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Definability in the Abstract Theory of Integration

by

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This is a sequel to the author's works [10], [11] and [12]; these are all concerned with the "definability problems" in certain areas of analysis. For the author's standpoint in this connection, we refer the reader to [10] and [11].

In the present paper we investigate the logical structure of the abstract theory of integration, and show that it is definable relative to the given spaces.

The definable formulation of the theory is mostly routine, and hence we restrict ourselves to giving a brief account of such a procedure in each topic. One needs some elaboration in defining the class of simple functions, which is necessary for the product integrals. This being done (Section 9) the Fubinis' theorem can be formulated in our theory (if we do not take the "quotient" modulo null sets). As is remarked in Chan [5], the unconditional Jordan's decomposition of a signed integral cannot be hoped for, but a "relative existence" of such a decomposition can be definably formulated (Section 11).

We have included only the basic properties of the abstract integral. We are planning to continue our study and formulate the theory of L^p-spaces in our system.

Mathematically we have almost exclusively followed [1]; its presentation of the subject is itself quite "definable," and it has been very helpful to our endeavors. We have also consulted [4] and [6]–[8]. The acquaintance with [9]–[11] is assumed throughout. The proof-theoretical background which is required here is exactly the same as that of [11], and so we do not repeat it. Mathematical notations are often preferred to the strictly formal ones as in [11]. Mathematical proofs are sketchy, but the emphasis is placed on the explicit expressions of the definable objects that are claimed to exist. The fact that they are indeed definable will not be pointed out each time.

The references [2], [3] and [5] are given for the reader's convenience. They deal with constructive measure theory in the framework of Bishop's constructive analysis.

§ 1. Axioms and the main theorem

DEFINITION 1.1. 1) Types are defined as in Definition 1.1 of [11] except that here we start with two atomic types, one for the rationals and one for the elements of a space.

2) The language is defined as in Definition 1.2 of [11]. The symbols for an integration space are X, L, J and eq $(X; \ , \ )$.

3) Definability, terms, formulas, abstracts, **min**, sequents and substitution are defined as in Definitions 1.3 and 1.4 of [11]. In particular, $X(x)$, $L(\phi)$ and $J(\phi, t)$ are atomic formulas for appropriate x, ϕ and t. The details will become clearer in Definition 1.3 below.

We follow the notational conventions in [10] and [11].

DEFINITION 1.2. We use abbreviated notations for some defined concepts. Some of them are adopted from [9], [10] and [11].

κ: a definable enumeration of the pairs of natural numbers

R(a): "a is a real number."

∞: $\{t\}(t=t)$, where t stands for the rationals.

ept or $-\infty$: the empty set

eR (a): $\mathrm{R}(a) \vee a=\infty \vee a=-\infty$ (a is an extended real.)

mp (ϕ, X) or mp (ϕ): $\forall x \mathrm{R}(\{t\}\phi(x, t))$

(ϕ is a real-valued function defined on X.)

ss (X, C) or ss (C): C is a subset of X.

$a\phi$: the multiple of ϕ by a real number a

$\Sigma[\Phi(i);\ i \leq n]$, $\Pi[\Phi(i);\ i \leq n]$, $\max[\Phi(i);\ i \leq n]$, $\min[\Phi(i);\ i \leq n]$: the sum, the product, the maximum and the minimum of n functions respectively

$\Sigma\{\alpha(i);\ i \leq n\}$, $\Sigma\{\alpha(i);\ i=1, 2, \cdots\}$: the finite sum and the infinite sum of the reals $\{\alpha(i)\}_i$ respectively; similarly for the product.

limsup $\{\alpha(i);\ i=1, 2, \cdots\}$: the limit superior of the sequence of reals $\{\alpha(i)\}$; similarly for liminf and lim.

We shall abbreviate these expressions, for example, to $\Sigma\{\alpha(i); i\}$ or even to $\Sigma\alpha(i)$.

DEFINITION 1.3. Axioms. The axiom sets $\mathscr{A}$ and $\mathscr{C}$ are those in Definition 1.6 of [11]. $\mathscr{A}$ stands for the set of axioms of arithmetic and $\mathscr{C}$ stands for the set of axioms of "definitions by definable induction" (abbreviated to DDI).

