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On a Fixed Point Theorem of Contractive Type

by

Cheng Chun CHANG

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The main purpose of this paper is to extend a fixed point theorem of Jungck [3] to a great general form.

In 1976, Jungck [3] proved a stronger form of the Banach contraction principle (motivated by a geometrical consideration).
His result is:

THEOREM 1. *Let (X, d) be a complete metric space. Let f and g be commuting continuous self-maps on X such that $g(X) \subset f(X)$. Suppose that there exists a constant $\alpha \in (0, 1)$ such that for all x, y in X,*

$$d(g(x), g(y)) \leqslant \alpha d(f(x), f(y)).$$

Then f and g have a unique common fixed point.

The following theorem is an extension of Theorem 1.

THEOREM 2. *Let (X, d) be a complete metric space. Let f, g and h be three self-maps on X such that*

(A) *f is continuous;*

(B) *f and g, f and h are commutative, $g(X) \subset f(X)$ and $h(X) \subset f(X)$;*

(C) *there exists a real-valued function ϕ: $[0, \infty)^5 \rightarrow [0, \infty)$*

such that

$$d(gx, hy) \leqslant \phi(d(fx, fy), d(fx, gx), d(fx, hy), d(fy, gx), d(fy, hy))$$

for all x, $y \in X$, where ϕ is upper semi-continuous from the right and non-decreasing in each coordinate variable such that $\phi(t, t, at, bt, t) < t$ for each $t > 0$ and $a \geqslant 0$, $b \geqslant 0$ with $a + b \leqslant 2$.

Then f, g and h have a unique common fixed point in X.

To prove Theorem 2 we need the following result [4].

LEMMA 1. *Suppose ψ: $[0, \infty) \rightarrow [0, \infty)$ is upper semi-continuous from right and non-decreasing. If for every $t > 0$, $\psi(t) < t$ then $\lim_{n\rightarrow\infty} \psi^n(t) = 0$, where $\psi^n(t)$ denotes the* *composition of $\psi(t)$ with itself n times.*

Proof of Theorem 2. Let $x_0 \in X$ be given. Construct a sequence $\{x_n\}$ defined by

$$gx_{2n}=fx_{2n+1},$$
$$hx_{2n+1}=fx_{2n+2}, \qquad n=0, 1, \cdots.$$

By (B), the iterates are well-defined. For simplicity of the notation, let

$$\alpha_n=d(fx_n, fx_{n+1}), \qquad n=0, 1, \cdots.$$

we claim that

$$\alpha_{2n+1} \leqslant \alpha_{2n} \qquad \text{for} \quad n=0, 1, \cdots.$$

Indeed, assuming that for some non-negative integer n,

$$\alpha_{2n+1} > \alpha_{2n}$$

Then by (C), we have

$$\begin{aligned}\alpha_{2n+1} &= d(fx_{2n+1}, fx_{2n+2}) \\ &= d(gx_{2n}, hx_{2n+1}) \\ &\leqslant \phi(d(fx_{2n}, fx_{2n+1}), d(fx_{2n}, gx_{2n}), \\ &\qquad d(fx_{2n}, hx_{2n+1}), d(fx_{2n+1}, gx_{2n}), d(fx_{2n+1}, hx_{2n+1})) \\ &\leqslant \phi(d(fx_{2n}, fx_{2n+1}), d(fx_{2n}, fx_{2n+1}), \\ &\qquad d(fx_{2n}, fx_{2n+1})+d(fx_{2n+1}, fx_{2n+2}), \\ &\qquad d(fx_{2n+1}, fx_{2n+1}), d(fx_{2n+1}, fx_{2n+2})) \\ &\leqslant \phi(\alpha_{2n}, \alpha_{2n}, \alpha_{2n}+\alpha_{2n+1}, 0, \alpha_{2n+1}) \\ &\leqslant \phi(\alpha_{2n+1}, \alpha_{2n+1}, 2\alpha_{2n+1}, 0, \alpha_{2n+1}) \\ &< \alpha_{2n+1}\end{aligned}$$

yielding a contradiction. A similar argument shows that $\alpha_{2n+2} \leqslant \alpha_{2n}$ for $n=0, 1, \cdots$. Thus $\{\alpha_n\}$ is decreasing. Since

$$\begin{aligned}\alpha_1 &= d(fx_1, fx_2) \\ &= d(gx_0, hx_1) \\ &\leqslant \phi(d(fx_0, fx_1), d(fx_0, gx_0), d(fx_0, hx_1), \\ &\qquad d(fx_1, gx_0), d(fx_1, hx_1)) \\ &\leqslant \phi(\alpha_0, \alpha_0, 2\alpha_0, 0, \alpha_0),\end{aligned}$$

it follows by induction that $\alpha_n \leqslant \psi^n(\alpha_0)$, where

$$\psi(t)=\mathrm{Max}\{\phi(t,\,t,\,2t,\,0,\,t),\ \phi(t,\,t,\,0,\,2t,\,t)\}$$

Thus by Lemma 1, we have

$$\lim_{n\to\infty}\alpha_n=0 \tag{1}$$

We show next that $\{fx_n\}$ is a Cauchy sequence. To show that $\{fx_n\}$ is Cauchy, in view of (1), it suffices to show that $\{fx_{2n}\}$ is Cauchy. Suppose that $\{fx_{2n}\}$ is not a Cauchy sequence. There is an $\varepsilon>0$ such that for each even integer $2\,k$, there are even integers $2m(k)$, $2n(k)$ such that

$$d(fx_{2m(k)},\,fx_{2n(k)})>\varepsilon \qquad \text{for} \quad 2m(k)>2n(k)>2k\,. \tag{2}$$

