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Centralizers of Finite Semicyclic Transformation Monoids

by

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1. Introduction

In this paper, the properties of centralizers of semicyclic transformation semigroups will be discussed. The basic theory of this paper is analogous to a folkloric theorem on centralizers of semiregular permutation groups which states that the centralizer of a semiregular permutation group G is isomorphic both to a certain wreath product and to a semigroup of row-monomial matrices over G^0 (Wells [14], Suzuki [12, p. 288], see also Burnside [1, §171]). The first result is a theorem which points out that the centralizer of a semicyclic transformation monoid (S, V) can be represented by the semigroup of matrices determined by some n-tuple of right congruences on S (Theorem 5.1). The second result is a theorem on the number of generators of a strictly cyclic transformation monoid. This theorem asserts that for a given finite monoid S there exists a strictly cyclic transformation monoid (T, V) such that the centralizer of (T, V) is isomorphic to S, and such that T is generated as a semigroup by at most two elements (Theorem 6.1). In the first five sections, we shall deal with a semicyclic representation of semigroups and a representation of centralizers of semicyclic transformation monoids. The last section contains the investigations of the centralizers of semiregular transformation monoids.

Throughout this paper the terms "semigroup" and "monoid" will mean a finite semigroup and a finite monoid, respectively.

When T is a transformation semigroup on a finite set V, we shall write T more concretely as (T, V). Henceforth, we call a transformation semigroup (T, V) a *t-semigroup* simply. If a t-semigroup (T, V) contains an identity permutation on V, then (T, V) is called a *t-monoid.*

If A and B are two sets, then $A-B$ denotes the set of elements of A not contained in B, and $|A|$ will denote as usual the cardinality of A.

Let $f: V_1 \to V_2$ and $g: V_2 \to V_3$ be mappings of V_1 and V_2, respectively. We read a product fg from left to right:

$$a(fg)=(af)g, \qquad a \in V_1 .$$

We shall sometimes write $a \cdot f$ and $f \cdot g$ instead of af and fg.

Let (T, V) be a t-semigroup. A non-empty subset U of V is called a block of (T, V) if

$$(U)T \subseteq U \quad \text{and} \quad (V-U)T \subseteq (V-U).$$

A block U of (T, V) is called a *minimal block* if for any block U' of (T, V) the implication $[U' \subseteq U \Rightarrow U' = U]$ holds.

If $\{V_i \mid i=1, 2, \cdots, n\}$ is the set of all distinct minimal blocks of a t-semigroup (T, V), then $V = \bigcup_{i=1}^{n} V_i$ and $V_i \cap V_j = \varnothing$ if $i \neq j$, where $\varnothing$ is the empty set (cf., [11]). Each f in T induces a transformation f_i on V_i (i.e., f_i is the restriction of f to V_i). The t-semigroup (T_i, V_i), $T_i = \{f_i \mid f \in T\}$, is called a *constituent* of T on V_i.

Let (T, V) be a t-semigroup. If there exists an element $s_0 \in V$ such that $(s_0)T = V$, then (T, V) is called a *strictly cyclic t-semigroup*. We call (T, V) a *semicyclic t-semigroup* if for every minimal block V_i of (T, V) the constituent (T_i, V_i) is strictly cyclic.

The t-semigroups (S, U) and (T, V) are equivalent if there exist an isomorphism $\alpha: S \to T$ and a bijection $\xi: U \to V$ such that $(uf)\xi = (u\xi) \cdot (f\alpha)$ for all $u \in U$ and all $f \in S$.

Let (S, V) be a t-semigroup and let $\mathscr{T}_V$ be the full transformation semigroup on V, then

$$\{f \in \mathscr{T}_V \mid fg = gf \text{ for all } g \in S\}$$

is called a *centralizer* of (S, V).

Let S be a semigroup. If D is a subset of S, we denote by $\langle D \rangle$ the subsemigroup of S generated by the elements of D. Let ρ be a relation on S. If $(a, b) \in \rho$, then we shall write $a\rho b$. We use the symbol ε for the equality relation on S. If ρ is a right congruence on S, we denote by a_ρ the ρ-class containing an element $a \in S$. The set of all such ρ-classes is denoted by S_ρ. An element e in S is called a left [resp. right] identity modulo ρ if $ea\rho a$ [resp. $ae\rho a$] for all $a \in S$. A right congruence ρ is said to be *modular* if there exists a left identity modulo ρ.

Let ρ be a right congruence on a semigroup S. For each x in S we define the transformation $(x)\pi_\rho$ on S_ρ by

$$(x)\pi_\rho: a_\rho \to (ax)_\rho, \qquad a_\rho \in S_\rho.$$

The correspondence $\pi_\rho: x \to (x)\pi_\rho$ establishes a homomorphism from S onto a t-semigroup $(S)\pi_\rho$ on S_ρ. If S has a left identity modulo ρ, then $((S)\pi_\rho, S_\rho)$ is a strictly cyclic t-semigroup. Conversely, if (T, V) is strictly cyclic, then there exists a modular right congruence ρ on T such that (T, V) is equivalent to $((T)\pi_\rho, T_\rho)$ ([13]).

2. Normalizers of right congruences

In Section 4 we define the (R, S)-matrix for a semigroup S and an n-tuple R of right congruences on S. For this purpose, in this section, we introduce the notion of normalizers of ordered pairs of right congruences and establish some of their properties. The definition and the fundamental properties of the normalizer $N(\rho)$ of a right congruence ρ on a semigroup have already been given in [8] and [2, p. 279]. First of all we recall the definition of a right conguence ρx ([2, p. 261]).

Definition 2.1. Let ρ be a right congruence on a semigroup S and x be an element of S, then the right conguence ρx on S is defined by for a, b in S, $a(\rho x)b$ if and only if $(xa)\rho(xb)$.

Definition 2.2. Let ρ and σ be right congruences on a semigroup S. We define the normalizer $N(\rho, \sigma)$ of the ordered pair (ρ, σ) to be the set of all $x \in S$ such that $\rho \subseteq \sigma x$:

$$N(\rho, \sigma) = \{x \in S \mid \rho \subseteq \sigma x\} .$$

We shall write $N(\rho)$ instead of $N(\rho, \rho)$ and call $N(\rho)$ a normalizer of ρ.