The axiom set $\mathscr{B}$ consists of the following.

1) The axioms on the space

$\forall x X(x)$; the equivalence relations on eq $(X; \ , \ )$.

We may write $x \in X$ for $X(x)$, and $x=y$ for eq $(X; x, y)$.

2) The axioms on L, the class of elementary functions. (We use ϕ, ψ, $\cdots$ for the members of L, and Φ, Ψ, $\cdots$ for the sequences from L.)

$$\forall\phi(L(\phi) \vdash \mathrm{mp}(\phi, X))$$

$L(\phi)$ may be written as $\phi \in L$.

1°. $$\forall a \in R \forall \phi \in L(a\phi \in L);$$

$$\forall n \forall \Phi(\forall i \leq n(\Phi(i) \in L) \vdash \Sigma[\Phi(i);\ i \leq n],\ \max[\Phi(i);\ i \leq n],\ \min[\Phi(i);\ i \leq n] \in L)$$

We shall abbreviate $\phi \in L \wedge \forall x(\phi(x) \geq 0)$ to $\phi \in L(+)$.

3) The axioms on J, the integral of the members of L

2°. $$\forall \phi \in L\ R(\{t\}J(\phi, t))$$

We write $J(\phi)$ for $\{t\}J(\phi, t)$.

3°. $$\forall a \in R \forall \phi \in L(J(a\phi) = aJ(\phi))\,;$$

$$\forall n \forall \Phi(\forall i \leq n(\Phi(i) \in L) \vdash J(\Sigma[\Phi(i);\ i \leq n] = \Sigma\{J(\Phi(i));\ i \leq n\})\,;$$

$$\forall \phi \in L(+)(J(\phi) \geq 0)$$

4°. $$\forall \Phi(\forall n(\Phi(n) \in L(+)) \wedge \forall n \forall x(\Phi(n+1, x) \leq \Phi(n, x))$$

$$\wedge \forall x(\lim\{\Phi(n, x);\ n = 1, 2, \cdots\} = 0) \vdash \lim\{J(\Phi(n));\ n = 1, 2, \cdots\} = 0)$$

DEFINITION 1.4. The definable predicate calculus $\mathscr{L}$ is defined as in Definition 1.5 of [11].

THEOREM. *Let $\Gamma \to \Delta$ be a sequent in our language which expresses a theorem of Daniell integral. Then $\mathscr{A}, \mathscr{B}, \mathscr{C}, \Gamma \to \Delta$ is provable in $\mathscr{L}$, hence without cuts. In this case we call $\Gamma \to \Delta$ a theorem of $\mathscr{J}$.*

The subsequent sections of this article are devoted to the proof of this theorem. The theorem above incorporated with the argument in Section 2 of [11] yields the

Conclusion (Relative soundness). The theory of Daniell integral is sound relative to $\mathscr{B}$, the axioms on X, L and J.

DEFINITION 1.5. Two concepts Σ and Σ' which can be formulated in our language and which may have some parameters, say Θ and Ξ respectively, are said to be "mutually definably interpretable" if there are definable Θ^* and Ξ^* such that

$$\Sigma(\Theta) \to \Sigma'(\Xi^*(\Theta)) \quad \text{and} \quad \Sigma'(\Xi) \to \Sigma(\Theta^*(\Xi))$$

are both theorems of $\mathscr{J}$. (See the Theorem above.)

The propositions below are immediate consequences of our axioms and definitions, and will be frequently used without specific references to them.

PROPOSITION 1.1. 1) *The definability property and the subset property are both preserved under the basic set theoretical operations.* (See Theorem 4 of [11].)

2) *The definability property is preserved under the following operations on the reals and the functions;* $a\phi$, Σ, Π, max, min, limsup, liminf, lim, *the absolute value*, ϕ^+ *and* ϕ^-, *where* ϕ^+ *and* ϕ^- *are the positive part and the negative part of* ϕ *respectively.* (See [9].)

