i)E_i-(m_0-m_2)E_0-(m_1-m_2)E_1 \\ &\geqq (d-3m_2)L-(m_0-m_2)E_0-(m_1-m_2)E_1 \\ &= (d-m_0-2m_2)L+(m_0-m_2)L'\geqq 0, \end{aligned}$$

where $L'=L-E_0-E_1$. Hence we have

$$P_{m_2}(C \,\&\, \boldsymbol{P}_2)>0.$$

This inequality contradicts the assumption $\kappa(C \,\&\, \boldsymbol{P}^2)=-\infty$. Therefore, we have done in the case of (i) of Theorem 1.2.

Step 3.

Case (i). $h=-1$, i.e., $3m_0\leqq d$;

In this case, $\kappa(1/m_0C \,\&\, \boldsymbol{P}^2)\geqq 0$, because

$$\begin{aligned} m_0K(S)+D &= -3m_0L+m_0R_\mu+dL-\sum m_iE_i \\ &= (d-3m_0)L-\sum(m_i-m_0)E_i \\ &\geqq (d-3m_0)L\geqq 0. \end{aligned}$$

Therefore we have $1/m_0\geqq 3/d_1$ by Step 1. Hence $d_1\geqq 3m_0$. Therefore,

$$d_1K(S)+3D=3(d-d_1)L-\sum(3m_i-d_1)E_i\geqq 3(d-d_1)L.$$

We have

$$\kappa(3/d_1C \,\&\, \boldsymbol{P}^2)=2.$$

This contradicts the hypothesis.

Case (ii). $h=0$, i.e., $3m_0>d\geqq 2m_1+m_0$;

In this case, we have $\kappa(1/m_1C \,\&\, \boldsymbol{P}^2)\geqq 0$. Indeed,

$$\begin{aligned} m_1K(S)+D &= (d-3m_1)L-\sum(m_i-m_1)E_i \\ &\geqq (d-3m_1)L-(m_0-m_1)E_0 \\ &= (m_0-m_1)(L-E_0)+(d-m_0-2m_1)L>0. \end{aligned}$$

Therefore, by Step 1, we have

$$1/m_1\geqq 3/d_1, \text{ i.e., } d_1\geqq 3m_1.$$

Therefore,

$$d_1K(S)+3D\geqq 3(d-d_1)L-(3m_0-d_1)E_0$$
$$=\alpha L+(3m_0-d_1)(L-E_0)\,,$$

where $\alpha=3d-2d_1-3m_0$. If $\alpha\geqq 0$, then $\kappa(3/d_1C\ \&\ \boldsymbol{P}^2)\geqq 1$, which is a contradiction. If $\alpha<0$, then there exists an effective divisor E such that

$$(3m_0-d_1)(L-E_0)\sim E-\alpha L\,.$$

Since $3m_0-d_1>0$, we have

$$1=\kappa((3m_0-d_1)(L-E_0))=2\,.$$

This is a contradiction.

Case (iii). $h=1$, i.e., $2m_1+m_0>d\geqq 2m_2+m_0$.

Since

$$m_2K(S)+D\geqq(d-3m_2)L-(m_0-m_2)E_0-(m_1-m_2)E_1\,,$$

and

$$(d-3m_2)-(m_0-m_2)=d-m_0-2m_2\geqq 0\,,$$

we have by Step 1,

$$\kappa(1/m_2C\ \&\ \boldsymbol{P}^2)\geqq 0\,,\quad\text{and}\quad 1/m_2\geqq 3/d_1\,.$$

Therefore,

$$d_1K(S)+3D\geqq 3(d-d_1)L-(3m_0-d_1)E_0-(3m_1-d_1)E_1\,.$$

Since $P_{d_1}(3/d_1C\ \&\ \boldsymbol{P}^2)=1$, we have

$$3(d-d_1)=3m_0-d_1=3m_1-d_1\,.$$

Therefore, we have

$$m_0=m_1\,, \tag{3}$$

and

$$3(d-m_0)=2d_1 \tag{4}$$

Therefore, there exists an integer η such that

$$d_1=3\eta\,, \tag{5}$$

and

$$d-m_0=2\eta\,. \tag{6}$$

If $\eta=1$, then $2=(d-1)(d-2)-2m_0(m_0-1)-2\varepsilon$ by the genus formula of Clebsch, where $\varepsilon=\sum_{i=2}^{k}m_i(m_i-1)/2$.