By well-ordering principle, for each even integer $2\,k$, let $2m(k)$ be the least even integer exceeding $2n(k)$ satisfying (2), that is,

$$d(fx_{2n(k)},\,fx_{2m(k)-2})\leqslant\varepsilon \quad \text{and (2) holds}\,. \tag{3}$$

Since

$$\begin{aligned}\varepsilon&<d(fx_{2n(k)},\,fx_{2m(k)})\\&\leqslant d(fx_{2n(k)},\,fx_{2m(k)-2})+\alpha_{2m(k)-2}+\alpha_{2m(k)-1}\,,\end{aligned}$$

we have by (2) and (3) that

$$\lim_{k\to\infty} d(fx_{2n(k)},\,fx_{2m(k)})=\varepsilon \tag{4}$$

By the triangle inequality, we have

$$|d(fx_{2n(k)},\,fx_{2m(k)-1})-d(fx_{2n(k)},\,fx_{2m(k)})|\leqslant\alpha_{2m(k)-1}$$

and

$$|d(fx_{2n(k)+1},\,fx_{2m(k)-1})-d(fx_{2n(k)},\,fx_{2m(k)})|\leqslant\alpha_{2m(k)-1}+\alpha_{2n(k)}\,.$$

Thus by (4),

$$d(fx_{2n(k)},\,fx_{2m(k)-1})\to\varepsilon$$

and

$$d(fx_{2n(k)+1},\,fx_{2m(k)-1})\to\varepsilon$$

By hypothesis (C), we have

$$\begin{aligned}d(fx_{2n(k)},fx_{2m(k)})&\leqslant d(fx_{2n(k)},\,fx_{2n(k)+1})+d(fx_{2n(k)+1},\,fx_{2m(k)})\\&\leqslant\alpha_{2n(k)}+\phi(d(fx_{2n(k)},\,fx_{2m(k)-1}),\,\alpha_{2n(k)},\\&\qquad \phi(d(fx_{2n(k)},\,fx_{2m(k)}),\,d(fx_{2m(k)-1},\,fx_{2n(k)+1}),\,\alpha_{2m(k)-1})\,,\end{aligned}$$

by upper semi-continuity of ϕ,

$$\varepsilon \leqslant \phi(\varepsilon, 0, \varepsilon, \varepsilon, 0) \leqslant \phi(\varepsilon, \varepsilon, \varepsilon, \varepsilon, \varepsilon) < \varepsilon \qquad \text{as} \quad k \to \infty ,$$

yielding a contradiction. Thus $\{fx_n\}$ is Cauchy. By completeness of X, $\{fx_n\}$ converges to a point $\zeta \in X$. Thus

$$\{gx_{2n}\} \quad \text{and} \quad \{hx_{2n+1}\}$$

also converge to ζ.

By the continuity of f,

$$f(fx_n) \to f\zeta ,$$
$$f(gx_{2n}) \to f(\zeta) \quad \text{and} \quad f(hx_{2n+1}) \to f(\zeta) .$$

It follows that

$$f(f\zeta) = f(g\zeta) = g(f\zeta) = g(g\zeta) = f(h\zeta) = h(f\zeta) = h(g\zeta) = g(h\zeta) = h(h\zeta) .$$

If

$$g\zeta \neq h(g\zeta) ,$$

then

$$d(g\zeta, h(g\zeta)) \leqslant \phi(d(f\zeta, f(g\zeta)), d(f\zeta, g\zeta), d(f\zeta, h(g\zeta)) ,$$
$$d(f(g\zeta), g\zeta), d(f(g\zeta), h(g\zeta)) \leqslant \phi(d(g\zeta, h(g\zeta)), 0, d(g\zeta, h(g\zeta)),$$
$$d(g\zeta, h(g\zeta)), 0) < d(g\zeta, h(g\zeta)), \qquad \text{a contradiction.}$$

Hence

$$g\zeta = h(g\zeta) .$$

Thus $g\zeta$ is a common fixed point of f, g and h.

Let u and v with $u \neq v$ such that u, v are common fixed points of f, g and h. Then by (C).

$$d(u, v) = d(gu, hv)$$
$$\leqslant \phi(d(fu, fv), d(fu, gu), d(fu, hv) ,$$
$$d(fv, gu), d(fv, hv)) \leqslant \phi(d(u, v), d(u, v) ,$$
$$d(u, v), d(u, v), d(u, v)) < d(u, v), \qquad \text{a contradiction .}$$

Therefore the proof is completed.

Remark. Theorem 2 extends an earlier result of [2]. Theorem 2 also extends an important fixed point theorem of Boyd and Wong [1]. In fact, our argument is motivated by Boyd and Wong's paper.

References

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Department of Mechanical Engineering
National Central University
Chung Li
Taiwan, R.O.C.