PROPOSITION 1.2. *The following are theorems of $\mathscr{J}$.*

1) $0 \in L$, *where we write* 0 *for* $\{x\}\{t\}(t<0)$.
2) $\phi \in L \rightarrow \phi^{+}, \phi^{-}, |\phi| \in L$.
3) *We write* $\phi \geq \psi$ *for* $\forall x(\phi(x) \geq \psi(x))$.

$$\phi \geq \psi \vdash J(\phi) \geq J(\psi); \qquad |J(\phi)| \leq J(|\phi|);$$

$$J(\phi^{+}) \leq J(|\phi|); \qquad J(\phi^{-}) \leq J(|\phi|); \qquad J(0)=0.$$

§ 2. The integral of elementary functions

Here and in the sections that follow, the propositions are meant to be the theorems of $\mathscr{J}$.

DEFINITION 2.1. We shall abbreviate $\forall n(\chi(n) \in L)$ to $\chi \subset L$.

1) $\mathrm{nls}\,(E, \chi)$: $\mathrm{ss}\,(X, E) \wedge \chi \subset L \wedge \forall n(\chi(n) \leq \chi(n+1))$

$$\wedge \forall x \in E \forall r > 0 \exists n(\chi(n, x) > r) \wedge \lim J(\chi(n)) \in R$$

(E is a null set by χ.)

2) $\mathrm{nls}'\,(E, \theta)$: $\mathrm{ss}\,(E) \wedge \theta \subset L \wedge \forall x \in E(\Sigma\theta(n, x)=\infty) \wedge \Sigma J(|\theta(n)|) \in R$
3) $\mathrm{nls}^{*}\,(E, \chi)$: $\chi \subset L(+) \wedge \mathrm{nls}\,(E, \chi)$

PROPOSITION 2.1. nls, nls$'$ *and* nls* *are all mutually definably interpretable. Thus, we are free to use any of these notions to our convenience.*

PROPOSITION 2.2. 1) $\mathrm{nls}\,(\mathrm{ept}, \{n\}0)$.
2) $\mathrm{nls}\,(E, \chi)$, $D \subset E \rightarrow \mathrm{nls}\,(D, \chi)$.
3) $\forall m\, \mathrm{nls}'\,(A(m), \Xi(m)) \rightarrow \mathrm{nls}'(\bigcup A, \chi^{*})$
for a definable χ^{*}, *where* $\bigcup A = \bigcup\{A(m); m=1, 2, \cdots\}$.

Proof of 3). We may assume

$$\Sigma\{J(|\Xi(m, n)|); n=1, 2, \cdots\} \leq \exp(2, -m)$$

for each m, where $\exp(a, b)$ denotes a^{b}. Define $\chi^{*}(j)=\Xi(m, n)$ where $\kappa(j)=(m, n)$.

DEFINITION 2.2. Let P be any formula in our language in which x is not bounded.

$$\mathrm{ae}\,(x, P, E, \chi): \mathrm{nls}\,(E, \chi) \wedge \forall x \notin E\, P(x)$$

PROPOSITION 2.3. 1) $\Phi \subset L(+)$, $\forall n(\Phi(n+1) \leq \Phi(n))$,

$$\mathrm{ae}\,(x, \lim\{\Phi(n, x); n\}=0, E, \chi) \rightarrow \lim J(\Phi(n))=0.$$

2) $\Phi \subset L(+)$, $\mathrm{ae}\,(x, \Sigma\Phi(n, x) \geq \psi(x), E, \chi) \rightarrow \Sigma J(\Phi(n)) \geq J(\psi)$.
3) $\mathrm{ae}\,(x, \Sigma|\Phi(n, x)|=0, E, \chi)$, $\Sigma J(|\Phi(n)|) \in R \rightarrow \Sigma J(\Phi(n))=0$.
4) $\Phi \subset L$, $\forall n(\Phi(n) \leq \Phi(n+1))$, $\lim J(\Phi(n)) \in R \rightarrow \mathrm{nls}\,(E^{*}, \Phi)$,
where $E^{*}=\{x; \lim \Phi(n, x)=0\}$.

