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p-Valent Regular Functions with Negative Coefficients

by

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Let T denote the class of functions

$$f(z)=a_pz^p-\sum_{n=1}^{\infty} a_{n+k}z^{n+k}, \qquad k\geq p\geq 1, \qquad a_{n+k}\geq 0, \qquad a_p>0$$

regular in the unit disc E. Let T_1, T_2 denote subclasses of T satisfying $f(z_0)=z_0^p$ and $f'(z_0)=pz_0^{p-1}$ ($z_0\neq 0$, $-1<z_0<1$) respectively. Properties of certain subclasses of T_1 and T_2 are investigated and sharp results are obtained.

1. Introduction

Let T denote the class of functions

$$f(z)=a_pz^p-\sum_{n=1}^{\infty} a_{n+k}z^{n+k}, \qquad p\geq 1, \qquad k\geq p, \qquad a_p>0 \qquad \text{and} \qquad a_{n+k}\geq 0,$$

regular in the unit disc $E=\{z: |z|<1\}$. Let $H=\{w$ regular in E: $w(0)=0$, $|w(z)|<1$, $z\in E\}$. Let

$$S_p(A, B)=\left\{f\in T: \frac{zf'(z)}{f(z)}=p\frac{1+Aw(z)}{1+Bw(z)},\ -1\leq A<B\leq 1,\ w\in H\right\}$$

and

$$K_p(A, B)=\{f\in T: zf'(z)/p\in S_p(A, B)\}\ .$$

In the sequel we write $S_p(A, B)=S$; $K_p(A, B)=K$. We observe that $S=S_p(A, B)$ and $K=K_p(A, B)$ are subclasses of T consisting of p-valently starlike functions and p-valently convex functions respectively. The definition of S implies that functions f in S satisfy $\mathrm{Re}\{zf'(z)/f(z)\}>0$, $z\in E$. Further for $f\in S$, $z=re^{i\theta}$, $r<1$,

$$\frac{1}{2\pi}\int_0^{2\pi}\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\}d\theta=\frac{p}{2\pi}\int_0^{2\pi}\mathrm{Re}\,\frac{1+Aw(z)}{1+Bw(z)}d\theta=\frac{p}{2\pi}2\pi=p\,,$$

since

$$\mathrm{Re}\,\frac{1+Aw(z)}{1+Bw(z)}$$

is a harmonic function in E with $w(0)=0$. This proves p-valence of $f \in S$. Similarly if $f \in K$, f is p-valently convex in E.

For a given real number z_0 $(-1<z_0<1)$ let T_1 and T_2 be the subclasses of T satisfying $f(z_0)=z_0^p$ and $f'(z_0)=pz_0^{p-1}$, $z_0 \neq 0$, respectively. Consider subclasses S_1, S_2, K_1 and K_2 of T defined as follows:

$$S_i=S \cap T_i\,; \qquad K_i=K \cap T_i\,, \qquad i=1,2\,.$$

Silverman [3] and [4] studied univalent functions with negative coefficients and extreme points of univalent functions with two fixed points. Gupta and Jain [1] and [2] studied certain classes of univalent functions with negative coefficients. In this paper we obtain necessary and sufficient conditions for functions to be in S, K, S_i and K_i, $i=1, 2$. We also determine radius of convexity for the classes S_i, $i=1, 2$. Further closure and distortion theorems are proved and it is also shown that these subclasses are closed under convex linear combinations. Our results generalise in various ways corresponding theorems in [4], which can be deduced for the choice $k=p=1$; $A=2\alpha-1$, $B=1$.

