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Direct Sums of Cyclic Summands

by

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Introduction

By group we will mean Abelian group. Erdös [3] proved that if H is a pure subgroup of a free group F, then H contains a direct summand K of F such that $r(K)=r(H)$. In § 1, we will show that if H is a pure subgroup of a direct sum of cyclic groups, then H contains a direct summand K of F such that the torsion-free rank of K and H are equal and, for all primes p, both the p-rank and final p-rank of K and H are equal. In the case that F is a direct sum of cyclic p-groups, and $H=\bigoplus_{x\in X}\langle x\rangle$ is any given decomposition of H, we may choose K to be generated by a subset of X.

Following [1] we call a p-group G, C-decomposable if G has a summand C such that C is a direct sum of cyclic groups and fin $r_p(G)=$ fin $r_p(C)$. In § 2, using the results of § 1, we will give another proof [see 6] that $p^{\omega+1}$-projective p-groups are C-decomposable. More generally, we will give another proof of the following [see 1]: Let G be a $p^{\omega+n}$-projective p-group such that $G[p^n]=S[p^n]\check{\oplus}P$ where S is a pure subgroup of G, both S and G/P are direct sums of cyclic groups, and the sum is direct as valuated groups. Then G is C-decomposable.

We will for the most part follow the notation of [2] and [5]. The symbol $\oplus_c$ will denote a direct sum of cyclic groups, and $\check{\oplus}$ denotes a direct sum as valuated p-groups where the valuation is given by the height function in the obvious containing group. The torsion-free rank of a group G will be denoted by $r_0(G)$ whereas the p-rank will be denoted by $r_p(G)$. Also fin $r_p(G)=\inf_n r(p^nG)$. If the meaning is clear, we shall drop the subscript. As usual, ω is the first infinite ordinal and $\omega^*=\omega-\{0\}$. A cardinal is the least ordinal of the given cardinality and an ordinal is the set of all smaller ordinals.

§ 1. We will first state our main theorem

THEOREM 1. *Let F be a direct sum of cyclic groups and H a pure subgroup of F. Then H contains a summand K of F such that $r_0(K)=r_0(H)$ and both $r_p(K)=r_p(H)$ and* fin $r_p(K)=$ fin $r_p(H)$ *for each prime p.*

The remainder of this section is devoted to the proof of this theorem.

We will prove the theorem first for direct sums of cyclic p-groups. The idea of the proof is as follows. Let $H=\bigoplus_{n\in\omega^*} H_n$ be a pure subgroup of $F=\bigoplus_{n\in\omega^*} F_n$ where F_n and H_n are direct sums of cyclic groups of order p^n. Then we can find a large enough summand of $H_n[p]$ contained in $\bigoplus_{n\le i\le m} F_i[p]$ for some $m\ge n$. This is the essence of Lemma 7. Our first lemma will reduce the problem to the case in which $r(F)=\text{fin } r(F)$.

LEMMA 2. *Let H be a pure unbounded subgroup of a direct sum of cyclic p-groups F. Then there exists decompositions $H=B\oplus R$ and $F=B\oplus K\oplus L$ for which B is bounded, $R\le K$ and* fin $r(R)=r(R)=r(K)=\text{fin } r(K)$.

Proof. Let k be a nonnegative integer and B a maximal p^k-bounded summand of H such that $H=B\oplus R$ and $r(R)=\text{fin } r(R)$. Since H is pure in F, it can be extended to a basic subgroup of F. Using Theorem 29.3 in [4] we obtain the decomposition $F=B\oplus G$ with $R\le G$. Since any infinite subgroup of a direct sum of cyclics can be embedded in a direct summand of the same rank, we can write $G=K\oplus L$ with $r\le K$ and $r(R)=r(K)$.