Proof. 1) We may assume $\chi(n) \geq 0$. Define

$$\eta(n, \varepsilon): (\Phi(n) - \varepsilon\chi(n))^+ ,$$

where ε stands for the positive rationals. Applying 4° to η and using

$$\Phi(n) = (\Phi(n) - \varepsilon\chi(n)) + \varepsilon\chi(n) \leq \eta(n, \varepsilon) + \varepsilon\chi(n) ,$$

we obtain the conclusion.

3) $\mathrm{ae}\,(x, \Sigma\{\Phi(n, x)^-; n \leq m\} \leq \Sigma\Phi(n, x)^+, E, \chi)$,
and hence 2) above and 2° yield $\Sigma J(\Phi(n)^-) = \Sigma J(\Phi(n)^+)$, from which follows the conclusion.

§ 3. Integrable functions

DEFINITION 3.1. $\mathrm{emp}\,(f)$: $\mathrm{mp}\,(f, X, \mathrm{eR})$.
(f is a map from X to the extended reals.)

$$\mathrm{itg}\,(f, \Phi, E, \chi)\text{: } \Phi \subset L \wedge \mathrm{ae}\,(x, f(x) = \Sigma\Phi(n, x), E, \chi) \ \wedge \Sigma J(|\Phi(n)|\} \in R .$$

(f is integrable with respect to Φ, E and χ.)

$$J^1(f, \Phi, E, \chi)\text{: } \mathrm{limsup}\,\{\Sigma(J(\Phi(i)); i \leq m\}; m = 1, 2, \cdots\} .$$

This may be abbreviated to $J^1(f)$, or even to $J(f)$. Notice that the definiens above is an extended real.

($J^1(f)$ is the Daniell integral of f with respect to Φ, E and χ if f is integrable.)

PROPOSITION 3.1. 1) $\psi \in L \rightarrow \mathrm{itg}\,(\psi, \Phi^*, \mathrm{ept}, \chi^*)$, *where* $\Phi^*(1) \equiv \psi$, $\Phi^*(n) \equiv 0$ *for* $n \geq 2$ *and* $\chi^*(n) \equiv 0$ *for all* n.

2) $\mathrm{itg}\,(f, \Phi, E, \chi) \rightarrow J(f) = \Sigma J(\Phi(n)) \in R$.

3) $\mathrm{itg}\,(f, \Phi, E, \chi), \mathrm{itg}\,(f, \Phi', E', \chi') \rightarrow J^1(f, \Phi, E, \chi) = J^1(f, \Phi', E', \chi')$.

Proof of 3). Define $A \equiv \{E, E', \mathrm{ept}, \mathrm{ept}, \cdots\}$ and $\Xi \equiv \{\chi, \chi', 0, 0, \cdots\}$, and apply 3) of Proposition 2.2 to these A and Ξ. Then $\mathrm{nls}'\,(\bigcup A, \chi^*)$ for a definable χ^*, or $\mathrm{nls}'\,(E \cup E', \chi^*)$, so that the premises of 3) of Proposition 2.3 are satisfied by $\Phi(n) - \Phi'(n)$, $E \cup E'$ and χ^*, and hence $\Sigma J(\Phi(n) - \Phi'(n)) = 0$, or $\Sigma(J(\Phi(n)) - J(\Phi'(n))) = 0$. This and 2) yield the conclusion.

Note. By virtue of 2) and 3) above, we may write $J(f) = \Sigma J(\Phi(n))$ when f is integrable (with respect to Φ, E and χ).

PROPOSITION 3.2. $\Phi \subset L$, $\quad \forall n(\Phi(n) \leq \Phi(n+1))$, $\quad \lim J(\Phi(n)) \in R$

$$\rightarrow \mathrm{ae}\,(x, \lim \Phi(n, x) \in R, E^*, \Phi)$$

$$\wedge\, [\mathrm{ae}\,(x, f(x) = \lim \Phi(n, x), E^*, \Phi) \vdash \mathrm{itg}\,(f, \Phi^*, E^*, \Phi) \wedge J(f) = \lim J\,(\Phi(n))] ,$$

where

$$E^* \equiv \{x; \forall r > 0\, \exists n \forall m \geq n \Phi(m, x) > r\}, \Phi^*(1) \equiv \Phi(1) \quad \textit{and} \quad \Phi^*(n+1) \equiv \Phi(n+1) - \Phi(n) .$$