Therefore, we have by (6)

$$(m_0-1)(m_0-2)+2\varepsilon=0\,.$$

Hence

$$\varepsilon=0\,,\quad \text{and}\quad m_0=2\,.$$

Again, we have $d(d-3)=4$ by the genus formula; thus

$$d=4\,. \tag{7}$$

If $P_1 \notin \boldsymbol{P}^2$, we choose two general points A and B on C, and applying $c(P_0, A, B)$ to $(C\ \&\ \boldsymbol{P}^2)$, we may assume $P_1 \in \boldsymbol{P}^2$. Then $c(P_0, P_1, A)_p C$ is nonsingular, where A is a general point of C. This is a contradiction. Therefore $\eta \geqq 2$. By the genus formula,

$$\begin{aligned} g &= (d_1-1)(d_1-2)/2 \\ &= (d-1)(d-2)/2 - m_0(m_0-1) - \varepsilon\,. \end{aligned}$$

By (5) and (6), and since $\varepsilon \geqq 0$, we have

$$\begin{aligned} 0 &\geqq m_0^2 - (4\eta-1)m_0 + \eta(5\eta-3) \\ &= (m_0-(2\eta-1/2))^2 + \eta(\eta-1) - 1/4 > 0\,. \end{aligned}$$

This is a contradiction. ■

5. The proof of corollaries

Corollaries 1.3 and 1.4 follow immediately from Theorem 1.2 and Lemma 4.2.

Proof of Corollary 1.5. Referring to the Enriques's criterion for ruled surfaces and Kuramoto [8] and Tsunoda [13], it suffices to check the next Proposition 5.2.

THEOREM 5.1 (Kuramoto-Tsunoda). *Let S be a normal complete algebraic surface, and D be a reduced effective and connected divisor on S. If $\bar{P}_{12}(S \setminus D)=0$, then $\bar{\kappa}(S \setminus D) = -\infty$.*

For a proof, see [8] and [13].

PROPOSITION 5.2. *Let S be a complete algebraic surface over k, and D be an irreducible curve on S. If $\kappa(D\ \&\ S) = -\infty$, then there exist a nonsingular complete curve B and a birational map $\mu\colon S \dashrightarrow B \times \boldsymbol{P}^1$ such that either there exists a section C of the first projection of $B \times \boldsymbol{P}^1$ such that $(D\ \&\ S) \sim (C\ \&\ B \times \boldsymbol{P}^1)$ or D is mapped to a point on $B \times \boldsymbol{P}^1$ by μ.*

*Proof.**) Considering Corollary 1.3, if we cannot contract D to a point on $B \times \boldsymbol{P}^1$ by any birational map $\mu\colon S \dashrightarrow B \times \boldsymbol{P}^1$, then we may assume S is an

*) This proof is due to Professor S. Iitaka. The original one of authors is longer and more complicated.

irrational ruled surface and there exists an irreducible curve C on $B\times \boldsymbol{P}^1$ with $(D\ \&\ S)\sim(C\ \&\ B\times \boldsymbol{P}^1)$. Taking a nonsingular model $(C^\# \ \&\ S^\#)\to(C\ \&\ B\times \boldsymbol{P}^1)$ and considering a fiber space $g: S^\#\to B\times \boldsymbol{P}^1\to B$, we have

