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# A Note on Zeros of Differential Polynomials in the Gamma Function

by

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## 1. Introduction

In a recent paper, M. Boshernitzan ([4], Prop. 10.1, p. 254) announced the following result concerning the Gamma function: If $P(u_0, u_1, \cdots, u_n)$ is a polynomial in $u_0, \cdots, u_n$, with real constant coefficients, then on some interval $(t_0, +\infty)$, the function, $P(\Gamma(t), \Gamma'(t), \cdots, \Gamma^{(n)}(t))$ has no zeros.

The object of the present note is to show that the situation concerning the zeros of $P(\Gamma(t), \Gamma'(t), \cdots, \Gamma^{(n)}(t))$ in the complex domain is far different. We consider broader classes of polynomials $P(z, u_0, u_1, \cdots, u_n)$, namely those polynomials in $u_0, \cdots, u_n$ of positive total degree in $u_0, \cdots, u_n$, whose coefficients are entire functions $\phi(z)$ satisfying the condition,

$$\log M(r, \phi)=o(r) \quad \text{as} \quad r\to\infty, \tag{1}$$

where $M(r, \phi)$ denotes the maximum modulus of $\phi(z)$. We show that in general, the meromorphic function,

$$f(z)=P(z, \Gamma(z), \Gamma'(z), \cdots, \Gamma^{(n)}(z)), \tag{2}$$

(which cannot be identically zero by an extension of a theorem of O. Hölder (see § 3)), does possess many zeros in the plane, although there are obviously exceptions (e.g. $\Gamma(z)$ itself has no zeros).

More specifically, we prove two results. The first considers the case where $P(z, u_0, \cdots, u_n)$ contains at least two nontrivial terms of different total degrees in $u_0, \cdots, u_n$. In this case, we show that the exponent of convergence ([10], p. 327) of the zero-sequence of $f(z)$ is 1. (In fact, more is true, namely if $n(r)$ denotes the number of zeros of $f(z)$ in $|z|\leq r$ (counting multiplicity), then $n(r)\neq o(r(\log r))$ as $r\to\infty$.) The second result considers the case where all terms of $P(z, u_0, \cdots, u_n)$ have the same total degree in $u_0, \cdots, u_n$. In this case, we construct an auxiliary entire function $h_P(z)$ which is simply a special linear combination (with integer coefficients) of the coefficients of $P(z, u_0, \cdots, u_n)$. We show that if $h_P(z)\not\equiv 0$, then

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again the exponent of convergence of the zero-sequence of $f(z)$ is 1, unless $P(z, u_0, \cdots, u_n)$ has the special form, $\phi(z)u_0^q$ where $q$ is the total degree of $P$. (In fact, we show that unless $P(z, u_0, \cdots, u_n)$ has this special form, we have $n(r) \neq o(r)$ as $r \to \infty$. Of course, if $P(z, u_0, \cdots, u_n)$ does have the special form $\phi(z)u_0^q$, then $n(r) = o(r)$ as $r \to \infty$ in view of (1), Jensen's formula ([10] p. 181), and the fact that $\Gamma(z)$ has no zeros.) In the exceptional case where $h_P(z) \equiv 0$, the situation concerning the distribution of zeros of $f(z)$ is still unclear (see § 7). However, we remark that the exceptional case $h_P(z) \equiv 0$ cannot occur for first-order differential polynomials $P(z, \Gamma(z), \Gamma'(z))$ (see § 7).

## 2.

We now state our main results

THEOREM. *Let $P(z, u_0, u_1, \cdots, u_n)$ be a polynomial in $u_0, u_1, \cdots, u_n$, whose coefficients are entire functions $\phi(z)$ satisfying Condition* (1), *and assume that $P(z, u_0, u_1, \cdots, u_n)$ is of positive total degree in $u_0, u_1, \cdots, u_n$. Let $f(z)$ be the meromorphic function defined in* (2) *and let $n(r)$ denote the number of zeros of $f(z)$ (counting multiplicity) in $|z| \leq r$ for $r > 0$. Then:*

(a) *If $P(z, u_0, u_1, \cdots, u_n)$ contains at least two nontrivial terms of different total degrees, then $n(r) \neq o(r(\log r))$ as $r \to \infty$ (and so the exponent of convergence of the zero-sequence of $f(z)$ is* 1).