DEFINITION 3.2. itg′ (f, Φ', E', χ'):

$$\Phi' \subset L \wedge \text{ae}\,(x, \lim \Phi'(n, x) = f(x), E', \chi')$$

$$\wedge \lim \{J(|\Phi'(n) - \Phi'(m)|);\ n, m = 1, 2, \cdots\} = 0$$

$J'(f)$: limsup $J(\Phi'(n))$

itg* (f, Ψ): $\Psi \subset L \wedge \Sigma J(|\Psi(n)|) < \infty$

$$\wedge \forall x(\Sigma |\Psi(n, x)| < \infty \vdash f(x) = \Sigma \Psi(n, x))$$

PROPOSITION 3.3. 1) itg′ $(f, \Phi', E', \chi') \rightarrow J'(f) = \lim J(\Phi'(n)) < \infty$.

2) *$J'(f)$ is independent of the parameters.*

3) itg *and* itg′ *are mutually definably interpretable with respect to* (Φ, E, χ) *and* (Φ', E', χ').

4) $J^1(f) = J'(f)$ *whenever either of* itg *or* itg′ *holds.*

5) (Mikusiński) itg *and* itg* *are mutually definably interpretable with respect to* (Φ, E, χ) *and* Ψ.

Proof. 3) Assume itg and put $\Phi'(n) \equiv \Sigma[\Phi(k); k \leq n]$, $E' \equiv E$ and $\chi' \equiv \chi$. Then itg′ (f, Φ', E', χ').

Assume itg′ and define $\nu(k)$ by

$$\nu(1) = \mathbf{min}\,(n, \forall m \geq n J(|\Phi'(n) - \Phi'(m)|) \leq 1/2)\,,$$

$$\nu(k+1) = \mathbf{min}\,(n, n > \nu(k) \wedge \forall m \geq n (J(|\Phi'(n) - \Phi'(m)|)$$

$$\leq \exp(2, -(k+1))))\,.$$

Then $\theta(k) \equiv \Phi'(\nu(k))$ is a subsequence of Φ', and

$$\forall m \geq \nu(k)(J(|\theta(k) - \theta(m)|) \leq \exp(2, -k))\,.$$

Put $E \equiv E'$, $\chi \equiv \chi'$, $\Phi(1) \equiv \theta(1)$ and $\Phi(k+1) \equiv \theta(k+1) - \theta(k)$.

5) Assume itg* (f, Ψ). Then ae $(x, \Sigma\Psi(n, x) = f(x), E^*, \chi^*)$, where $E^* \equiv \{x;\ \Sigma |\Psi(n, x)| < \infty\}$ and $\chi^*(n) \equiv \Sigma[|\Psi(k)|; k \leq n]$. So itg (f, Ψ, E^*, χ^*).

Assume itg (f, Φ, E, χ) and define

$$D \equiv \{x;\ \Sigma |\Phi(n, x)| < \infty \wedge f(x) \neq \Sigma\Phi(n, x)\}\,.$$

Then nls (D, χ). Define Ψ by $\Psi(3n-2) \equiv \Phi(n)$, $\Psi(3n-1) \equiv \chi(n)$ and $\Psi(3n) \equiv -\chi(n)$. For this Ψ, $\Sigma |\Psi(n, x)| < \infty$ implies $x \notin D$, and hence $f(x) = \Sigma\Psi(n, x)$.

By virtue of the propositions above, we may use either one of itg, itg′ and itg*.

§4. Some properties of the Daniell integral

PROPOSITION 4.1. 1) $1° \sim 3°$ *in the axiom set* $\mathscr{B}$ (*Definition* 1.3) *are satisfied by the integrable functions.* ($4°$ *will be proved later.*)

2) *The properties which correspond to those in Proposition* 1.2 *hold for* itg *and* J^1.

Proof. We work two of the claimed properties as examples.