2. The main lemmas

In this section we determine necessary and sufficient conditions for functions to be in S, K, S_i and K_i, $i=1, 2$. We now introduce the following notations for brevity.

$$n+k=m\,, \quad m(B+1)-p(A+1)=C_m\,, \quad p(B-A)=D\,, \sum_{m=k+1}^{\infty}=\sum\,, \quad \sum_{m=k}^{\infty}=\sum\nolimits_1\,,$$

$$C_m-Dz_0^{m-p}=E_m \quad \text{and} \quad C_{k+1}=F\,.$$

LEMMA 2.1. *Let $f \in T$. Then $f \in S$ if and only if*

$$\sum C_m a_m \le D a_p\,. \tag{2.1}$$

Proof. Suppose $f \in S$. Then

$$zf'(z)/f(z)=p\frac{1+Aw(z)}{1+Bw(z)}\,, \qquad -1 \le A<B \le 1\,, \qquad w(z)\in H\,, \qquad z \in E\,.$$

That is,

$$w(z)=\frac{p-zf'(z)/f(z)}{Bzf'(z)/f(z)-Ap}\,, \qquad w(0)=0$$

and

$$|w(z)|=\left|\frac{zf'(z)-pf(z)}{Bzf'(z)-Apf'(z)}\right| = \left|\frac{\sum(m-p)a_m{}^{zm}}{Da_pz^p-\sum(Bm-Ap)a_mz^m}\right|<1\,.$$

Thus

$$\text{Re}\left\{\frac{\sum(m-p)a_mz^m}{Da_pz^p-\sum(Bm-Ap)a_mz^m}\right\}<1\,.$$

Take $z=r$ with $0<r<1$. Then, for sufficiently small r, the denominator of (2.2) is positive and so it is positive for all r with $0<r<1$, since $w(z)$ is regular for $|z|<1$. Then (2.2) gives

$$\sum(m-p)a_m r^m < Da_p r^p - \sum(Bm-Ap)a_m r^m ,$$

that is, $\sum[m(B+1)-p(A+1)]a_m r^m < Da_p r^p$, that is, $\sum C_m a_m r^m < Da_p r^p$, and (2.1) follows on letting $r \to 1$.

Conversely, for $|z|=r$, $0<r<1$, we have, since $r^m<r^p$,

$$\sum[m(B+1)-p(A+1)]a_m r^m = \sum C_m a_m r^m < r^p \sum C_m a_m < Da_p r^p \qquad \text{by (2.1).}$$

So we have

$$|\sum(m-p)a_m z^p| \leqq \sum(m-p)a_m r^m < Da_p r^p - \sum(Bm-Ap)a_m r^m$$
$$\leqq |Da_p z^p - \sum(Bm-Ap)a_m z^m| .$$

This proves that $zf'(z)/f(z)$ is of the form

$$p\,\frac{1+Aw(z)}{1+Bw(z)}$$

with $w \in H$. Therefore $f \in S$ and the proof is complete.

LEMMA 2.2. *Let $f \in T$. Then $f \in K$ if and only if*

$$\sum \frac{m}{p} C_m a_m \leq Da_p . \tag{2.3}$$

Proof. $f \in K$ if and only if $zf'/p \in S$ and hence the lemma follows from the Lemma 2.1.

THEOREM 2.1. *Let $f \in T_1$. Then $f \in S_1$ if and only if*

$$\sum E_m a_m \leq D . \tag{2.4}$$

Proof. Suppose $f \in S_1$. Then for fixed z_0 $(-1<z_0<1)$, $f(z_0)=a_p z_0^p - \sum a_m z_0^m$, $k \geq p$. Since $f(z_0)/z_0^p = 1 = a_p - \sum a_m z_0^{m-p}$, we get $a_p = 1 + \sum a_m z_0^{m-p}$, $f \in S_1$ implies $f \in S$ and so Lemma 2.1 is applicable for f. Therefore (2.1) holds and by substituting the value of a_p in (2.1) we get (2.4).

Conversely, let (2.4) be satisfied. Since $f(z_0)=z_0^p$, we have $a_p = 1 + \sum a_m z_0^{m-p}$. (2.4) gives (2.1) by substituting $\sum a_m z_0^{m-p} = a_p - 1$. From Lemma 2.1 it follows that $f \in S$ and already $f \in T_1$. Therefore $f \in T_1 \cap S = S_1$.