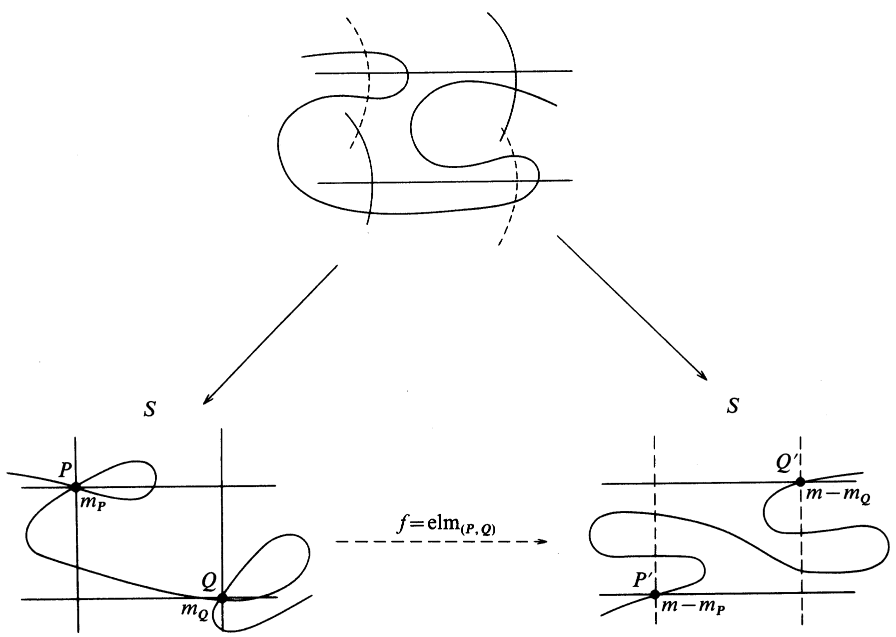
$$\begin{aligned}\kappa(D\ \&\ S)&=\kappa(D^\#\ \&\ S^\#)\\&=\bar{\kappa}(S^\#\setminus D^\#)\\&\geqq\bar{\kappa}(S^\#_x\setminus D^\#_x)+\kappa(B)\end{aligned}$$

by the addition formula of Kawamata [7], where $S^\#_x$ is a general fiber of g and $D^\#_x=D^\#|_{S_x^\#}$. Since $S^\#_x=\boldsymbol{P}^1$, $\kappa(B)\geqq 0$ and $\kappa(D\ \&\ S)=-\infty$, we have $\bar{\kappa}(S^\#_x\setminus D^\#_x)=-\infty$ and so $D^\#_x$ is a point. Therefore C is a section of the first projection of $B\times \boldsymbol{P}^1$. ∎

Proof of Corollary 1.6. Considering Theorem 1.1 and Corollary 1.5, we may assume $S=B\times \boldsymbol{P}^1$ and $g\geqq 1$ and $\kappa(C\ \&\ S)=0$, where $g=g(B)$.

Step 1. Let P and Q be two points on S which are neither on the same section nor on the same fiber. Then the elementary transformation $f=\mathrm{elm}_{(P,Q)}$ is a birational automorphism of S, and if C has multiplicities m_P and m_Q at P and Q, respectively, and $C\cdot b=m$, where b is a fiber of $B\times \boldsymbol{P}^1\to B$, and let $C'=f_p(C)$, then C' has multiplicities $m-m_P$ and $m-m_Q$ at corresponding points P' and Q', respectively.

The proof is easy.



Step 2. Considering a nonsingular model

$$f:(C^* \ \& \ S^*)\to(C \ \& \ B\times \boldsymbol{P}^1)\,,$$

and a fiber space

$$g: S^* \xrightarrow{\ f\ } B\times \boldsymbol{P}^1 \xrightarrow{\ p\ } B\,,$$

where $g=p\circ f$, we have

$$\bar{\kappa}(S^*_x\setminus C^*_x)+1\geqq 0=\kappa(C \ \& \ S)$$
$$=\bar{\kappa}(S^*\setminus C^*)\geqq\bar{\kappa}(S^*_x\setminus C^*_x)+\kappa(B)$$

by the easy addition theorem [3, p. 338] and the addition formula as in the proof of Proposition 5.2, we have

$$\bar{\kappa}(S^*_x\setminus C^*_x)=\kappa(B)=0\,.$$

Therefore $C\cdot b=2$ and $g=1$, where b is a fiber of p.