PROPOSITION 2.1. *Let ρ and σ be right congruences on a semigroup S. Then*

(1) *If $N(\rho, \sigma)$ is not the empty set, then $N(\rho, \sigma)$ is a union of σ-classes.*

(2) *If $\rho \supseteq \sigma$, then $N(\rho) \supseteq N(\rho, \sigma)$.*

(3) *If $\rho \subseteq \sigma$, then $(N(\rho) \cup N(\sigma)) \subseteq N(\rho, \sigma)$.*

(4) *If $\rho \subseteq \sigma$ and there exists a left congruence τ on S such that $\rho \subseteq \tau \subseteq \sigma$, then $N(\rho, \sigma) = S$. In particular, $N(\varepsilon, \rho) = S$ for all right congruences ρ on S.*

(5) *$N(\rho)$ is a subsemigroup of S. If S is a monoid, then $N(\rho)$ is a submonoid of S.*

(6) *The restriction of ρ to $N(\rho)$ is a congruence on $N(\rho)$.*

The assertion (5) and (6) will be found in [2, p. 279].

If X and Y are subsets of a semigroup S, we write $X \circ Y$ for the subset consisting of all products xy with $x \in X$ and $y \in Y$.

PROPOSITION 2.2. *Let ρ and σ be right congruences on a semigroup S. Then x is an elements of $N(\rho, \sigma)$ if and only if $x_\sigma \circ y_\rho \subseteq (xy)_\sigma$ for all $y \in S$.*

Definition 2.3. Let ρ and σ be right congruences on a semigroup S. If $x \in N(\rho, \sigma)$, then the mapping $\xi_{\rho\sigma}(x)$ from S_ρ to S_σ is defined as follows:

$$\xi_{\rho\sigma}(x): y_\rho \rightarrow (xy)_\sigma , \qquad y_\rho \in S_\rho .$$

PROPOSITION 2.3. *Let ρ, σ and τ be right congruences on a semigroup S. Then*

(1) *If $x \in N(\rho, \sigma)$ and $y \in N(\sigma, \tau)$, then $yx \in N(\rho, \tau)$ and*

$$\xi_{\rho\sigma}(x) \cdot \xi_{\sigma\tau}(y) = \xi_{\rho\tau}(yx) .$$

(2) *If $x \in N(\rho, \sigma)$ and $y_\sigma = x_\sigma$, then $\xi_{\rho\sigma}(x) = \xi_{\rho\sigma}(y)$.*

(3) *Let x, $y \in N(\rho, \sigma)$. If there exists a right identity modulo σ then $\xi_{\rho\sigma}(x) = \xi_{\rho\sigma}(y)$ if and only if $x_\sigma = y_\sigma$.*

(4) *If S is a monoid and x, $y \in N(\rho, \sigma)$, then $\xi_{\rho\sigma}(x) = \xi_{\rho\sigma}(y)$ if and only if $x_\sigma = y_\sigma$.*

Definition 2.4. Let ρ and σ be two right congruences on a semigroup S and let e be a left identity modulo ρ. Then we define the subset $N_e(\rho, \sigma)$ of $N(\rho, \sigma)$ to be the set of all $x \in N(\rho, \sigma)$ such that $xe\sigma x$.

If $\rho = \sigma$, then we shall write $N_e(\rho)$ instead of $N_e(\rho, \rho)$.

PROPOSITION 2.4. *Let ρ and σ be two right congruences on a semigroup S and let e be a left identity modulo ρ. Then*

(1) *$N_e(\rho, \sigma)$ is a union of σ-classes.*

(2) *$N_e(\rho)$ is a left ideal of $N(\rho)$* [2, p. 279].

(3) *If $x \in N(\rho, \sigma)$, then $xe \in N_e(\rho, \sigma)$ and $\xi_{\rho\sigma}(x)=\xi_{\rho\sigma}(xe)$.*

(4) *If $x, y \in N_e(\rho, \sigma)$, then $\xi_{\rho\sigma}(x)=\xi_{\rho\sigma}(y)$ if and only if $x_\sigma=y_\sigma$.*

(5) *If, in addition, e is also a right identity modulo σ, then $N_e(\rho, \sigma)=N(\rho, \sigma)$.*

Proof. (1). Let $x \in N_e(\rho, \sigma)$ and $y \in x_\sigma$, then $y \in N(\rho, \sigma)$ by (1) of Proposition 2.1. Since $ye\sigma xe$, $xe\sigma x$ and $x\sigma y$, we have that $ye\sigma y$ and $y \in N_e(\rho, \sigma)$.

(3). Let $x \in N(\rho, \sigma)$ and $a, b \in S$. Then

$$a\rho b \Rightarrow ea\rho eb \Rightarrow ea(\sigma x)eb \Rightarrow xea\sigma xeb \Rightarrow a(\sigma xe)b\,.$$

Hence $xe \in N(\rho, \sigma)$. $e^2\rho e$ implies $(xe)e\sigma xe$, and so $xe \in N_e(\rho, \sigma)$. Moreover, for any $a_\rho \in S_\rho$,

$$(a_\rho)\cdot\xi_{\rho\sigma}(x)=((ea)_\rho)\cdot\xi_{\rho\sigma}(x)=(xea)_\sigma=(a_\rho)\cdot\xi_{\rho\sigma}(xe)\,.$$

(4). If $x_\sigma=y_\sigma$, then $\xi_{\rho\sigma}(x)=\xi_{\rho\sigma}(y)$ by Proposition 2.3.

Conversely, if $\xi_{\rho\sigma}(x)=\xi_{\rho\sigma}(y)$, then $(e_\rho)\cdot\xi_{\rho\sigma}(x)=(e_\rho)\cdot\xi_{\rho\sigma}(y)$. Hence $(xe)_\sigma=(ye)_\sigma$. Since $xe\sigma x$ and $ye\sigma y$, we have $x_\sigma=y_\sigma$.

(5). If $x \in N(\rho, \sigma)$, then $xe \in N_e(\rho, \sigma)$ from (3). Since e is also a right identity modulo σ, we have $xe\sigma x$. Therefore we must have that $x \in (xe)_\sigma \subseteq N_e(\rho, \sigma)$.

Q.E.D.

THEOREM 2.1. *Let ρ and σ be two right congruences on a semigroup S and f be a mapping from S_ρ to S_σ. If ρ is modular, then for all $y \in S$,*

$$(y)\pi_\rho\cdot f=f\cdot(y)\pi_\sigma$$

holds if and only if $f=\xi_{\rho\sigma}(x)$ for some $x \in N_e(\rho, \sigma)$, where e is a left identity of S modulo ρ.