COROLLARY 2.1. *Let $f(z)=a_p z^p - \sum a_m z^m$, $k \geq p$, be in the class S_1. Then*

$$a_m \leq D/E_m , \qquad m \geq k+1 .$$

Equality holds for functions of the form

$$f(z) = \frac{(C_m z^p - Dz^m)}{E_m} .$$

THEOREM 2.2. *Let $f \in T_1$. Then $f \in K_1$ if and only if*

$$\sum\left(\frac{m}{p}C_m - Dz_0^{m-p}\right)a_m \leq D\,. \tag{2.5}$$

Proof. Suppose $f \in K_1$. Then for fixed z_0, we have $a_p = 1 + \sum a_m z_0^{m-p}$.

By definition f also belongs to K and so Lemma 2.2 holds for f and by substituting the value of a_p in (2.3) we get (2.5). Conversely, let (2.5) be satisfied. Since $f \in T_1$ we have $a_p = 1 + \sum a_m z_0^{m-p}$. By substituting $\sum a_m z_0^{m-1} = a_p - 1$ in (2.5) we conclude that $f \in K$ and this completes the proof.

THEOREM 2.3. *Let $f \in T_2$. Then $f \in S_2$ if and only if*

$$\sum\left(C_m - \frac{m}{p}Dz_0^{m-p}\right)a_m \leq D\,. \tag{2.6}$$

Proof. For fixed z_0 $(-1 < z_0 < 1)$, $f'(z_0) = pa_p z_0^{p-1} - \sum ma_m z_0^{m-1}$, $k \geq p$. Since

$$f'(z_0)/pz_0^{p-1} = 1 = a_p - \sum \frac{m}{p} a_m z_0^{m-p}\,,$$

we have

$$a_p = 1 + \sum \frac{m}{p} a_m z_0^{m-p}\,.$$

$f \in S_2$ implies $f \in S$ and so Lemma 2.1 holds for f and hence by substituting the value of a_p in (2.1) we obtain (2.6).

Conversely, let (2.6) hold. Since $f \in T_2$, we have

$$\sum \frac{m}{p} a_m z_0^{m-p} = a_p - 1\,.$$

By substituting the value of

$$\sum \frac{m}{p} a_m z_0^{m-p}$$

in (2.6) we get (2.1) from which we conclude that $f \in S$. Hence $f \in S \cap T_2$.

THEOREM 2.4. *Let $f \in T_2$. Then $f \in K_2$ if and only if*

$$\sum \frac{m}{p} E_m a_m \leq D\,. \tag{2.7}$$

Proof. Suppose $f \in K_2$. Since $f \in T_2$, we have

$$a_p = 1 + \sum \frac{m}{p} a_m z_0^{m-p}$$

and since $f \in K$ condition (2.3) is applicable and hence by substituting the value of a_p in (2.3) we get (2.7).

Conversely, let (2.7) hold. By hypothesis $f \in T_2$ which implies that

$$\sum \frac{m}{p} a_m z_0^{m-p} = a_p - 1$$

and substituting in (2.7) we conclude that $f \in K$. Hence the proof of the theorem is complete.

3. The radius of p-valent convexity of the classes S_i, $i=1, 2$

THEOREM 3.1. *Let $f \in T$. If $f \in S_1$ or S_2, then f is p-valently convex in the disc*

$$|z| < r = r(A, B) = \inf_m \left[\left(\frac{p}{m}\right)^2 \frac{C_m}{D} \right]^{1/(m-p)}.$$

The bound is sharp.