Step 3. If $C\cdot b=2$, then $\kappa(C \ \& \ S)=1$.

Let $C\approx 2a+nb$, where a is a section over B, and $\approx$ means numerical equivalence. Since C is irreducible and $C\cdot a=n$, we have $n>0$. Applying Step 1 to C successively, we may assume C has only ordinary double points, say $P_1\cdots$, P_e, on a section. Therefore we have

$$n\geqq 2e\,.$$

Let $S'=\boldsymbol{P}_B(\mathcal{O}_B\oplus\mathcal{O}_B(e))$, $\eta=\mathrm{elm}_{(P_1,\dots,P_e)}$, and $\eta_p(C)=C_e$. Then C_e is nonsingular, and

$$K(S')+C_e\approx(2g-2-e+n)b=(n-e)b>0\,.$$

Hence, there exists F on B such that $\deg(F)>0$ and

$$K(S')+C_e=p^*F\,,$$

where $p\colon B\times\boldsymbol{P}^1\to B$ is the first projection. So,

$$h^0(K(S')+C_e)=h^0(p^*F)\geqq h^0(F)=\deg(F)>0$$

by the Riemann-Roch Theorem for curves. Thus we have an effective divisor $E\in|K(S')+C_e|$ with $E\neq 0$ and $E^2=0$. Therefore we have $\kappa(D \ \& \ S)=1$ by a result of Kawamata [5]. This is a contradiction. The converse is obvious. ■

Proof of Corollary 1.7. At first, we state the following Lemma.

LEMMA 5.3. *Let C be an irreducible plane curve which has degree d and multiple ponits* P_0, P_1, P_2, $\cdots$ *possibly including infinitely near singular points with multiplicities* m_i. *If* $m_0\geqq m_1\geqq m_2\geqq\cdots$, *and* $m_0+2m_2\leqq d$, *then*

$$P_{m_2}(1/m_2C \ \& \ \boldsymbol{P}^2)>0\,.$$

Proof. Considering the nonsingular model

$$\mu\colon (D\ \&\ S)\to(C\ \&\ \boldsymbol{P}^2)\,,$$

we have

$$\begin{aligned} m_2K(S)+D &= -3m_2L+m_2R_\mu+dL-\sum m_iE_i \\ &\geqq (d-3m_2)L-(m_0-m_2)E_0-(m_1-m_2)E_1 \\ &= (d-m_0-2m_2)L+(m_0-m_2)(L-E_0)-(m_1-m_2)E_1 \\ &\geqq (d-m_0-2m_2)L+(m_1-m_2)(L-E_0-E_1)\geqq 0\,. \end{aligned}$$

■

Considering a birational pair $(C\ \&\ \boldsymbol{P}^2)$ as in the statement of Corollary 1.7, and suppose this pair satisfies the condition (i) or (ii) of Theorem 1.2. If (i) is true, then

$$\kappa(C\ \&\ \boldsymbol{P}^2)\geqq\kappa(1/m_2C\ \&\ \boldsymbol{P}^2)\geqq 0$$

by the above Lemma 5.3, a contradiction. Therefore this pair satisfies (ii) of Theorem 1.2. If $h\geqq 2$, then $2m_2>d-m_0$; thus

$$d\geqq m_0+m_1+m_2\geqq m_0+2m_2>d\,,$$

This is a contradiction. Therefore, we have $h<2$. But this also contradicts Step 3 of the proof of Theorem 1.2. Moreover, (ii) of Corollary 1.7 corresponds to (7) in Step 3 of the same proof. ■

Proof of Corollary 1.8. Assume $(C\ \&\ \sum_e)$ satisfies the conditions (i), (ii) or (iii). It is easy to check that $(C\ \&\ \sum_e)\sim$(a nonsingular curve & $\boldsymbol{P}^2$) if $(C\ \&\ \sum_e)$ satisfies (i) or (ii) of Corollary 1.8.