1. itg is closed under the finite sum; namely, assume

$$\forall i \leq m \operatorname{itg}(F(i), \Theta(i), A(i), \Xi(i)).$$

We define ρ, D and θ as follows. $\rho(n) \equiv \Sigma[\Theta(i, n); i \leq m]$ for every n, $D \equiv \bigcup\{A(i); i \leq m\}$ and $\theta(n) \equiv \Xi(i, j)$ where $\kappa(n) = (i, j)$. Then

$$\begin{aligned} J(\Sigma[F(i); i \leq m]) &= \Sigma J(\rho(n)) \\ &= \Sigma\{J(\Sigma[\Theta(i, n); i \leq m]); n = 1, 2, \cdots\} \\ &= \Sigma\{J(F(i)); i \leq m\}, \end{aligned}$$

and so $\operatorname{itg}(\Sigma[F(i); i \leq m], \rho, D, \theta)$.

2. itg is closed under the finite maximum; namely, assume

$$\forall i \leq m \operatorname{itg}(F(i), \Theta(i), A(i), \Xi(i)).$$

We shall define Θ^*, A^* and Ξ^*, and put

$$H(j) \equiv \operatorname{itg}(\max[F(i); i \leq j], \Theta^*(j), A^*(j), \Xi^*(j)),$$

so that $\forall j \leq m H(j)$.

Notice that in general

$$\max(f, g) = f \vee g = 1/2(f+g) + 1/2|f-g|,$$

and hence, if the proposition is assumed to hold for $+$, $-$, $|\ |$ and the scalar product, then there must be definable Φ, E and χ with the appropriate parameters so that

$$\operatorname{itg}(f, \Phi_1, E_1, \chi_1), \qquad \operatorname{itg}(g, \Phi_2, E_2, \chi_2) \rightarrow \operatorname{itg}(f \vee g, \Phi, E, \chi).$$

Write $\Theta^*[j]$ for $\{k\}$ $(k < j \wedge \Theta^*(k))$. Similarly for $A^*[j]$ and $\Xi^*[j]$. Now, if we abbreviate the totality of

$$\max[F(i); i \leq j-1], F(j), \Theta^*[j], A^*[j], \Xi^*[j], \Theta(j), A(j), \Xi(j)$$

to Ψ, and if we let H_1, H_2 and H_3 be the defining formulas of Φ, E and χ respectively, then Θ^*, A^* and Ξ^* are defined by the DDI-axioms applied to $H_1(\Psi)$, $H_2(\Psi)$ and $H_3(\Psi)$ respectively; Θ^*, A^* and Ξ^* are regarded as the DDI-predicates. $\forall j \leq m H(j)$ can be proved by induction on j applied to $H(j)$, which is a definable formula. (In fact this is a simultaneous definition of Θ^*, A^* and Ξ^*, but the original form of DDI can be easily adjusted to this case.)

PROPOSITION 4.2. *There is a definable* Ψ^* *with parameters* f, Φ, E, χ, ε, *such that*

$$\begin{aligned}\text{itg}\,(f,\ \Phi,\ E,\ \chi),\qquad & \varepsilon>0\rightarrow \Psi^*\subset L\\ & \wedge\,\text{ae}\,(x,\ \Sigma\Psi^*(n,\ x)=f(x),\ E,\ \chi)\\ & \wedge\,\Sigma\{J(|\Psi(n)|);\ n\geq 2\}\leq\varepsilon\wedge J(|f-\Psi(1)|)\leq\varepsilon\\ & \wedge\,\Sigma J(|\Psi(n)|)\leq J(|f|)+2\varepsilon\,.\end{aligned}$$

Proof. Define

$$\begin{aligned} m_0 &\equiv m_0(f,\ \Phi,\ E,\ \chi,\ \varepsilon)\\ &=\mathbf{min}\,(m,\ \Sigma\{J(|\Phi(k)|);\ k\geq m+2\}\leq\varepsilon)\,,\end{aligned}$$

and

$$\begin{aligned}\Psi^*(1)&\equiv\Sigma[\Phi(k);\ k\leq m_0+1]\,,\\ \Psi^*(n)&\equiv\Phi(m_0+n)\qquad\text{for}\quad n\geq 2\,.\end{aligned}$$