Proof. It is easy to see that if $x \in N(\rho, \sigma)$ and $f=\xi_{\rho\sigma}(x)$, then $(y)\pi_\rho\cdot f=f\cdot(y)\pi_\sigma$ for all $y \in S$. Hence it suffices to treat the "only if" part. Assume that $(e_\rho)f=z_\sigma$, then for any $y \in S$,

$$(y_\rho)f=((e_\rho)\cdot(y)\pi_\rho)f=((e_\rho)f)\cdot(y)\pi_\sigma=(z_\sigma)\cdot(y)\pi_\sigma=(zy)_\sigma\,.$$

Since f is a mapping of S_ρ, the equality $w_\rho=y_\rho$ implies $(w_\rho)f=(y_\rho)f$ and so $(zw)_\sigma=(zy)_\sigma$. Hence $z \in N(\rho, \sigma)$ and $f=\xi_{\rho\sigma}(z)$. This proves that $f=\xi_{\rho\sigma}(ze)$ and $ze \in N_e(\rho, \sigma)$ by (3) of Proposition 2.4. Q.E.D.

PROPOSITION 2.5. *Let ρ and σ be two right congruences on a semigroup S, and let $C_S(\pi_\rho, \pi_\sigma)$ be the set of all mappings f from S_ρ to S_σ such that $(x)\pi_\rho\cdot f=f\cdot(x)\pi_\sigma$ for all $x \in S$. If ρ is modular, then we have*

(1) $|C_S(\pi_\rho, \pi_\sigma)| \leqq |S_\sigma|$.

(2) *In particular, the order of the centralizer of* $((S)\pi_\rho, S_\rho)$ *is less than or equal to* $|S_\rho|$.

(3) *Let e be a left identity modulo* ρ. *If S is commutative and* $\rho \subseteq \sigma$, *in addition if e is also a left identity (equivalently a right identity) modulo* σ, *then* $|C_S(\pi_\rho, \pi_\sigma)| = |S_\sigma|$.

(4) *If S is commutative, then* $((S)\pi_\rho, S_\rho)$ *is its own centralizer.*

Proof. (1). This is true by Theorem 2.1 and Proposition 2.4.

(2). Follows from (1).

(3). Since ρ is a congruence, $N(\rho, \sigma) = S$ by (4) of Proposition 2.1. From (5) of Proposition 2.4 we have $N_e(\rho, \sigma) = S$. Since $|C_S(\pi_\rho, \pi_\sigma)|$ is equal to the number of σ-classes of $N_e(\rho, \sigma)$, we have (3).

(4). From (3) we have that $|C_S(\pi_\rho, \pi_\rho)| = |S_\rho|$. Since $(S)\pi_\rho$ is commutative, $C_S(\pi_\rho, \pi_\sigma) \supseteq (S)\pi_\rho$. Thus

$$|S_\rho| = |C_S(\pi_\rho, \pi_\rho)| \geqq |((S)\pi_\rho, S_\rho)| \geqq |S_\rho| . \qquad \text{Q.E.D.}$$

It is known that $N_e(\rho)/\rho$ is anti-isomorphic to the centralizer of $((S)\pi_\rho, S_\rho)$ (cf. [2, Theorem 11.28]). The assertions (2) and (4) of Proposition 2.5 will be found, respectively, in [8] and [4, Theorem 5.4.1] in terms of endomorphism semigroups of cyclic automata.

3. Semicyclic *t*-semigroups

Let S be a semigroup and V be a finite set. A homomorphism α from S into the full transformation semigroup on V is called a representation of S by a t-semigroup on V. If α is a bijection, then α is said to be faithful. We say α is a semicyclic [resp. strictly cyclic] representation if $(S)\alpha$ is a semicyclic [resp. strictly cyclic] t-semigroup.

Tully [13] presented the following result:

A necessary and sufficient condition for a semigroup S to have a strictly cyclic faithful representation is that there exists a modular right congruence ρ such that ρ contains no left congruence except for the equality relation.

In this section we shall consider the above result.

Definition 3.1. Let $R = (\rho_1, \rho_2, \cdots, \rho_n)$ be an n-tuple of right congruence on a semigroup S. The condition

$$\bigcap_{x \in S} \bigcap_{i=1}^{n} \rho_i x = \varepsilon$$

is called 'Separation condition' or 'SP-condition' simply.

PROPOSITION 3.1. *Let* ρ *be a right congruence on a semigroup S. If* ρ *is modular, then the following two conditions are equivalent.*

(1) $\bigcap_{x \in S} \rho x$ *is the equality relation* ε. *In other words,* $R = (\rho)$ *satisfies the SP-condition.*

(2) ρ *contains no left congruence except for the equality relation* ε.

Proof. (2) ⇒ (1). It follows from Lemma 3 of [9]. Q.E.D.

Definition 3.2. Let $R=(\rho_1, \rho_2, \cdots, \rho_n)$ be an n-tuple of right congruences on a semigrouop S, and let 0 be a symbol such that $0 \notin \Delta$, where $\Delta=\bigcup_{i=1}^{n} S_{\rho_i}$. If a vector (X_i) $(1 \leqq i \leqq n)$ satisfies the following conditions, then (X_i) is called an (R, S)-vector:

(1) For each i $(1 \leqq i \leqq n)$, the component X_i is an element of the set $S_{\rho_i} \cup \{0\}$.

(2) There exists a unique number k $(1 \leqq k \leqq n)$ such that $X_k \neq 0$.

By $V(R, S)$ we denote the set of all (R, S)-vectors. From the definition, an (R, S)-vector is a monomial vector and the cardinality of $V(R, S)$ is equal to $\sum_{i=1}^{n} |S_{\rho_i}|$.

Definition 3.3. Let $R=(\rho_1, \rho_2, \cdots, \rho_n)$ be an n-tuple of right congruences on a semigroup S. For each $x \in S$ we define the transformation $(x)\pi_R$ on $V(R, S)$ by

$$(x)\pi_R : (0, \cdots, a_{\rho_i}, \cdots, 0) \rightarrow (0, \cdots, (ax)_{\rho_i}, \cdots, 0) .$$

PROPOSITION 3.2. *If (S, V) is a semicyclic t-semigroup, then there exists an n-tuple $R=(\rho_1, \rho_2, \cdots, \rho_n)$ of modular right congruences on S, such that R satisfies the SP-condition, and such that the t-semigroup $((S)\pi_R, V(R, S))$ is equivalent to (S, V).*

Proof. Let $\{V_i \mid i=1, 2, \cdots, n\}$ be the set of all minimal blocks of (S, V). Then $V=\bigcup_{i=1}^{n} V_i$ and for each V_i there exists an element $s_{i0} \in V_i$ such that $(s_{i0})S=V_i$. The relation ρ_i $(1 \leqq i \leqq n)$ on S is defined by for $x, y \in S$,

$$x \rho_i y \quad \Leftrightarrow \quad (s_{i0})x=(s_{i0})y .$$

This ρ_i is a modular right congruence on S. Let $a, b \in S$. If

$$a\left(\bigcap_{x \in S} \bigcap_{i=1}^{n} \rho_i x\right) b ,$$

then we have that $xa\rho_i xb$ for all $x \in S$ and all i $(1 \leqq i \leqq n)$. This yields that $((s_{i0})x)a=((s_{i0})x)b$ for all $x \in S$ and all i $(1 \leqq i \leqq n)$. Hence $a=b$.