In the case (iii), $e=0$ or 2. We choose a general point P on C. Let $C^*=(\mathrm{elm}_P)_p(C)$, and a_1 be the canonical section of $\sum_1$, where elm_P: $\sum_e\to\sum_1$ is an elementary transformation. Then, the pair $(C^*\ \&\ \sum_1)$ is nonsingular and $C^*\cdot a_1=1$. Therefore, contracting a_1, we have the required nonsingular birational pair $(C'\ \&\ \boldsymbol{P}^2)$.

Conversely, we assume $(C\ \&\ \sum_e)\sim(C'\ \&\ \boldsymbol{P}^2)$ and both C and C' are nonsingular.

We continue the proof by examining the following two cases.

(I) $e=0$.

(II) $e\geqq 2$.

Case (I). Assume $C\approx ma+nb$, where $\approx$ means numerical equivalence, and m, $n\geqq 0$, and a and b denote a section and a fiber of $\sum_0$, respectively. Taking a general point P on $\sum_0$, let $(\mathrm{elm}_P)_p(C)=C^*$, and a_1 be the canonical section of $\sum_1$. Since C^* has an m-ple point and $C^*\cdot a_1=n$, contracting a_1, we have $(C'\ \&\ \boldsymbol{P}^2)$, where C' has two distinct multiple points with multiplicities m and n, respectively. Assume $m\geqq n$, and that C' has degree d, and consider a nonsingular model

$$\mu: (D \& S) \to (C' \& \boldsymbol{P}^2) .$$

Since

$$K(S)+D=(d-3)L-(m-1)E_0-(n-1)E_1 ,$$

if $\kappa(C' \& \boldsymbol{P}^2)=-\infty$, we have by (i) of Theorem 1.2,

$$(d-3)<(m-1) .$$

Hence, $d \leqq m+1$, we obtain

$$d=m+1 \quad \text{and} \quad n=1 .$$

Therefore C is a section of the second projection of $\sum_0=\boldsymbol{P}^1 \times \boldsymbol{P}^1$.

When $\kappa(C \& \sum_e) \geqq 0$, let d_1 be the virtual degree of C. Then $d_1 K(S)+3D= 3(d-d_1)L-(3m-d_1)E_0-(3n-d_1)E_1$, and

$$\dim |d_1 K(S)+3D|=0$$

by (ii) of the Theorem 1.2. Therefore, we have

$$3(d-d_1)=3m-d_1=3n-d_1 .$$

Thus we have

$$m=n , \quad \text{and} \quad 3(d-m)=2d_1 .$$

Particularly, there exists an integer η such that

$$d_1=3\eta , \quad \text{and} \quad d-m=2\eta .$$

If $\eta=1$, then C is an elliptic curve and $d=m+2$. Hence we have by the genus formula of Clebsch

$$m=n=2 .$$

This corresponds to (iii) of Corollary 1.8.

If $\eta \geqq 2$, we can derive a contradiction as in the case (iii) of Step 3 of the proof of Theorem 1.2.

Case (II). Assume $C \approx ma+nb$, where m and $n \geqq 0$, and a is the canonical section of $\sum_e$ with $a^2=-e$, and b is a fiber of $\sum_e$. Then, $K(\sum_e) \approx -2a-(e+2)b$. If $\kappa(C \& \sum_e)=-\infty$, then we have

$$m<2 , \quad \text{or} \quad n<e+2 ,$$

because

$$K(\textstyle\sum_e)+C \approx (m-2)a+(n-e-2)b .$$

If $m \geqq 2$, then $n<e+2$. Therefore, by the adjunction formula of C,

$$\begin{aligned}-2&=(K(\textstyle\sum_e)+C)\cdot C\\&=-em(m-2)+m(n-e-2)-m(e+2)\\&<-em(m-2)+2(e+2)(m-1)-m(e+2)\\&=(m-2)(-e(m-1)+2)\,.\end{aligned}$$