Remark 3. Let $\lambda_1<\lambda_2$ be infinite cardinals. Then there exists a cardinal ρ such that ρ is not cofinal with ω and $\lambda_1<\rho\le\lambda_2$. To see this, let ρ be the successor of λ_1. Since $\lambda_1<\rho\le\lambda_2$ and any infinite successor cardinal is regular [Theorem 8.6 in 2], we have the desired conclusion.

Remark 4. *Let F be an unbounded direct sum of cyclic groups with* fin $r(F)=r(F)=\lambda>\aleph_0$. Fix a decomposition $F=\bigoplus_{i\in\omega^*} F_i$ where F_i is a direct sum of cyclic groups of order p^i. If λ is not confinal with ω, then

(5) there exists a sequence of positive integers $k_0<k_1<k_2<\cdots$ such that $r(F_{k_i})=\lambda$ for all $i\in\omega$.

If λ is cofinal with ω and (5) does not hold, then,

(6) there exists a sequence of positive integers $k_0<k_1<k_2<\cdots$ such that $r(F_{k_i})=\lambda_i$ for $i\in\omega$ where $\{\lambda_i\}_{i\in\omega}$ is a strictly increasing sequence of cardinals and lim $\lambda_i=\lambda$.

The following lemma is the key to our next theorem.

LEMMA 7. *Let F be an unbounded direct sum of cyclic p-groups with* fin $r(F)=r(F)$. *Fix a decomposition of F, say $F=\bigoplus_{n\in\omega^*} F_n$, where F_n is a direct sum of cyclic groups of order p^n. Let $H=\bigoplus_{\alpha\in\lambda}\langle x_\alpha\rangle$ be a pure subgroup of F where λ is a cardinal and $o(x_\alpha)=p^k$ for a fixed positive integer k.*

(i) *If λ is finite then $H[p]\le\bigoplus_{k\le n\le m} F_n[p]$ for some positive integer $m\ge k$.*

(ii) *If λ is not cofinal with ω then $H=H_1\oplus H_2$ such that $H_1=\bigoplus_{\alpha\in X}\langle x_\alpha\rangle$ for some subset X of λ with $|X|=\lambda$ and $H_1[p]\leq\bigoplus_{k\leq n\leq m}F_n[p]$ for some integer $m\geq k$.*

(iii) *If λ is cofinal with ω and ρ is any cardinal with $\rho<\lambda$, then $H=H_1\oplus H_2$ such that $H_1=\bigoplus_{\alpha\in X}\langle x_\alpha\rangle$ with $|X|\geq\rho$ and $H_1[p]\leq\bigoplus_{k\leq n\leq m}F_n[p]$ for some integer $m\geq k$.*

Proof. Case (i) is clear. For each integer $m\geq k$ let X_m be the set of all $\alpha\in\lambda$ such that $\langle x_\alpha\rangle[p]\subseteq\bigoplus_{k\leq n\leq m}F_n$. Since $\bigcup_{k\leq m<\omega}X_m=\lambda$, we must have, for some m, $|X_m|=\lambda$ if λ is not cofinal with ω. If λ is cofinal with ω and $\lambda_1<\lambda_2<\lambda_3<\cdots$ is a sequence of cardinals converging to λ, then we have for each λ_i a positive integer m such that $|X_m|\geq\lambda_i$. Hence we can obtain the desired conclusion.

THEOREM 8. *Let H be a pure subgroup of a direct sum of cyclic p-groups F. Then H contains a summand K of F with $\operatorname{fin} r(K)=\operatorname{fin} r(H)$ and $r(K)=r(H)$. Moreover, if $H=\bigoplus_{x\in X}\langle x\rangle$ is any given decomposition of H, we can choose K to be generated by a subset of X.*

Proof. If H is bounded then H itself is a summand of F, so we assume H is unbounded. In addition, by Lemma 2, we may assume $\operatorname{fin} r(H)=r(H)=r(F)=\operatorname{fin} r(F)=\lambda\geq\aleph_0$. Let $F=\bigoplus_{n\in\omega^*}F_n$ and $H=\bigoplus_{n\in\omega^*}H_n$ be decompositions where F_n and H_n are direct sums of cyclic groups of order p^n. Also for each positive integer n, fix a decomposition of H_n, say $H_n=\bigoplus_{\alpha\in X_n}\langle h_\alpha\rangle$. Since $r(H)=\operatorname{fin} r(H)=\lambda$, there exists a sequence of integers $k_0<k_1<\cdots$ such that:

Case 1. λ is not cofinal with ω and $r(H_{k_i})=\lambda$ for all $i\in\omega$.