Proof. To prove the theorem it is enough to show that

$$\left| 1 + \frac{zf''(z)}{f'(z)} - p \right| \leq p \qquad \text{for } |z| < r(A, B), \qquad z \in E.$$

Now

$$\left| \frac{(1-p)f'(z) + zf''(z)}{f'(z)} \right| = \left| \frac{-\sum m(m-p)a_m z^{m-1}}{pa_p z^{p-1} - \sum ma_m z^{m-1}} \right|$$

$$\leq \frac{\sum \frac{m}{p}(m-p)a_m |z|^{m-p}}{|a_p - \sum \frac{m}{p} a_m |z|^{m-p}|}. \tag{3.1}$$

Consider the values of z for which

$$|z| \leq \inf_m \left[\left(\frac{p}{m}\right)^2 \frac{C_m}{D} \right]^{1/(m-p)}$$

so that

$$|z|^{m-p} \leq \left(\frac{p}{m}\right)^2 \frac{C_m}{D}$$

holds. Then

$$\sum \frac{m}{p} a_m |z|^{m-p} \leq \sum \frac{m}{p} \frac{p^2}{m^2} \frac{C_m}{D} a_m .$$

Now

$$\sum \frac{m}{p} a_m |z|^{m-p} < a_p$$

provided

$$\sum \frac{p}{m}\frac{C_m}{D} a_m < a_p ,$$

which holds, if

$$\sum \frac{C_m}{D} a_m < a_p = 1 + \sum a_m z_0^{m-p} .$$

This is equivalent to $\sum E_m a_m \leq D$, which is true by Theorem 2.1 or Theorem 2.3. Hence we can rewrite the denominator of the right hand side of inequality (3.1), for the considered values of z, using the fact that

$$a_p > \sum \frac{m}{p} a_m | z |^{m-p} .$$

Thus

$$\left| \frac{(1-p)f'(z) + zf''(z)}{f'(z)} \right| \leq \frac{\sum \frac{m}{p} (m-p) a_m | z |^{m-p}}{a_p - \sum \frac{m}{p} a_m | z |^{m-p}} \leq p \quad \text{if}$$

$$\sum m(m-p) a_m | z |^{m-p} \leq p^2 a_p - \sum pm a_m | z |^{m-p} . \tag{3.2}$$

If $f \in S_1$, (3.2) is equivalent to

$$\sum \left(\frac{m}{p}\right)^2 a_m | z |^{m-p} \leq a_p = 1 + \sum a_m z_0^{m-p} ,$$

that is,

$$\sum \left[\left(\frac{m}{p}\right)^2 | z |^{m-p} - z_0^{m-p} \right] a_m \leq 1 . \tag{3.3}$$

Again if $f \in S_2$ (3.2) is equivalent to

$$\sum \left[\left(\frac{m}{p}\right)^2 | z |^{m-p} - \frac{m}{p} z_0^{m-p} \right] a_m \leq 1 . \tag{3.4}$$

In view of Theorem 2.1, $f \in S_1$ if and only if

$$\sum \left(\frac{C_m}{D} - z_0^{m-p} \right) a_m \leq 1 .$$

Hence inequality (3.3) is true if

$$\left(\frac{m}{p}\right)^2 | z |^{m-p} - z_0^{m-p} \leq \frac{C_m}{D} - z_0^{m-p} ,$$

for all m, that is, if

$$|z|\leq\left[\left(\frac{p}{m}\right)^2\frac{C_m}{D}\right]^{1/(m-p)},$$

for all m. Again in view of Theorem 2.3, $f\in S_2$ if and only if

$$\sum\left(\frac{C_m}{D}-\frac{m}{p}\,z_0^{m-p}\right)a_m\leq 1\,.$$

Inequality (3.4) is true if

$$\left(\frac{m}{p}\right)^2|z|^{m-p}-\frac{m}{p}z_0^{m-p}\leq\frac{C_m}{D}-\frac{m}{p}\,z_0^{m-p},$$

for all m, that is, if

$$|z|\leq\left[\left(\frac{p}{m}\right)^2\frac{C_m}{D}\right]^{1/(m-p)},$$

for all m. Then result is sharp with the extremal function

$$f_m(z)=\frac{C_mz^p-Dz^m}{E_m},\qquad m=k+1,\qquad k+2,\cdots,k\geq p\,.$$

Remark 3.1. The conclusion of Theorem 3.1 is independent of the point z_0.