Put $V_i=\{s_{ij} \mid j=0, 1, \cdots, |V_i|-1\}$, and define the mapping α from V to $V(R, S)$ by

$$\alpha : s_{ij} \rightarrow (0, \cdots, x_{\rho_i}, \cdots, 0) \qquad \text{if} \quad (s_{i0})x=s_{ij} .$$

Then α is a bijection. $\pi_R : S \rightarrow (S)\pi_R$ is an isomorphism and $(s)x\alpha=(s\alpha)\cdot(x)\pi_R$ for all $s \in V$ and all $x \in S$. Q.E.D.

By an argument similar to the previous one we have the following theorem.

THEOREM 3.1. *A necessary and sufficient condition for a semigroup S to have a semicyclic faithful representation is that there exists an n-tuple $R=(\rho_1, \rho_2, \cdots, \rho_n)$ of modular right congruences on S such that R satisfies the SP-condition.*

4. (*R*, *S*)-matrices

In the next section we shall give a representation of centralizers of semicyclic t-monoid. For this purpose, in this section, we define the (R, S)-matrix.

Notation. Let S be a semigroup and $R=(\rho_1, \rho_2, \cdots, \rho_n)$ be an n-tuple of right congruences on S. If no confusion will arise, we write $\xi_{ij}(x)$ instead of $\xi_{\rho_i\rho_j}(x)$ for all $x\in N(\rho_i, \rho_j)$ $(i, j=1, 2, \cdots, n)$.

Definition 4.1. Let $R=(\rho_1, \rho_2, \cdots, \rho_n)$ be an n-tuple of right congruences on a semigroup S. We set

$$\Gamma'=\{\xi_{ij}(x)\,|\,x\in N(\rho_i, \rho_j)\ \ i, j=1, 2, \cdots, n\}.$$

Let 0 be a symbol such that $0\notin\Gamma'\cup(\bigcup_{i=1}^{n}S_{\rho_i})$. In the set $\Gamma=\Gamma'\cup\{0\}$, we introduce the operation $(\cdot)$ as follows:

(1) $0\cdot\xi_{ij}(x)=\xi_{ij}(x)\cdot 0=0$ and $0\cdot 0=0$.

(2) $\xi_{ij}(x)\cdot\xi_{pq}(y)=\begin{cases}\xi_{iq}(yx) & \text{if } j=p,\\ 0 & \text{if } j\neq p.\end{cases}$

Definition 4.2. Let $R=(\rho_1, \rho_2, \cdots, \rho_n)$ be an n-tuple of right congruences on a semigroup S. An $n\times n$ matrix $X=(X_{ij})$ is called an (R, S)-matrix if it satisfies the following conditions:

(1) For all i and j $(i, j=1, 2, \cdots, n)$, $X_{ij}\in\Gamma$.

(2) For each i $(i=1, 2, \cdots, n)$, there exists a unique number k such that $X_{ik}\neq 0$ (i.e., X is a row-monomial matrix over Γ).

(3) If $X_{ik}\neq 0$, then $X_{ik}=\xi_{ik}(x)$ for some $x\in N(\rho_i, \rho_k)$.

By $M(R, S)$ we denote the set of all (R, S)-matrices.

Definition 4.3. Let $X=(X_{ij})$ and $Y=(Y_{ij})$ be two elements of $M(R, S)$. The product XY is an (R, S)-matrix $Z=(Z_{ij})$, where $Z_{ij}=\sum_{k=1}^{n}X_{ik}Y_{kj}$ for all i and j $(1\leqq i, j\leqq n)$.

Remark. $0+X_{ik}=X_{ik}+0=X_{ik}$ for all $X_{ik}\in\Gamma$. The elements of $M(R, S)$ might be regarded as a matrix over the semigroup ring $K[\Gamma]$, where K is a field of characteristic zero.

Definition 4.4. Let $X=(X_i)$ be an (R, S)-vector and let $Y=(Y_{ij})$ be an (R, S)-matrix. The product XY is an (R, S)-vector $Z=(Z_i)$, where $Z_i=\sum_{k=1}^{n}X_kY_{ki}$ for all i $(1\leqq i\leqq n)$ and

(1) $X_kY_{ki}=(x_{\rho_k})\xi_{ki}(y)=(yx)_{\rho_i}$ if $X_k=x_{\rho_k}$ and $Y_{ki}=\xi_{ki}(y)$.

(2) $X_kY_{ki}=0$ if $X_k=0$ or $Y_{ki}=0$.

(3) $0+X_j=X_j+0=X_j$ for all X_j in S_{ρ_j}.

To make the definitions clearer we include the following example.

Example. Let S be a monoid such that the multiplication table for S is

	e	a	b	c
e	e	a	b	c
a	a	a	b	a
b	b	a	b	a
c	c	a	b	c .

The right congruences ρ and σ are given by the following partitions of S:

$$\rho:\ e_\rho=\{e, b\}\ a_\rho=\{a, c\}\,, \qquad \sigma:\ e_\sigma=\{e, c\}\ a_\sigma=\{a, b\}\,.$$