Case 2. λ is cofinal with ω and $r(H_{k_i})=\lambda$ for all $i\in\omega$.

Case 3. λ is cofinal with ω and $r(H_{k_i})=\lambda_i$ where $\{\lambda_i\}_{i\in\omega}$ is a strictly increasing sequence of cardinals with $\lim_i\lambda_i=\lambda$.

We will prove the theorem for Case 1 and then indicate the slight changes needed for Cases 2 and 3. We will choose inductively a subsequence of $\{k_i\}_{i\in\omega}$, say $\{n_i\}_{i\in\omega}$ such that $H_{n_i}=U_i\oplus L_i$ with

(a) $r(U_i)=\lambda$ and

(b) $U_i[p]\leq\bigoplus_{n_i\leq n<n_{i+1}}F_n[p]$.

Let $n_0=k_0$ and assume that n_i has been defined where $n_i=k_j$ for some $j\in\omega$. By Lemma 7(ii), there exists a decomposition $H_{n_i}=U_i\oplus L$ and an integer $m\geq n_i$ such that $r(U_i)=\lambda$ and $U_i[p]\subseteq\bigoplus_{n_i\leq n\leq m}F_n[p]$. Let j be the least integer such that $k_j>m$. Let $n_{i+1}=k_j$. Hence by induction we have the subsequence $\{n_i\}_{i\in\omega}$ with the desired properties. Let $K=\bigoplus_{i\in\omega}U_i$.

For each $i\in\omega$ let M_i be a pure subgroup of $\bigoplus_{n_i\leq n<n_{i+1}}F_n$ supported by $U_i[p]$. Then M_i is a summand of this bounded group. Thus $M=\bigoplus_{i\in\omega}M_i$ is a summand of F

with $M[p]=K[p]$. By [Theorem 16 in 7], K is a summand of F which proves Case 1.

For Case 2, let $\{\lambda_i\}_{i\in\omega}$ be a strictly increasing sequence of cardinals converging to λ. Assuming that n_i has been defined, we have by Lemma 4(iii), $H_{n_i}=U_i\oplus L$ such that $r(H_{n_i})=\lambda_i$ and a positive integer m such that, etc.

For Case 3, we let $n_0=k_1$ and choose n_i inductively with $H_{n_i}=U_i\oplus L$ where $r(U_i)=\lambda_i$.

That K is generated by a subset of X follows from the way the summand was chosen in Lemma 7.

At this point we would like to discuss the extension of Theorem 8 to arbitrary direct sums of cyclic groups.