PROPOSITION 4.3 (Beppo-Levi theorem). *There are definable E^*, χ^* and θ^* (with parameters F, Φ, A, Ξ, g) such that*

$$\begin{aligned}&\forall n\,\text{itg}\,(F(n),\ \Phi(n),\ A(n),\ \Xi(n))\,,\qquad \Sigma J(|F(n)|)<\infty\\ &\rightarrow\text{ae}\,(x,\ \Sigma F(n,\ x)<\infty,\ E^*,\ \chi^*)\\ &\wedge\,[\forall x\notin E^*(g(x)=\Sigma F(n,\ x)\vdash\text{itg}\,(g,\ \theta^*,\ E^*,\ \chi^*)\wedge J(g)=\Sigma J(F(n))]\,.\end{aligned}$$

Proof. Let Ψ^* be the object defined in Proposition 4.2 above, and define

$$\begin{aligned}&\sigma(n,\ k)\equiv\Psi^*(F(n),\ \Phi(n),\ A(n),\ \Xi(n),\ \exp(2,\ -n-1),\ k)\,,\\ &\theta^*(l)\equiv\sigma(n,\ k)\quad\text{where}\quad\kappa(l)=(n,\ k)\,,\\ &E^*\equiv\{x;\ \Sigma\,|\,\theta^*(l,\ x)\,|\ \text{diverges}\}\cup\bigcup A\,,\\ &\chi^*(j)\equiv\Xi(n,\ m)\quad\text{where}\quad\kappa(j)=(n,\ m)\,.\end{aligned}$$

From Proposition 4.3 follows

PROPOSITION 4.4. *There are E^* and χ^* such that*

$$\begin{aligned}&\text{itg}\,(f,\ \Phi,\ E,\ \chi)\,,\qquad J(|f|)=0\\ &\rightarrow\text{ae}\,(x,\ f(x)=0,\ E^*,\ \chi^*)\,.\end{aligned}$$

PROPOSITION 4.5 (The monotone convergence theorem; the increasing case). *There are Ψ^*, E^* and χ^* such that*

$$\begin{aligned}&\forall n\,\text{itg}\,(F(n),\ \Theta(n),\ A(n),\ \Xi(n))\,,\\ &\text{“}\{F(n)\}_n\ \textit{is increasing,}\text{”}\\ &\exists r\forall n(J(F(n))\leq r)\ \rightarrow\text{ae}\,(x,\ \lim F(n,\ x)<\infty,\ E^*,\ \chi^*)\\ &\wedge\,[\text{ae}\,(x,\ g(x)=\lim F(n,\ x),\ E^*,\ \chi^*)\vdash\ \text{itg}\,(g,\ \Psi^*,\ E^*,\ \chi^*)\wedge J(g)=\lim J(g(n))].\end{aligned}$$

Proof. Put $G(n) \equiv F(n+1) - F(n)$, $\theta(n, 2j+1) \equiv \Xi(n, j)$ and $\theta(n, 2j) \equiv \Xi(n+1, j)$. Then, by virtue of Proposition 4.1,

$$\text{itg}\,(G(n),\ \{\Theta(n+1,\ k) - \Theta(n,\ k)\}_k,\ A(n+1) \cup A(n),\ \theta(n))\,.$$

$$\Sigma[G(n);\ n \leq m-1] = F(m) - F(1)$$

and

$$\Sigma G(n,\ x) = \lim\,(F(m,\ x) - F(1,\ x))$$

if the limit of either side exists.

$$\exists r \forall m (\Sigma\{J(\,|\,G(n)\,|\,);\ n \leq m-1\} \leq r + |\,J(F(1))\,|\,)\,,$$

and so $\Sigma J(\,|\,G(n)\,|\,)$ converges and ae $(x,\ \Sigma G(n,\ x) < \infty,\ E^*,\ \chi^*)$ by Proposition 4.3, where E^* and χ^* are defined in terms of $\{A(n+1) \cup A(n)\}_n$ and θ. Thus,

$$x \notin E^* \rightarrow \lim F(n,\ x) = \Sigma G(n,\ x) + F(1,\ x)\,,$$

or ae $(x,\ \lim F(n,\ x) < \infty,\ E^*,\ \chi^*)$. The latter assertion then follows from Proposition 4.3.