4. Closure theorems

In this section we prove that the classes S_i and K_i, $i=1$, 2, are closed under convex linear combinations and also show that the functions of these classes can be expressed in a particular form.

THEOREM 4.1. *The class S_1 is closed under convex linear combination.*

Proof. Let $f(z)=a_pz^p-\sum a_mz^m$, $k\geq p$ and $g(z)=b_pz^p-\sum b_mz^m$, $k\geq p$, be any two functions of the class S_1. For λ such that $0\leq\lambda\leq 1$, it suffices to show that $h(z)=(1-\lambda)f(z)+\lambda g(z)$, $z\in E$, is also a function of S_1. Now $h(z)=[(1-\lambda)a_p+\lambda b_p]z^p-\sum[(1-\lambda)a_m+\lambda b_m]z^m$. Applying Theorem 2.1 to f, $g\in S_1$, we have

$$\sum E_m((1-\lambda)a_m+\lambda b_m)=(1-\lambda)\sum E_ma_m+\lambda\sum E_mb_m\leq(1-\lambda)D+\lambda D=D\,.$$

Also $h(z_0)=(1-\lambda)f(z_0)+\lambda g(z_0)=(1-\lambda)z_0^p+\lambda z_0^p=z_0^p\Rightarrow h\in T_1$. Hence by Theorem 2.1, $h\in S_1$.

THEOREM 4.2. *S_2 is closed under convex linear combination.*

THEOREM 4.3. *K_i, $i=1$, 2, are closed under convex linear combinations.*

THEOREM 4.4. *Define $f_k(z)=z^p$ and $f_m(z)=(C_mz^p-Dz^m)/E_m$, $m\geq k+1$, $k\geq p$. Then $f\in S_1$ if and only if f is of the form $f(z)=\sum_1\lambda_mf_m(z)$, $z\in E$ where $\lambda_m\geq 0$ for $m\geq k$ and $\sum_1\lambda_m=1$.*

Proof. Suppose

$$f(z)=\sum_1 \lambda_m f_m(z)=\lambda_k f_k(z)+\sum \lambda_m f_m(z)$$
$$=\lambda_k z^p+\sum \frac{C_m z^p - Dz^m}{E_m}\lambda_m$$
$$=\left[\lambda_k+\sum \frac{C_m}{E_m}\lambda_m\right]z^p-\sum \frac{\lambda_m D z^m}{E_m}.$$

Now

$$\sum \frac{E_m}{D}a_m=\sum \frac{E_m}{D}\frac{D}{E_m}\lambda_m=\sum \lambda_m=1-\lambda_k\le 1.$$

Further $f_m(z_0)=z_0^p$. Therefore

$$f(z_0)=\Sigma_1 \lambda_m f_m(z_0)=\Sigma_1 \lambda_m z_0^p=z_0^p \Sigma_1 \lambda_m=z_0^p.$$

This proves that $f\in T_1$. Hence by Thoerem 2.1, $f\in S_1$. Conversely, let $f\in S_1$. Then $a_p=1+\sum a_m z_0^{m-p}$. Setting

$$\lambda_m=\frac{E_m}{D}a_m,\qquad m\ge k+1 \qquad \text{and} \qquad \lambda_k=a_p-\frac{1}{D}\sum C_m a_m$$

we note that $\lambda_k\ge 0$ by Lemma 2.1. Now we have

$$f(z)=a_p z^p-\sum a_m z^m=\lambda_k z^p+\sum \lambda_m z^p\left[1+\frac{(z_0^{m-p}-z^{m-p})a_m}{\lambda_m}\right]$$
$$=\lambda_k z^p+\sum \frac{C_m z^p-Dz^m}{E_m}\lambda_m$$
$$=\lambda_k z^p+\sum \lambda_m f_m(z)=\sum_1 \lambda_m f_m(z).$$

Hence the theorem.

We can prove the following theorems in a similar manner.