Proof (of Theorem 1). Let H be a pure subgroup of a direct sum of cyclic groups F. Decompose $H=H_0\oplus H_t$ where H_0 is torsion free and H_t is torsion. Let F_t be the torsion subgroup of F, and let σ: $F\to F/F_t$ be the natural homomorphism. It is easily shown that $H_0\oplus F_t$ is pure in F and, since F_t is pure in F, we have $\sigma(H_0)$ pure in F/F_t. By a lemma of Erdös (see [3] or Lemma 51.2 in [4]), we can write $F/F_t=\bar{K}\oplus\bar{R}$ where $\bar{K}$ is a subgroup of $\sigma(H)$ with $r(\bar{K})=r(\sigma(H))$. Decompose $\bar{K}=\bigoplus_{\alpha\in X}\langle x_\alpha\rangle$ and $\bar{R}=\bigoplus_{\alpha\in Y}\langle x_\alpha\rangle$. For each $\alpha\in X$ we may choose $y_\alpha\in H$ such that $\sigma(y_\alpha)=x_\alpha$ and for each $\alpha\in Y$ we may choose $y_\alpha\in F$ such that $\sigma(y_\alpha)=x_\alpha$. Let $K_0=\bigoplus_{\alpha\in X}\langle y_\alpha\rangle$ and $R=\bigoplus_{\alpha\in Y}\langle y_\alpha\rangle$. Since F_t is pure in F and F/F_t is free, we have $F=K_0\oplus R\oplus F_t$ with $K\leq H$ and $r(K)=r(H)$. Next we decompose $F_t=\bigoplus F_p$ and $H_t=\bigoplus H_p$ into their primary components. Since $H_p\subseteq F_p$, we have, by Theorem 8, that each H_p has a summand K_p of the desired rank which is also a summand of F_p. Thus, setting $K=K_0\oplus(\bigoplus K_p)$, we have H containing K a summand of F with $r_0(K)=r_0(H)$, and for all primes p, both $r_p(K)=r_p(H)$ and fin $r_p(K)=$fin $r_p(H)$.

§2

A well-known problem in the theory of abelian p-groups is to determine whether a given group G is C-decomposable. In [1] several necessary and sufficient conditions are given for a $p^{\omega+n}$-projective p-group G to be C-decomposable. One of the conditions is that $G[p^n]=S[p^n]\check{\oplus}P$ where S is a pure subgroup of G and both S and G/P are direct sums of cyclic groups. Using the results of §1, we will give another proof of the sufficiency. That $p^{\omega+1}$-projective p-groups are C-decomposable (see [6] for a different proof) will follow as a corollary.

We will need several lemmas. Using the notation in the preceding paragraph, Lemma 9 will show that we can find a summand C of G such that $C=\bigoplus_c$ and fin $r(C)=$fin $r(S)$. Lemma 11 will be used to show that G is C-decomposable in the case that fin $r(S)<$fin $r(G)$. Lemma 12 reduces the problem to that case $r(S)=$fin $r(S)$.

LEMMA 9. *Let H be a subgroup of a group G such that G/H is a direct sum of* *cyclic groups. Suppose that there exists a pure subgroup S of G such that S is a direct sum of cyclic groups,* $S\cap H=0$, *and the natural map* $\pi\colon G\to G/H$ *preserves heights of elements of S. Then there exists a subgroup T of S such that T is a summand of G,* $r_0(T)=r_0(S)$, *and both* $r_p(T)=r_p(S)$ *and* $\operatorname{fin} r_p(T)=\operatorname{fin} r_p(S)$ *for all primes p.*

Proof. Since π preserves heights of elements of S and $S\cap H=0$, $S\cong\pi(S)=(S+H)/H$ is a pure subgroup of G/H. Hence by Theorem 1 there exists a subgroup T of S such that $\pi(T)$ is a direct summand of G/H, $r_0(T)=r_0(S)$, and for all primes p, $r_p(T)=r_p(S)$ and $\operatorname{fin} r_p(T)=\operatorname{fin} r_p(S)$. By Lemma 6 in [8], T is a direct summand of G.

COROLLARY 10. *Let G be a torsion group such that* G/G^1 $(G^1=\bigcap_n nG)$ *is a direct sum of cyclic groups. Let S be a pure subgroup of G and a direct sum of cyclic groups. Then S contains a summand T of G with* $r_p(T)=r_p(S)$ *and* $\operatorname{fin} r_p(T)=\operatorname{fin} r_p(S)$ *for all primes p.*

LEMMA 11. *Let G be a p-group such that* $\operatorname{fin} r(G)>\aleph_0$. *Let H be a subgroup of G such that H is a direct sum of cyclic groups and* $r(G/H)<\operatorname{fin} r(G)$. *Then there exists a summand C of G such that C is a direct sum of cyclic groups and* $\operatorname{fin} r(C)=\operatorname{fin} r(G)$.