(b) *Suppose every nontrivial term of $P(z, u_0, \cdots, u_n)$ is of total degree $q$ for some fixed $q > 0$, say*

$$P(z, u_0, \cdots, u_n) = \sum_{i_0 + \cdots + i_n = q} \phi_{i_0 i_1 \cdots i_n}(z) u_0^{i_0} \cdots u_n^{i_n}. \tag{3}$$

*Let $d$ denote the maximum of the numbers, $i_0 + 2i_1 + \cdots + (n+1)i_n$ for which $\phi_{i_0 i_1 \cdots i_n}(z) \not\equiv 0$, and form the entire function,*

$$h_p(z) = \sum \phi_{i_0 i_1 \cdots i_n}(z)((-1)1!)^{i_1}((-1)^2 2!)^{i_2} \cdots ((-1)^n n!)^{i_n}, \tag{4}$$

*where the sum is extended over all $(i_0, \cdots, i_n)$ satisfying $i_0 + 2i_1 + \cdots + (n+1)i_n = d$. Assume that $h_P(z) \not\equiv 0$. Then either $P(z, u_0, \cdots, u_n)$ has the special form $\phi_{q,0,\cdots,0}(z)u_0^q$ or we have $n(r) \neq o(r)$ as $r \to \infty$ (in which case the exponent of convergence of the zero-sequence of $f(z)$ is* 1).

## 3. Preliminaries

For a meromorphic function $g(z)$ on the plane, we will use the standard notation for the Nevanlinna functions $m(r, g)$, $N(r, g)$, and $T(r, g)$ introduced in ([8], pp. 6, 12) (see also [5], p. 3), including the notation $n(r, g)$ to denote the number of poles (counting multiplicity) of $g(z)$ in $|z| \leq r$.

We will also use the notation $\Psi = \Gamma'/\Gamma$, where $\Gamma$ is Euler's $\Gamma$-function. We will require the following facts from ([8], p. 17) and ([3], p. 62):

(5) $$T(r, \Gamma)=(r(\log r)/\pi)+O(r) \quad \text{as} \quad r\to\infty\,,$$

and

(6) $$T(r, \Psi)=r+o(r) \quad \text{as} \quad r\to\infty\,.$$

Finally, we will require the following extension of a theorem of O. Hölder which is proved in ([3], §§5, 8, 15) (see also [2]):

EXTENSION OF HÖLDER'S THEOREM. *Let* $R(z, u_0, \cdots, u_n)$ *be a polynomial in* $u_0, \cdots, u_n$ *with coefficients which are entire functions* $\phi(z)$ *satisfying Condition* (1). *Then if either* $y=\Gamma(z)$ *or* $y=\Psi(z)$ *is a solution of the differential equation* $R(z, y, y', \cdots, y^{(n)})=0$, *then all coefficients of* $R(z, u_0, \cdots, u_n)$ *must vanish identically.*

(*The original result due to Hölder considered the case where the coefficients of* $R(z, u_0, \cdots, u_n)$ *are polynomials* [7].)

## 4. Proof of Part (a)

For each nonnegative integer $j$, let $P_j(z, u_0, \cdots, u_n)$ denote the homogeneous part of $P(z, u_0, \cdots, u_n)$ of total degree $j$ in $u_0, \cdots, u_n$, so that,

(7) $$P(z, u_0,\cdots, u_n)=\sum_{j=0}^{m} P_j(z, u_0, \cdots, u_n)\,,$$

where by hypothesis $P_m$ is not the zero polynomial and $m\geq 1$. Now it is easy to verify (see [5], p. 73) that if $y(z)$ is a meromorphic function and $w=y'/y$, then for $k=1, 2,, \cdots$, the function $y^{(k)}/y$ can be written as a polynomial in $w, w', \cdots, w^{(k-1)}$ with constant coefficients. It easily follows that for each $j=0, 1, \cdots, m$, we can write,

(8) $$P_j(z, y, y', \cdots, y^{(n)})/y^j=P_j^*(z, w, w', \cdots, w^{(n-1)})\,,$$

where $P_j^*(z, w, w', \cdots, w^{(n-1)})$ is a polynomial in $w, w', \cdots, w^{(n-1)}$, whose coefficients are entire functions satisfying the Condition (1). Applying this to the case where $y=\Gamma$ and $w=\Psi$, it follows from (2) and (7) that,

(9) $$f(z)=\sum_{j=0}^{m} R_j(z)(\Gamma(z))^j\,,$$

where,

(10) $$R_j(z)=P_j^*(z, \Psi(z), \Psi'(z), \cdots, \Psi^{(n-1)}(z))\,.$$

In view of (1), (6) and the elementary rules for calculating with the Nevanlinna characteristic (see [5] or [8]), we have