Hence we have

$$2>(m-2)(e(m-1)-2)\geqq 0\,.$$

If $m-2=e(m-1)-2=1$, then $m=3$ and $2e=3$, a contradiction. Therefore, $m-2=0$ or $e(m-1)-2=0$, i.e., $e=2/(m-1)=2$. Hence we have $m=2$, and $C\sim 2a+(e+1)b$ by the adjunction formula. Therefore $0\leqq C\cdot a=1-e\leqq -1$, a contradiction. If $m=1$, then C is a section of $\sum_e$, and if $m=0$, then C is a fiber of $\sum_e$.

Hereafter, we assume $\kappa(C\ \&\ \sum_e)\geqq 0$. In this case, C has the virtual degree d_1 with $d_1\geqq 3$. Let $\Gamma=d_1K(\sum_e)+3C\sim(3m-2d_1)a+(3n-(e+2)d_1)b=\alpha a+\beta b$, where

$$\alpha=3m-2d_1\,,\quad\text{and}\quad\beta=3n-(e+2)d_1\,.$$

Recalling $\kappa(3/d_1C\ \&\ \sum_e)=0$ and the Zariski decomposition of divisors (see Kawamata [6] or Miyanishi [9, p. 129, Chapter II]), we divide the proof into the following two cases.

$$(\mathrm{II}_-)\quad \Gamma\cdot a<0\,,$$

and

$$(\mathrm{II}_+)\quad \Gamma\cdot a\geqq 0\,.$$

Case (II_-). Let $\Gamma'=\Gamma+(\Gamma\cdot a/e)a=(\beta/e)a+\beta b$.

Then, $\Gamma'^2=\beta^2/e\geqq 0$. Since the intersection matrix of Γ' is negative definite, we have $\beta=0$, i.e., $3n=(e+2)d_1$, and $0<-\Gamma\cdot a=\alpha e=(3m-2d_1)e$, i.e.,

$$3m>2d_1\,.$$

On the other hand, $0\leqq C\cdot a=-em+n$, i.e.,

$$n\geqq em\,.$$

Therefore, $(e+2)d_1=3n\geqq 3em>2ed_1$, we have

$$2>e\,.$$

This is a contradiction.

Case (II_+). Assume $\Gamma\cdot b=\alpha<0$.

Since $P_{d_1}(3/d_1C\ \&\ \sum_e)=1$, we can take an effective divisor $E\in|\Gamma|$ such that $\mathrm{supp}\,(E)\supset b$. Let $E=E'+kb$ for some $k>0$, where $\mathrm{supp}\,(E')\not\supset b$. Then, $0>E\cdot b=E'\cdot b\geqq 0$, which is a contradiction. Therefore we have $\Gamma\cdot b=\alpha\geqq 0$. Hence Γ is already numerically semi-positive, i.e., arithmetically effective. Therefore we have

$$\Gamma \approx 0 .$$

Thus, we obtain $N\Gamma \sim 0$ for sufficiently large $N>0$, where $\sim$ means linear equivalence. Therefore, $\alpha=\beta=0$, and so

$$3m=2d_1 ,$$

and

$$3n=(e+2)d_1 .$$

Therefore, we have an integer η such that

$$d_1=3\eta , \tag{7}$$

$$m=2\eta , \tag{8}$$

and

$$n=(e+2)\eta . \tag{9}$$

Let g be the geometric genus of C. By the adjunction formula, we have

$$\begin{aligned} 2g-2=d_1^2-3d_1 &= (K(\Sigma_e)+C)\cdot C \\ &= -em(m-2)+m(n-e-2)+n(m-2) . \end{aligned} \tag{10}$$

Hence, by (7), (8), (9) and (10), we have

$$\eta=1 .$$

Therefore

$$g=1 ,$$

$$m=2 ,$$

and

$$n=e+2 .$$

Since, $0 \leqq C\cdot a=2-e$, we have

$$e \leqq 2 .$$

This case corresponds to (iii) of Corollary 1.8. ∎

6. A counter example to the proper birational invariance of κ (D & X)

Here, we show that $\kappa(D$ & $X)$ and $P_m(D$ & $X)$ are not proper birational invariants of the noncomplete variety $X\setminus D$ in general.