Proof. Write $G=L+H$ where L is a subgroup of G generated by a set of coset representatives of G/H. Fixing a decomposition of H as a direct sum of cyclic groups we can decompose H into $C\oplus D$ where D is exactly those cyclic summands in the decomposition of H containing a nonzero component of an element of $L\cap H$. Note that $G=(L+D)\oplus C$. Since $r(L+D)\aleph_0=r(L)\cdot\aleph_0=r(G/H)\cdot\aleph_0<\operatorname{fin} r(G)$, it follows that $\operatorname{fin} r(C)=\operatorname{fin} r(G)$.

LEMMA 12. *Suppose that* $G[p^n]=S[p^n]\check{\oplus}P$ *(direct as valuated groups where the valuation of elements are heights in G), where S is a pure subgroup of G and both S and G/P are direct sums of cyclic groups. If* $S=T\oplus S'$ *where T is bounded, then* $G/(P\oplus T[p^n])$ *is a direct sum of cyclic groups.*

Proof. Since $T\check{\oplus}P$ is a valuated direct sum, $(T\oplus P)/P$ is pure in G/P. Since T is bounded, $(T\oplus P)/P$ is a bounded pure subgroup of G/P and hence a direct summand of G/P. Writing $G/P=(T+P)/P\oplus R/P$, we see that

$$
\begin{aligned}
G/(T[p^n]+P) &\cong (G/P)/((T[p^n]+P)/P)\\
&\cong ((T+P)/P)/((T[p^n]+P)/P)\oplus R/P\\
&\cong p^nT\oplus R/P=\oplus_c\,.
\end{aligned}
$$

The following theorem is the (c) implies (a) part of Theorem 8 in [1].

THEOREM 13. *Let G be a* $p^{\omega+n}$*-projective p-group such that* $G[p^n]=S[p^n]\check{\oplus}P$ *(direct as valuated groups) where S is a pure subgroup of G and both S and G/P are direct sums of cyclic groups. Then there exists a summand C of G such that C is a direct sum of cyclic groups and* $\operatorname{fin} r(C)=\operatorname{fin} r(G)$.

Proof. Let T be a subgroup of G generated by a pure independent set maximal with respect to the property that $T[p^n] \subseteq P$. Then since $S \oplus P$ is a valuated direct sum, it follows that $B = S \oplus T$ is a basic subgroup of G. Before proceeding, we want to note that Lemma 12 allows us to assume that $r(S) = \text{fin}\, r(S)$.

Case 1. $\text{fin}\, r(S) < \text{fin}\, r(G)$. Pick H maximal disjoint from S containing P. By our choice of H, it is neat and thus $(G/H)[p] \cong G[p]/H[p] = G[p]/P[p] \cong S[p]$. Thus $r(G/H) = r(S) = \text{fin}\, r(S) < \text{fin}\, r(G)$. Also $H/H[p^n] = H/P \leq G/P = \bigoplus_c$. Since $p^n H \cong H/H[p^n] = \bigoplus_c$, we have $H = \bigoplus_c$. By Lemma 11, we have our desired summand C.

Case 2. $\text{fin}\, r(S) = \text{fin}\, r(G)$. Since $P \oplus S[p^n]$ is a valuated direct sum, the natural map π: $G \to G/P$ preserves heights of elements of S. Hence the result follows from Lemma 9.

COROLLARY 14 (Fuchs and Irwin [6]). *If G is a $p^{\omega+1}$-projective p-group, then there exists a summand C of G such that C is a direct sum of cyclic groups and* $\text{fin}\, r(C) = \text{fin}\, r(G)$.

Proof. From Theorem 1 of [6], $G[p] = S \check{\oplus} P$ where S is a pure subgroup of G and both S and G/P are direct sums of cyclic groups. Hence the corollary follows from Theorem 13.

Using the remarks in [6, pp. 465–466], the proof of the corollary can be reduced to Case 2 in the proof of Theorem 13.

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