(11) $$T(r, R_j)=O(r) \quad \text{as} \quad r\to\infty \qquad \text{for} \quad j=0, 1, \cdots, m\,.$$

In view of (9), we have

$$(12) \qquad f'(z)=\sum_{j=0}^{m} Q_j(z)(\Gamma(z))^j\,, \quad \text{where} \quad Q_j=R'_j+jR_j\Psi\,,$$

and thus,

$$(13) \qquad T(r, Q_j)=O(r) \quad \text{as} \quad r\to\infty \qquad \text{for} \quad j=0, 1, \cdots, m\,.$$

We now assume that the conclusion of Part (a) fails to hold, i.e., assume that $n(r, 1/f)=o(r(\log r))$ as $r\to\infty$. Of course, from (2) and (6) we also have $n(r, f)=O(r)$ as $r\to\infty$. Thus, if we set $D=f'/f$, then it follows (see [9], p. 63]) that,

$$(14) \qquad T(r, D)=o(r(\log r)) \qquad \text{as} \quad r\to\infty\,,$$

and we have from (9) and (12) that,

$$(15) \qquad \sum_{j=0}^{m} (Q_j-DR_j)\Gamma^j\equiv 0\,.$$

Since $T(r, Q_j-DR_j)=o(r(\log r))$ as $r\to\infty$ for $j=0, 1, \cdots, m$ from (11), (13), (14), it now follows that,

$$(16) \qquad Q_j-DR_j\equiv 0 \qquad \text{for} \quad j=0, 1, \cdots, m\,,$$

for in the contrary case it would follow from (15) and the elementary rules for calculating with the Nevanlinna characteristic (see [6], p. 108]) that $T(r, \Gamma)$ would be $o(r(\log r))$ as $r\to\infty$ in contradiction to (5).

From (16), we obtain,

$$(17) \qquad d(R_j\Gamma^j/f)/dz\equiv 0 \qquad \text{for} \quad j=0, 1, \cdots, m\,,$$

from which we conclude that,

$$(18) \qquad R_j=c_jf/\Gamma^j \qquad \text{for} \quad j=0, 1, \cdots, m\,,$$

where $c_j$ is a constant.

We observe first that not all the $c_j$ can be zero, for in the contrary case all $R_j\equiv 0$ and so $f(z)\equiv 0$ in contradiction to the extension of Holder's theorem (§ 3). We next observe that exactly one of the $c_j$ can be nonzero, for if $c_j\neq 0$ and $c_k\neq 0$ where $j<k$, then from (18) we would obtain $\Gamma^{k-j}=(c_kR_j/c_jR_k)$ from which it would follow using (11) that $T(r, \Gamma)=O(r)$ as $r\to\infty$ in contradiction to (5). Hence, there is an index $q$ such that $0\leq q\leq m$ with the property that $c_q\neq 0$ and $c_j=0$ if $j\neq q$. Thus from (18), we have $R_j(z)\equiv 0$ for $j\neq q$, and in view of (10) it now follows from the extension of Hölder's theorem (§ 3) that for $j\neq q$, all coefficients of $P_j^*(z, w, w', \cdots, w^{(n-1)})$ as a polynomial in $w, w', \cdots, w^{(n-1)}$, vanish identically. It is shown in ([1], § 4 (b), p. 56) that this implies that all coefficients of $P_j(z, u_0, \cdots, u_n)$ as a polynomial in $u_0, \cdots, u_n$, must vanish identically for $j\neq q$. From (7), we then see that $P=P_q$ which is contrary to the hypothesis that $P$ contains at least two terms of different total degrees in $u_0, \cdots, u_n$. This contradiction establishes the conclusion of Part (a) that $n(r, 1/f)\neq o(r(\log r))$ as $r\to\infty$. From this and ([9] § 14, p. 27), we can conclude that the exponent of

convergence of the zero-sequence of $f(z)$ is at least 1. However, since $f(z)$ is of order at most 1, the exponent of convergence of the zero-sequence has the same property by ([9], p. 31), and so is precisely 1.