THEOREM 4.5. *Define*

$$f_k(z)=z^p \qquad \textit{and} \qquad f_m(z)=\frac{C_m z^p-Dz^m}{C_m-\dfrac{m}{p}Dz_0^{m-p}},\qquad m\ge k+1,\qquad k\ge p.$$

Then $f\in S_2$ *if and only if* f *is of the form* $f(z)=\sum_1 \lambda_m f_m(z)$, $z\in E$ *where* $\lambda_m\ge 0$ *for* $m\ge k$ *and* $\sum_1 \lambda_m=1$.

THEOREM 4.6. *Define*

$$f_k(z)=z^p \qquad \textit{and} \qquad f_m(z)=\frac{mC_m z^p-pDz^m}{mC_m-pDz_0^{m-p}},\qquad m\ge k+1,\qquad k\ge p.$$

Then $f \in K_1$ if and only if f is of the form $f(z)=\sum_1 \lambda_m f_m(z)$, $z \in E$ where $\lambda_m \geq 0$ for $m \geq k$ and $\sum_1 \lambda_m = 1$.

THEOREM 4.7. *Define*

$$f_k(z)=z^p \qquad \text{and} \qquad f_m(z)=\frac{C_m z^p - pDz^m/p}{E_m}, \qquad m \geq k+1, \qquad k \geq p.$$

Then $f \in K_2$ if and only if f is of the form $f(z)=\sum_1 \lambda_m f_m(z)$, $z \in E$ where $\lambda_m \geq 0$ for $m \geq k$ and $\sum_1 \lambda_m = 1$.

5. Distortion theorems

THEOREM 5.1. *Let $f \in S_1$, $0 < z_0 < 1$, then for $|z| = r < 1$,*

$$\frac{Fr^p - Dr^{k+1}}{F - Dz_0^{k+1-p}} \leq |f(z)| \leq \frac{Fr^p + Dr^{k+1}}{F - Dz_0^{k+1-p}},$$

$$\frac{pFr^{p-1} - (k+1)Dr^k}{F - Dz_0^{k+1-p}} \leq |f'(z)| \leq \frac{pFr^{p-1} + (k+1)Dr^k}{F - Dz_0^{k+1-p}}.$$

These bounds are sharp.

Proof. Since $f \in S_1$, we have $a_p = 1 + \sum a_m z_0^{m-p}$. By Theorem 2.1, $f \in S_1$ if and only if

$$\sum \frac{E_m}{D} a_m \leq 1. \tag{5.1}$$

Now E_m increases with m and $E_{k+1} > 0$ and $a_m \geq 0$. So the left-hand member of (5.1) is a series of non-negative terms. Therefore

$$\frac{F - Dz_0^{k+1-p}}{D} \sum a_m \leq \sum \frac{E_m}{D} a_m \leq 1 \qquad \text{by (5.1)}.$$

That is,

$$\sum a_m \leq \frac{D}{F - Dz_0^{k+1-p}}.$$

Now

$$|f(z)| \leq a_p |z|^p + \sum a_m |z|^m \leq r^p(a_p + r^{k+1-p} \sum a_m), \qquad |z| = r. \tag{5.2}$$

By substituting the values of a_p and $\sum a_m$ in (5.2), we obtain

$$|f(z)| \leq r^p \left[1 + (z_0^{k+1-p} - r^{k+1-p}) \frac{D}{F - Dz_0^{k+1-p}} \right]$$

$$= \frac{Fr^p + Dr^{k+1}}{F - Dz_0^{k+1-p}}.$$

Similarly

$$|f(z)| \geq \frac{Fr^p - Dr^{k+1}}{F - Dz_0^{k+1-p}}.$$

Since E_m/m increases with m,

$$\frac{m}{E_m} \leq \frac{k+1}{E_{k+1}}$$

and again by (5.1) we have

$$\frac{1}{p}\sum ma_m = \frac{1}{p}\sum \frac{m}{E_m}(E_m a_m) \leq \frac{k+1}{pE_{k+1}}\sum E_m a_m \leq \frac{D(k+1)}{pE_{k+1}} \leq \frac{D(k+1)}{p(E - Dz_0^{k+1-p})}.$$