We can construct a counter example as follows. Let $X=\boldsymbol{P}^2$ and $D=C_1+C_2+C_3+L_1+L_2+L_3$, where C_i and L_j denote six lines in general position. Let

$$\mu: (D^* \& S^*) \to (D \& \boldsymbol{P}^2)$$

be a nonsingular model of $(D \& \boldsymbol{P}^2)$, where μ is a composition of blowings up of all double points of D, and let

$$D^* = C_1^* + C_2^* + C_3^* + L_1^* + L_2^* + L_3^*,$$

where

$$C_i^* = \mu^p(C_i) \quad \text{and} \quad L_i^* = \mu^p(L_i) \qquad \text{for each} \quad i.$$

Let $L_i \cap L_j = P_k$, where $\{i, j, k\} = \{1, 2, 3\}$, and we apply the standard Cremona transformation $c = c(P_1, P_2, P_3)$ to $\boldsymbol{P}^2$, and we can factor c like $c = g \circ f^{-1}$ is the composition of blowings up with centers P_1, P_2 and P_3, and g is the contraction of L'_1, L'_2 and L'_3, where $L'_i = f^p(L_i)$ for $i = 1, 2, 3$. Let $f^{-1}(P_i) = E_i$, $g_* C_i = \bar{C}_i$, $g_* E_i = \bar{E}_i$, and $A = \sum \bar{C}_i + \sum \bar{E}_i$.

$$h = c\,|_{\boldsymbol{P}^2 \setminus D}: \boldsymbol{P}^2 \setminus D \to \boldsymbol{P}^2 \setminus A$$

is an isomorphism, still more a proper birational morphism. We show that

$$\kappa(D \& \boldsymbol{P}^2) = 0 \quad \text{and} \quad \kappa(A \& \boldsymbol{P}^2) = -\infty$$

as follows.

$$(K(S) + D^*) \cdot C_i^* = -2 \quad \text{and} \quad C_i^{*2} = -4.$$

Hence by a standard process of Zariski decomposition for $K(S) + D^*$ (see Miyanishi [9, Section 3 of Chapter I and Section 1 of Chapter II]), we have

$$K(S) + D^* - (C_i^*/2),$$

and applying the standard process to all C_i^* and L_i^*, we have the semi positive part

$$K(S) + (D^*/2).$$

Since D has only double points as its singularity, we can contract all exceptional curves of the first kind with respect to μ. It follows that we have relatively minimal model $\boldsymbol{P}^2$ and $K(\boldsymbol{P}^2) + D/2$, and $2(K(\boldsymbol{P}^2) + D/2) \sim -6L + 6L = 0$, thus we obtain

$$\kappa(D \& \boldsymbol{P}^2) = 0.$$

On the other hand, let

$$\eta: (A^* \& S) \to (A \& \boldsymbol{P}^2)$$

be a nonsingular model of $(A \& \boldsymbol{P}^2)$, where η is the composition of blowings up of all multiple points of A, where

$$A^* = \sum C_i^* + \sum E_i^*, \quad \text{and} \quad E_i^* = \eta^p(E_i).$$

Since

$$(C_i^*)^2=(E_i^*)^2=-1\ ,$$

and

$$(K(S)+A^*)\cdot C_i^*=(K(S)+A^*)\cdot E_i^*=-2\ ,$$

for each i, we can contract all C_i^* and E_i^* by standard process of Zariski decomposition. Hence, we obtain relatively minimal model $\boldsymbol{P}^2$ and $K(\boldsymbol{P}^2)$, we have

$$\kappa(A\ \&\ \boldsymbol{P}^2)=-\infty\ .$$

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This is the paper in which the author bound his previous papers [11] and [12] together.

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