Then

$$N(\rho)=\{e, b\}\,, \qquad \xi_{\rho\rho}(e)=\begin{pmatrix} e_\rho & a_\rho \\ e_\rho & a_\rho \end{pmatrix}.$$

$$N(\rho, \sigma)=\{a, b\}\,, \qquad \xi_{\rho\sigma}(a)=\begin{pmatrix} e_\rho & a_\rho \\ a_\sigma & a_\sigma \end{pmatrix}.$$

$$N(\sigma)=S\,, \qquad \xi_{\sigma\sigma}(e)=\begin{pmatrix} e_\sigma & a_\sigma \\ e_\sigma & a_\sigma \end{pmatrix}, \qquad \xi_{\sigma\sigma}(a)=\begin{pmatrix} e_\sigma & a_\sigma \\ a_\sigma & a_\sigma \end{pmatrix}.$$

$$N(\sigma, \rho)=\phi\,.$$

Let $R=(\rho, \sigma)$, then

$$V(R, S)=\{(e_\rho, 0)\ (a_\rho, 0)\ (0, e_\sigma)\ (0, a_\sigma)\}$$

and

$$M(R, S)=\left\{\begin{bmatrix} \xi_{\rho\rho}(e) & 0 \\ 0 & \xi_{\sigma\sigma}(e) \end{bmatrix}, \begin{bmatrix} \xi_{\rho\rho}(e) & 0 \\ 0 & \xi_{\sigma\sigma}(a) \end{bmatrix}, \begin{bmatrix} 0 & \xi_{\rho\sigma}(a) \\ 0 & \xi_{\sigma\sigma}(e) \end{bmatrix}, \begin{bmatrix} 0 & \xi_{\rho\sigma}(a) \\ 0 & \xi_{\sigma\sigma}(a) \end{bmatrix}\right\}.$$

$$\begin{bmatrix} \xi_{\rho\rho}(e) & 0 \\ 0 & \xi_{\sigma\sigma}(a) \end{bmatrix} \cdot \begin{bmatrix} 0 & \xi_{\rho\sigma}(a) \\ 0 & \xi_{\sigma\sigma}(a) \end{bmatrix} = \begin{bmatrix} 0 & \xi_{\rho\sigma}(a) \\ 0 & \xi_{\sigma\sigma}(a) \end{bmatrix}$$

$$(e_\rho, 0)\begin{bmatrix} 0 & \xi_{\rho\sigma}(a) \\ 0 & \xi_{\sigma\sigma}(a) \end{bmatrix}=(0, a_\sigma)\,.$$

5. Representation of centralizers of semicyclic *t*-monoid

In this sections we show the main theorem of this paper.

THEOREM 5.1. *If S is a monoid and $R=(\rho_1, \rho_2, \cdots, \rho_n)$ is an n-tuple of right congruences on S, then the t-monoid $M(R, S)$ on $V(R, S)$ is the centralizer of the t-monoid $(S)\pi_R$ on $V(R, S)$.*

Proof. Let C be the centralizer of $((S)\pi_R, V(R, S))$. We show that $M(R, S) \subseteq C$. To prove this, take an element

$$(0, \cdots, a_{\rho_i}, \cdots, 0) \in V(R, S)$$

and an element $X = (X_{ij}) \in M(R, S)$. If $X_{ik} = \xi_{ik}(x)$ for some $x \in N(\rho_i, \rho_k)$, then we have that for all $y \in S$,

$$\begin{aligned}((0, \cdots, a_{\rho_i}, \cdots, 0)X) \cdot (y)\pi_R &= (0, \cdots, (xa)_{\rho_k}, \cdots, 0) \cdot (y)\pi_R \\ &= (0, \cdots, (xay)_{\rho_k}, \cdots, 0) \\ &= (0, \cdots, (ay)_{\rho_i}, \cdots, 0)X \\ &= ((0, \cdots, a_{\rho_i}, \cdots, 0) \cdot (y)\pi_R)X .\end{aligned}$$

Thus we obtain $M(R, S) \subseteq C$.

Next, we shall show that the reverse inclusion holds. Let e be an identity of S and let f be an element of C. If

$$(0, \cdots, e_{\rho_i}, \cdots, 0)f = (0, \cdots, x_{\rho_j}, \cdots, 0) ,$$

then for any $a \in S$ we have

$$\begin{aligned}(0, \cdots, a_{\rho_i}, \cdots, 0)f &= ((0, \cdots, e_{\rho_i}, \cdots, 0) \cdot (a)\pi_R)f \\ &= ((0, \cdots, e_{\rho_i}, \cdots, 0)f) \cdot (a)\pi_R \\ &= (0, \cdots, (xa)_{\rho_j}, \cdots, 0) .\end{aligned}$$

If a and b are elements in S such that $a\rho_i b$, then

$$\begin{aligned}(0, \cdots, (xa)_{\rho_j}, \cdots, 0) &= (0, \cdots, a_{\rho_i}, \cdots, 0)f \\ &= (0, \cdots, b_{\rho_i}, \cdots, 0)f \\ &= (0, \cdots, (xb)_{\rho_j}, \cdots, 0) .\end{aligned}$$

Therefore if $a\rho_i b$, then $xa\rho_j xb$ and $x \in N(\rho_i, \rho_j)$. If $x_{\rho_j} = y_{\rho_j}$, then $\xi_{ij}(x) = \xi_{ij}(y)$ from (4) of Proposition 2.3. Thus f is expressible in the form

$$(0, \cdots, a_{\rho_i}, \cdots, 0)f = (0, \cdots, \underset{\substack{j\text{-th} \\ \downarrow}}{(a_{\rho_i})\xi_{ij}(x)}, \cdots, 0) .$$

For each $f \in C$ we consider the $n \times n$ matrix $f_M = (X_{ij})$ as follows:

$$X_{ij} = \begin{cases} \xi_{ij}(x) & \text{if } (0, \cdots, e_{\rho_i}, \cdots, 0)f = (0, \cdots, x_{\rho_j}, \cdots, 0) \\ 0 & \text{otherwise.} \end{cases}$$

Then f_M is an (R, S)-matrix and $vf = vf_M$ for all $v \in V(R, S)$. This means that $f = f_M$ as a transformation and hence $M(R, S)$ contains C. Q.E.D.

Let S be a t-submonoid of the centralizer of a t-semigroup (T, V), then T is a subsemigroup of the centralizer of (S, V). If (S, V) is semicyclic, then (S, V) is equivalent to $(S)\pi_R$ on $V(R, S)$ by Proposition 3.2, where R is some n-tuple of right congruences on S. Therefore the centralizer of (S, V) is equivalent to $(M(R, S), V(R, S))$ by Theorem 5.1. Thus (T, V) is equivalent to a subsemigroup of $M(R, S)$. Therefore we have the following theorem.

THEOREM 5.2. *Let (S, V) be a submonoid of the centralizer of a t-semigroup (T, V). If (S, V) is semicyclic, then there exists an n-tuple $R=(\rho_1, \rho_2, \cdots, \rho_n)$ of right congruences on S, such that R satisfies the SP-condition, and such that (T, V) is equivalent to a subsemigroup of $(M(R, S), V(R, S))$.*

Since any permutation group (G, V) is a semicyclic t-monoid, from Theorem 5.2 we obtain the next theorem.