## 5. Proof of Part (b)

To prove Part (b), we assume that $P(z, u_0, \cdots, u_n)$ has the form (3), and without loss of generality we may assume that $P(z, u_0, \cdots, u_n)$ actually depends on $u_n$, that is,

(19) $$\partial P/\partial u_n \not\equiv 0 \quad \text{as polynomials in} \quad u_0, \cdots, u_n .$$

Now $\Gamma(z)$ has simple poles at the nonpositive integers, and for each $m=0, 1, \cdots$, let $\alpha_m$ denote the residue of $\Gamma(z)$ at the point $z=-m$. Hence, the Laurent expansion of $\Gamma(z)$ around $z=-m$ is of the form,

(20) $$\Gamma(z)=\alpha_m(z+m)^{-1}+A_m(z) ,$$

where $A_m(z)$ is analytic on $|z+m|<1$, from which it follows that for $j=1, 2, \cdots$, we have

(21) $$\Gamma^{(j)}(z)=(-1)^j(j!)\alpha_m(z+m)^{-(j+1)}+A_m^{(j)}(z) ,$$

on $0<|z+m|<1$. Substituting these expansions into the right side of (2), it follows from (3) that on $0<|z+m|<1$, we have,

(22) $$f(z)=h_P(z)\alpha_m^q(z+m)^{-d}+B_m(z) ,$$

where $h_P(z)$ and $d$ are defined in the statement of Part (b), and where $B_m(z)$ is analytic on $0<|z+m|<1$ and has at most a pole of order $d-1$ at $z=-m$.

From the hypothesis, we have $h_P(z)\not\equiv 0$, and we now assume that $n(r, 1/f)$ (which is $n(r)$) satisfies,

(23) $$n(r, 1/f)=o(r) \quad \text{as} \quad r\to\infty .$$

We now analyze the poles of $f(z)$. We observe first that since the coefficients of $P(z, u_0, \cdots, u_n)$ are entire functions, the poles of $f(z)$ can occur only at nonpositive integers. Let $E_1$ denote the set of all nonpositive integers which are not zeros of $h_P(z)$, and let $E_2$ denote the remaining nonpositive integers. From (22), any element of $E_1$ gives rise to a pole of $f(z)$ of multiplicity $d$. Let $F_1(z)$ denote the canonical product having a zero of multiplicity $d$ at each element of $E_1$ (and no other zeros), and consider the sequence of poles of the meromorphic function $g=fF_1$. In view of (22), this sequence (if it is not empty) consists of elements of $E_2$ which can appear at most $d-1$ times (or not at all). Hence, if we let $F_2(z)$ denote the canonical product with simple zeros at the elements of $E_2$, then

(24) $$n(r, g)\leq(d-1)n(r, 1/F_2) .$$

Now, by construction, every element of $E_2$ is a zero of $h_P(z)$ and hence,

(25) $$n(r,\ 1/F_2) \leq n(r, 1/h_P)\ .$$

Since $h_P$ is a linear combination of entire functions satisfying Condition (1), it also satisfies this condition, and hence in view of Jensen's formula, we have $n(r,\ 1/h_P) = o(r)$ as $r \to \infty$. In view of (24) and (25), we thus obtain,

(26) $$n(r,\ g) = o(r) \quad \text{and} \quad n(r,\ 1/F_2) = o(r) \qquad \text{as} \quad r \to \infty\ .$$

Finally, we observe that any zero of $g$ must be a zero of $f$ of the same multiplicity, and so from (23), we have,

(27) $$n(r,\ 1/g) = o(r) \qquad \text{as} \quad r \to \infty\ .$$

Now the entire function $F_1 F_2^d$ has zeros at precisely the non-positive integers, each of multiplicity $d$. This is the same zero-sequence possessed by the entire function $\Gamma^{-d}$, and since both functions are of order 1 (by (5) and ([10], (9.4), p. 330)), we have by the Hadamard factorization theorem ([10], p. 332) that,

(28) $$\Gamma(z)^{-d} = F_1(z)(F_2(z))^d e^{az+b}\ ,$$

for some constants $a$ and $b$.

Since $f = g/F_1$, it follows from (28) that,

(29) $$(f'/f) - d(\Gamma'/\Gamma) = \psi\ , \qquad \text{where} \quad \psi = (g'/g) + d(F_2'/F_2) + a\ .$$

In view of (26) and (27) (and the fact that $F_2$ has no poles), it follows from ([9], p. 63) that $T(r,\ g'/g)$ and $T(r,\ F_2'/F_2)$ are each $o(r)$ as $r \to \infty$. Since $a$ is a constant, we see that the meromorphic function $\psi(z)$ has the property that $T(r,\ \psi) = o(r)$ as $r \to \infty$. Hence by a theorem of Miles ([8], p. 372–373), we can write $\psi = \psi_1/\psi_2$, where $\psi_1$ and $\psi_2$ are entire functions each satisfying Condition (1), and $\psi_2 \not\equiv 0$.