Since $f \in S_1$, we have

$$|f'(z)| \leq pa_p r^{p-1} + \sum ma_m r^{m-1}$$

$$= pr^{p-1}\left(a_p + \sum \frac{m}{p} a_m r^{m-p}\right), \qquad \text{where} \quad r = |z|. \tag{5.3}$$

By substituting the values of

$$a_p \quad \text{and} \quad \sum \frac{m}{p} a_m$$

in (5.3), we obtain

$$|f'(z)| \leq pr^{p-1}\left(1 + \frac{Dz_0^{k+1-p} + (k+1)Dr^{k+1-p}/p}{F - Dz_0^{k+1-p}}\right)$$

$$= \frac{pFr^{p-1} + (k+1)Dr^k}{F - Dz_0^{k+1-p}}.$$

Similarly the lower bound can also be obtained. The result is sharp with equality for functions of the form

$$F(z) = (Fz^p - Dz^{k+1})/(F - Dz_0^{k+1-p}).$$

THEOREM 5.2. *If $f \in S_2$, $0 < z_0 < 1$, then for $|z| = r < 1$*

$$\frac{p(Fr^p - Dr^{k+1})}{pF - (k+1)Dz_0^{k+1-p}} \leq |f(z)| \leq \frac{p(Fr^p + Dr^{k+1})}{pF - (k+1)Dz_0^{k+1-p}},$$

$$\frac{p(pFr^{p-1} - (k+1)Dr^k)}{pF - (k+1)Dz_0^{k+1-p}} \leq |f'(z)| \leq \frac{p(pFr^{p-1} + (k+1)Dr^k)}{pF - (k+1)Dz_0^{k+1-p}}.$$

Equality holds for functions of the form

$$f(z) = p(Fz^p - Dz^{k+1})/(pF - (k+1)Dz_0^{k+1-p}).$$

Proof follows from Theorems 2.3 and 5.1.

THEOREM 5.3. *If $f\in K_1$, $0<z_0<1$, then for $|z|=r<1$*

$$\frac{(k+1)Fr^p-Dpr^{k+1}}{(k+1)F-Dpz_0^{k+1-p}}\le|f(z)|\le\frac{(k+1)Fr^p+Dpr^{k+1}}{(k+1)F-Dpz_0^{k+1-p}},$$

$$\frac{(k+1)(pFr^{p-1}-Dpr^k)}{(k+1)F-Dpz_0^{k+1-p}}\le|f'(z)|\le\frac{(k+1)(pFr^{p-1}+Dpr^k)}{(k+1)F-Dpz_0^{k+1-p}}.$$

Equality holds for functions of the form

$$f(z)=\frac{(k+1)Fr^p-Dpz^{k+1}}{(k+1)F-Dpz_0^{k+1-p}}.$$

Proof follows from Theorems 2.2 and 5.1.

THEOREM 5.4. *If $f\in K_2$, $0<z_0<1$, then for $|z|=r<1$*

$$\frac{Fr^p-\frac{Dp}{k+1}r^{k+1}}{F-Dz_0^{k+1-p}}\le|f(z)|\le\frac{Fr^p+\frac{Dp}{k+1}r^{k+1}}{F-Dz_0^{k+1-p}},$$

$$\frac{p(Fr^{p-1}-Dr^k)}{F-Dz_0^{k+1-p}}\le|f'(z)|\le\frac{p(Fr^{p-1}+Dr^k)}{F-Dz_0^{k+1-p}}.$$

These results are sharp with equality holding for functions of the form

$$f(z)=\frac{Fz^p-\frac{Dp}{k+1}z^{k+1}}{F-Dz_0^{k+1-p}}.$$

Proof follows from Theorems 2.4 and 5.1.

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