THEOREM 5.3. *If a permutation group (G, V) is a subgroup of the centralizer of a t-semigroup (T, V), then there exists an n-tuple R of right congruences on G, such that R satisfies the SP-condition, and such that (T, V) is equivalent to a subsemigroup of $(M(R, G), V(R, G))$.*

Let G be a group and H be its subgroup. Define the relation ρ on G by for $a, b \in G$, $a\rho b$ if and only if $Ha=Hb$. Then ρ is a right congruence on G. Conversely, all right congruences on G are obtained from the decomposition of G into cosets with respect to some subgroup H. Consequently, the properties of right congruences on a group G can be written in terms of subgroup of G. We list those correspondences:

$$R=(\rho_1, \rho_2, \cdots, \rho_n) \quad \leftrightarrow \quad R=(H_1, H_2, \cdots, H_n),$$

where H_i is a subgroup of G which corresponds to a right congruence ρ_i on G $(1 \leqq i \leqq n)$.

$$a_{\rho_i}, \quad (a \in G) \quad \leftrightarrow \quad H_i a, \quad (a \in G)$$

$$(0, \cdots, a_{\rho_i}, \cdots, 0) \in V(R, G) \quad \leftrightarrow \quad (0, \cdots, H_i a, \cdots, 0)$$

$$\rho_i x, \quad (x \in G) \quad \leftrightarrow \quad x^{-1} H_i x, \quad (x \in G)$$

$$N(\rho_i, \rho_j)=\{x \in G \mid \rho_i \subseteq \rho_j x\} \quad \leftrightarrow \quad \{x \in G \mid H_i \subseteq x^{-1} H_j x\}$$

$$\bigcap_{x \in G} \bigcap_{i=1}^{n} \rho_i x=\varepsilon \quad \leftrightarrow \quad \bigcap_{x \in G} \bigcap_{i=1}^{n} x^{-1} H_i x=\{e\},$$

where e is the identity of G.

$$x_{\rho_j} \circ y_{\rho_i} \subseteq (xy)_{\rho_j} \quad \leftrightarrow \quad (H_j x)(H_i y) \subseteq H_j xy$$

$$\xi_{ij}(x): a_{\rho_i} \to (xa)_{\rho_j} \quad \leftrightarrow \quad *(H_j x): H_i a \to H_j xa$$

$$i \cdots \begin{bmatrix} \cdots & \xi_{ij}(x) \\ & \vdots \end{bmatrix} \in M(R, G) \quad \leftrightarrow \quad i \cdots \begin{bmatrix} \cdots & *(H_j x) \\ & \vdots \end{bmatrix} .$$
$$\qquad\qquad j \qquad\qquad\qquad\qquad\qquad\qquad j$$

The (R, G)-vector $(0, \cdots, H_i a, \cdots, 0)$ and (R, G)-matrix correspond to "the generalized group-vector" and "the generalized group-matrix" of Ito [6], respectively. Theorem 5.3 is essentially in [6], though it is stated in terms of automorphism groups of automata.

6. Centralizers of semiregular t-monoid

Let S be a semigroup and $R=(\rho_1, \rho_2, \cdots, \rho_n)$ be an n-tuple of right congruences on S. In this section, for the special case in which S is a monoid and all ρ_i in R are the equality relation ε on S we consider the monoid $M(R, S)$ and its submonoids.

To each $x \in S$ we assign the transformation $*x$: $y \to xy$ $(y \in S)$. Then $*S$, where $*S=\{*x \mid x \in S\}$, forms a t-monoid on S. If we will not make a distinction between a one-element set and the single element it contains, then $(*S, S)$ is regarded as the centralizer of $((S)\pi_\varepsilon, S_\varepsilon)$.

We recall that if S is a monoid and $\rho_i=\varepsilon$ for all i $(1 \leqq i \leqq n)$, then

(1) $N(\rho_i, \rho_j)=S$ for all i and j $(1 \leqq i, j \leqq n)$.

(2) $a_{\rho_i}=a_\varepsilon=\{a\}=a$ for all $a \in S$ and i $(1 \leqq i \leqq n)$.

(3) $\xi_{ij}(x)=\xi_{ij}(y)$ if and only if $x=y$.

(4) $\xi_{ij}(x)=*x$ for all $x \in S$ and i, j $(1 \leqq i, j \leqq n)$.

Therefore $M(R, S)$ is a set of all $n \times n$ row-monomial matrices over the semigroup $*S \cup \{0\}$, and $V(R, S)$ is a set of all monomial vectors of order n over the set $S \cup \{0\}$.

Notation. Let S be a monoid and n be a positive integer. By $M(n, *S)$ we denote the set of all $n \times n$ row-monomial matrices over $*S \cup \{0\}$, and by $V(n, S)$ we denote the set of all monomial vectors of order n over $S \cup \{0\}$.

Definition 6.1. Let S be a semigroup and n be a positive integer. For each $x \in S$ the transformation $(x)\pi_\varepsilon^n$ on $V(n, S)$ is defined by

$$(x)\pi_\varepsilon^n\colon (0, \cdots, y_i, \cdots, 0) \quad \to \quad (0, \cdots, y_i x, \cdots, 0)$$

for all $(0, \cdots, y_i, \cdots, 0) \in V(n, S)$.

Definition 6.2. Let (T, V) be a semicyclic t-monoid and let $\{V_1, V_2, \cdots, V_n\}$ be the set of all minimal blocks of (T, V). A semicyclic t-monoid (T, V) is said to be semiregular if $|T|=|V_i|$ for all $i=1, 2, \cdots, n$.

PROPOSITION 6.1. *Let (T, V) be a semiregular t-monoid with n minimal blocks. Then (T, V) is equivalent to $((T)\pi_\varepsilon^n, V(n, T))$.*

PROPOSITION 6.2. *Let S be a monoid and n be a positive integer, then $(M(n, *S), V(n, S))$ is the centralizer of $((S)\pi_\varepsilon^n, V(n, S))$.*

Before studying the t-monoid $M(n, *S)$ we describe the relationship between the t-monoid $M(n, *S)$ and the wreath product.