Now, let $R(z,\ u_0,\ u_1,\ \cdots,\ u_{n+1})$ be the polynomial in $u_0,\ u_1,\ \cdots,\ u_{n+1}$ defined by

(30) $$R = (\partial P/\partial z) + \sum_{j=0}^{n} (\partial P/\partial u_j) u_{j+1}\ ,$$

so that in view of (2) we have,

(31) $$f'(z) = R(z,\ \Gamma(z),\ \Gamma'(z),\ \cdots,\ \Gamma^{(n+1)}(z))\ .$$

We remark that the coefficients of $R$ are entire functions satisfying Condition (1).

In view of (29), (2), and (31), it easily follows that if $Q(z,\ u_0,\ u_1,\ \cdots,\ u_{n+1})$ is defined by,

(32) $$Q = \psi_2(z) u_0 R - \psi_2(z) d u_1 P - \psi_1(z) u_0 P\ ,$$

then $Q(z,\ u_0,\ u_1,\ \cdots,\ u_{n+1})$ is a polynomial in $u_0,\ u_1,\ \cdots,\ u_{n+1}$, whose coefficients are entire functions satisfying Condition (1), and $Q$ has the property that,

(33) $$Q(z,\ \Gamma(z),\ \Gamma'(z),\ \cdots,\ \Gamma^{(n+1)}(z)) \equiv 0\ .$$

By the extension of Hölder's theorem (§3), all coefficients of $Q$ must vanish identically, so $Q(z, u_0, \cdots, u_{n+1})$ is the zero polynomial in $u_0, \cdots, u_{n+1}$. Hence $\partial Q/\partial u_{n+1}$ is also the zero polynomial in $u_0, \cdots, u_{n+1}$. But if $n$ is not zero, then the only terms of $Q$ involving $u_{n+1}$ occur in $\psi_2(z)u_0R$, and we have from (30) that $\partial Q/\partial u_{n+1}$ is $\psi_2(z)u_0(\partial P/\partial u_n)$. If this were the zero polynomial in $u_0, u_1, \cdots, u_{n+1}$, it would obviously contradict (19). Hence $n$ must be zero, so that $P$ is a polynomial in the single indeterminate $u_0$ whose coefficients are entire functions satisfying Condition (1). Since all terms of $P$ have degree $q$, it now follows that $P$ is of the form $\phi(z)u_0^q$ which proves Part (b). (Note that if $n(r)\neq o(r)$ as $r\to\infty$, then the exponent of convergence of the zero-sequence of $f$ is equal to 1 by the same argument used at the end of the proof of Part (a).)

## 6. Remark

We point out here that our main result remains true if the coefficients of $P(z, u_0, \cdots, u_n)$ are permitted to be meromorphic functions $\psi(z)$ satisfying the condition $T(r, \psi)=o(r)$ as $r\to\infty$. This follows easily since by [8], p, 372–373), any such function $\psi$ can be written as the quotient of entire functions satisfying Condition (1). Putting all terms in $P$ over the product of the denominators, and noting that $n(r, 1/\phi)=o(r)$ as $r\to\infty$ for any entire function $\phi$ satisfying Condition (1) (by Jensen's formula), the assertion then follows from the Theorem of § 2.

## 7. Remark

In the exceptional case where $h_P(z)\equiv 0$ in Part (b) of the theorem, the proof breaks down because an analysis of the sequence of poles of $f$ becomes very complicated (see (22)). Hence, we are unable to obtain an explicit representation for the canonical product whose zero-sequence is the sequence of poles of $f$, and such a representation was crucial in our proof. Even in the simplest case where $h_P(z)\equiv 0$, i.e., where $P(z, u_0, u_1, u_2)$ is $u_0u_2-2u_1^2$, it is easy to see that an analysis of the sequence of poles of $f(z)$ would involve determining the first coefficient (and possibly even the first three coefficients) in the power series expansion of the functions $A_m(z)$ in (20) around $z=-m$ for each $m=0, 1, 2, \cdots$.

However, in the case where $n=1$, we remark that the case $h_P(z)\equiv 0$ cannot occur. In this case, $h_P(z)$ is a nonzero constant multiple of a nonzero coefficient since the only index $(i_0, i_1)$ for which $i_0+i_1=q$ and $i_0+2i_1=d$ is $(2q-d, d-q)$.

## 8. Remark

The results of § 2 cannot be greatly improved for the following reasons: In Part (a), we always have $n(r)=O(r(\log r))$ as $r\to\infty$ in view of (5). In Part (b), we always have $n(r)=O(r)$ as $r\to\infty$ because when $P$ is homogeneous, there is only one term in the summation in (9), and so the assertion follows from (11).

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