Let S be a monoid and (T, V) be a full transformation semigroup on the set $V=\{1, 2, \cdots, n\}$. The wreath product $(*S, S)wr(T, V)$ of $(*S, S)$ by (t, V) is a set

$$\{(f, t) \mid t \in T,\ f \text{ is a mapping from } V \text{ to } *S\}$$

with the multiplication

$$(f_1, t_1)(f_2, t_2)=(f_3, t_1 t_2),$$

where $(i)f_3=(i)f_1(it_1)f_2$ for all $i \in V$.
The action of $(*S, S)wr(T, V)$ on the set $S \times V$ is given by

$$((s, i))(f, t)=((s)\cdot(i)f, (i)t).$$

If $(f, t) \in (*S, S)wr(T, V)$, then define the matrix $M(f, t)$ by

$$M(f, t)_{ij}=\begin{cases}(i)f & \text{if } j=(i)t, \\ 0 & \text{otherwise}.\end{cases}$$

Then α: $(f, t) \to M(f, t)$ is an isomorphism from $(*S, S)wr(T, V)$ onto $M(n, *S)$ (Wells [14], Lallement [10, p. 86]). Define the mapping α: $S \times V \to V(n, S)$ by

$$\beta: (s, i) \to (0, \cdots, \underset{\substack{\uparrow \\ i\text{-th}}}{s}, \cdots, 0).$$

Then, by the morphism (α, β), $(*S, S)wr(T, V)$ on $S \times V$ is equivalent to $M(*S, S)$ on $V(n, S)$. As is well known, the order of $M(n, *S)$ is $(n|S|)^n$.

Now, let X be an element in $M(n, *S)$. By X_{ij} we denote the (i, j)-component of X. Let D be a subset of $M(n, *S)$, we put

$$\hat{D}_{ij}=\left(\bigcup_{X \in D}\{X_{ij}\}\right)-\{0\}$$

and

$$\hat{D}=\left(\bigcup_{X \in D}\bigcup_{i,j=1}^{n}\{X_{ij}\}\right)-\{0\}=\bigcup_{p,q=1}^{n}\hat{D}_{pq}.$$

PROPOSITION 6.3. *Let S be a monoid and n be a positive integer, in addition, let T be a subsemigroup of $M(n, *S)$. Then the t-semigroup $(T, V(n, S))$ is strictly cyclic if and only if there exists a number p such that $\hat{T}_{pq}=*S$ for all $q=1, 2, \cdots, n$.*

Proof. Proof of the "only if" part. Let V_i be the set of all vectors $(0, \cdots, x, \cdots, 0)$ with $x \in T$ in the i-th component and 0 otherwise. Since $(T, V(n, S))$ is strictly cyclic, there exist a number p and a vector

$$u=(0, \cdots, g, \cdots, 0) \in V_p$$

such that $(u)T=V(n, S)$. If $\hat{T}_{pq}\neq {}^*S$ for some q $(1\leqq q\leqq n)$, then $\{x \mid {}^*x\in \hat{T}_{pq}\}\neq S$. Since S is finite, $\{xg \mid {}^*x\in \hat{T}_{pq}\}$ is a proper subset of S. Consequently, if

$$a\in(S-\{xg \mid {}^*x\in \hat{T}_{pq}\})$$

and

$$v=(0, \cdots, a, \cdots, 0)\in V_q,$$

then $v\notin(u)T$. This is a contradiction.

Proof of the "if" part. Let e be the identity of S and let $u=(0, \cdots, e, \cdots, 0)\in V_p$. Since $\hat{T}_{pq}={}^*S$, for a given element $v=(0, \cdots, a, \cdots, 0)\in V_q$ there exists an element (X_{ij}) in T such that $X_{pq}={}^*a$ and $u(X_{ij})=v$. Q.E.D.

Suppose that $(T, V(n, S))$ is a strictly cyclic subsemigroup of $M(n, {}^*S)$. If T is generated by a subset D of T, then every element X in T can be written as a product of some elements of D. Thus each non-zero component ${}^*x$ of X is a product ${}^*x_1{}^*x_2\cdots{}^*x_k$, where ${}^*x_i\in\hat{D}$ $(1\leqq i\leqq k)$. Since $\hat{T}_{pq}$ is contained in the subsemigroup $\langle\hat{D}\rangle$ and ${}^*S=\hat{T}_{pq}$ by Proposition 6.3, the monoid ${}^*S$ is generated by $\hat{D}$. This means that $\hat{D}$ contains a generating system of ${}^*S$ and

$$|\hat{D}|\geqq\mu(S),$$

$$\text{where } \mu(S)=\min\{|U| \,|\, U\subseteq S, \langle U\rangle=S\}.$$

PROPOSITION 6.4. *Let (S, V) be the centralizer of a strictly cyclic t-semigroup (T, V). If (S, V) is semiregular, then*

$$|D|\geqq\frac{\mu(S)\cdot|S|}{|V|},$$

where D is a generating system of T and $\mu(S)=\min\{|U| \,|\, U\subseteq S, \langle U\rangle=S\}$.

Proof. If (S, V) has n minimal blocks, then (S, V) is equivalent to $((S)\pi_\varepsilon^n, V(n, S))$ by Proposition 6.1. From Proposition 6.2 the centralizer of (S, V) is equivalent to $M(n, {}^*S)$ on $V(n, S)$. Thus (T, V) is equivalent to a strictly cyclic t-semigroup of $M(n, {}^*S)$. Suppose that $H\subseteq M(n, {}^*S)$ and $(H, V(n, S))$ is equivalent to (T, V). If D is a generating system of T, then there exists a generating system D_H of H such that $|D|=|D_H|$. If $X\in M(n, {}^*S)$, then the number of nonzero components in X is n because X is a row-monomial matrix. Since $|\hat{D}_H|\geqq\mu(S)$, we have

$$\mu(S)\leqq|\hat{D}_H|=\left|\left(\bigcup_{X\in D_H}\bigcup_{i,j=1}^{n}\{X_{ij}\}\right)-\{0\}\right|\leqq|D_H|\cdot n.$$

It implies that $\mu(S)\cdot|S|\leqq n\cdot|S|\cdot|D_H|=n\cdot|S|\cdot|D|$. Since $|V|=n\cdot|S|$, we obtain $\mu(S)\cdot|S|\leqq|V|\cdot|D|$. Q.E.D.

THEOREM 6.1. *For a given monoid S there exists a t-monoid (T, V) which satisfies the following conditions:*

(1) *(T, V) is strictly cyclic.*
(2) *The centralizer of (T, V) is isomorphic to S.*
(3) *T is generated as a semigroup by at most two elements.*

Proof. To each $x \in S$ we assign the transformation

$$x^*: a \to ax, \qquad (a \in S).$$

Then (S^*, S), $S^* = \{x^* \mid x \in S\}$, is the centralizer of $(^*S, S)$ because S is a monoid. It is obvious that $(^*S, S)$ satisfies the conditions (1) and (2) of our theorem. If S is generated by a single element (i.e., if S is cyclic), then $(^*S, S)$ satisfies the condition (3). Therefore we suppose that S is not cyclic semigroup.

Let D be a generating system of S such that $|D| = n$ $(n \geqq 2)$ and $D = \{a_1, a_2, \cdots, a_n\}$. Let $X = (X_{ij})$ be an element of $M(n, {}^*S)$ such that $X_{i1} = {}^*a_i$ $(1 \leqq i \leqq n,\ a_i \in D)$ and let $Y = (Y_{ij})$ be an element of $M(n, {}^*S)$ such that $Y_{i,(i)\alpha} = {}^*e$ $(1 \leqq i \leqq n)$, where e is an identity of S and α is a permutation such that $(i)\alpha = i+1$ for $i = 1, 2, \cdots, n-1$ and $(n)\alpha = 1$. Therefore

$$X = \begin{bmatrix} {}^*a_1 & 0 \cdots 0 \\ {}^*a_2 & 0 \cdots 0 \\ & \cdots \\ {}^*a_n & 0 \cdots 0 \end{bmatrix}, \qquad Y = \begin{bmatrix} 0 & {}^*e & 0 \cdots 0 \\ 0 & 0 & {}^*e \cdots 0 \\ & & \cdots \\ {}^*e & 0 & 0 \cdots 0 \end{bmatrix}.$$

Put $T = \langle X, Y \rangle$, then T is a submonoid of $M(n, {}^*S)$ because Y^n is an identity of $M(n, {}^*S)$. For any fixed k $(1 \leqq k \leqq n)$ we define the subset $T_{(k)}$ of T as follows:

$$T_{(k)} = \{Z \in T \mid Z = (Z_{ij}),\ Z_{ik} \neq 0 \text{ for all } i = 1, 2, \cdots, n\}.$$

Therefore, if $Z \in T_{(k)}$ and $r \neq k$, then the r-th column of Z is a zero vector.

(1). We shall show that $(T, V(n, S))$ is strictly cyclic. From Proposition 6.3 it suffices to show that $\hat{T}_{1q} = {}^*S$ for all $q = 1, 2, \cdots, n$. Since the set of all (1, 1)-components of $X, YX, \cdots, Y^{n-1}X$ is equal to $\{{}^*a \mid a \in D\}$, we have

$$\hat{T}_{11} \supseteq \{{}^*a \mid a \in D\}.$$

${}^*S$ is generated by $\{{}^*a \mid a \in D\}$ and $T_{(1)}$ contains the subsemigroup of T generated by the set $\{X, YX, \cdots, Y^{n-1}X\}$. Therefore, we have that $\hat{T}_{11} = {}^*S$, so that for an arbitrary ${}^*x$ in ${}^*S$ there exists an element $Z \in T_{(1)}$ such that (1, 1)-component is ${}^*x$. Consequently, by considering ZY^q $(1 \leqq q \leqq n)$ we have $\hat{T}_{1q} = {}^*S$ for all $q = 1, 2, \cdots, n$. Note that $Z = ZY^n \in T_{(1)}, ZY \in T_{(2)}, \cdots, ZY^{n-1} \in T_{(n)}$.

(2). Now we shall prove that the monoid S is isomorphic to the centralizer of $(T, V(n, S))$. Let ${}^*x$ be an arbitrary element of ${}^*S$, then there exists an element $Z = (Z_{ij}) \in T_{(q)}$ with $Z_{1q} = {}^*x$. Observe the matrices $YZ, Y^2Z, \cdots, Y^{n-1}Z$, then we have that for all ${}^*x \in {}^*S$ and for all integrers p and q $(1 \leqq p, q \leqq n)$ there exists an element $Z = (Z_{ij}) \in T_{(q)}$ such that $Z_{pq} = {}^*x$. Let V_i be the set of all vectors with $a \in S$ in the i-th component and 0 otherwise, and let $(C, V(n, S))$ be the centralizer of $(T, V(n, S))$. By

Proposition 6.2, C contains $(S)\pi^n_\varepsilon$. We need to show that $C \subseteq (S)\pi^n_\varepsilon$.

Let $f \in C$ and let $u=(0, \cdots, e, \cdots, 0) \in V_p$. Suppose that $uf=v$, where $v=(0, \cdots, y, \cdots, 0) \in V_q$. For any $x \in S$ there exists an element Z in $T_{(p)}$ such that its (p, p)-component is $*x$. Note that $(V_q)T_{(p)} \subseteq V_p$ for all p and q, then, since $fZ=Zf$ and for $w=(0, \cdots, x, \cdots, 0)$ in V_p the equality $w=uZ$ holds, we have

$$wf=uZf=ufZ=vZ \in V_p .$$

This means that $(V_p)f \subseteq V_p$ for $p=1, 2, \cdots, n$. Furthermore, we have that for all $x \in S$ and $(0, \cdots, x, \cdots, 0) \in V_p$,

$$(0, \cdots, x, \cdots, 0)f=(0, \cdots, xy, \cdots, 0) \in V_p$$

holds. Let

$$u_i=(0, \cdots, e, \cdots, 0) \in V_i$$

for $i=1, 2, \cdots, n$. Assume that

$$u_1 f=(a, \cdots, 0, \cdots, 0)$$

and

$$u_q f=(0, \cdots, b, \cdots, 0) \in V_q ,$$

where $q \neq 1$. Since the $(1, q)$-component of Y^{q-1} is $*e$ and $u_1 Y^{q-1}=u_q$, the equalities

$$\begin{aligned} u_q f=u_1 Y^{q-1} f &= u_1 f Y^{q-1} \\ &=(a, \cdots, 0, \cdots, 0) Y^{q-1} \\ &=(0, \cdots, a, \cdots, 0)\ (\in V_q) \end{aligned}$$

are valid. Thus we get $a=b$, that is, $f=(a)\pi^n_\varepsilon$. Therefore we conclude that $C \subseteq (S)\pi^n_\varepsilon$.
Q.E.D.

Remark. Let (T, V) be a t-semigroup. If for any pair of elements $s, t \in V$ there exists an element $x \in t$ such that $sx=t$, then (T, V) is called a transitive t-semigroup. The following result have already been given in [7] (in terms of automorphism groups of automata).

Let G be a group, then there exists a transitive t-semigroup (T, V) such that the centralizer of (T, V) is isomorphic to G, and such that T is generated by at most two elements.

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