The monotone convergence theorem for the decreasing case can be stated and proved in a similar manner. This and Proposition 4.1 imply

PROPOSITION 4.6 (The continuity property). itg *and* J^1 *satisfy* 4°.

PROPOSITION 4.7 (Fatou's lemma). *There are* Ψ^*, E^* *and* χ^* *such that*

$$\forall n\ \text{itg}\,(F(n),\ \Theta(n),\ A(n),\ \Xi(n))\,,$$

$$\forall n\ \text{ae}\,(x,\ F(n,\ x) \geq 0,\ A(n),\ \Xi(n))\,,$$

$$\begin{aligned}\text{liminf}\,J(F(n)) < \infty \rightarrow\ & \text{ae}\,(x,\ \text{liminf}\,F(n,\ x) < \infty,\ E^*,\ \chi^*)\\ & \wedge [\text{ae}\,(x,\ g(x) = \text{liminf}\,(F(n,\ x),\ E^*,\ \chi^*)\\ & \vdash \text{itg}\,(g,\ \Psi^*,\ E^*,\ \chi^*) \wedge J(g) \leq \text{liminf}\,J(F(n))]\,.\end{aligned}$$

Proof. We may assume $F(n,\ x) \geq 0$ everywhere. Define

$$G(n,\ x) = \inf\,\{F(j,\ x);\ j \geq n\}$$

and

$$\iota(n,\ k) \equiv \min\,[F(j);\ n \leq j \leq n+k-1]\,.$$

$\iota(n,\ k) \in \text{itg}$ for each $(n,\ k)$ (by 1 of Proposition 4.1), and $\{\iota(n,\ k)\}_k$ is decreasing for each n. The assertion is then obtained by repeated applications of Proposition 4.5 (the decreasing case).

PROPOSITION 4.8 (Lebesgue: the dominated convergence theorem). *There are definable* ρ^*, E^* *and* χ^* *such that*

$$\begin{aligned}
&\forall n \,\mathrm{itg}\,(F(n),\ \Theta(n),\ A(n),\ \Xi(n))\,,\\
&\mathrm{ae}\,(x,\ \lim F(n,\ x)<\infty,\ E,\ \chi)\,,\\
&\mathrm{itg}\,(g,\ \Psi,\ E',\ \chi')\,,\\
&\forall n \,\mathrm{ae}\,(x,\ |F(n,\ x)|\le g(x),\ B(n),\ \Lambda(n))\\
&\mathrm{ae}\,(x,\ f(x)=\lim F(n,\ x),\ E,\ \chi)\\
\to &\mathrm{itg}\,(f,\ \rho^*,\ E^*,\ \chi^*)\wedge J(f)=\lim J(F(n))\,.
\end{aligned}$$

§5. Comparison theorems

We shall henceforth dispense with the explicit denotation of the parameters unless it is essential.

THEOREM 5.1 (Comparison theorem). *Let i be either 1 or 2, let $\mathscr{B}_i$ denote the axiom set $\mathscr{B}$ on (X, L_i, J_i) and let L_i^1 and J_i^1 be respectively the family of L_i-integrable functions and the L_i-integral. (See Definitions 1.3 and 3.1.) We assume $\mathscr{A}$, $\mathscr{B}_1$, $\mathscr{B}_2$ and $\mathscr{C}$ as the axioms.*

Suppose $L_1 \subset L_2^1$ and $\forall\phi(J_1(\phi)=J_2^1(\phi))$. Then $L_1^1\subset L_2^1$, and J_1^1 and J_2^1 coincide on L_1^1.

Proof. Suppose $L_1^1(f,\ \Phi,\ E,\ \chi)$. We may assume $L_1^*(f,\ \theta)$ for some θ (Proposition 3.3); hence $\theta\subset L_1\subset L_2^1$. So,

$$\forall n L_2^1(\theta(n), \Theta(n), A(n), \Xi(n))\,,\quad \Sigma J_1(|\theta(n)|)<\infty\,,$$

$$\forall x(\Sigma|\theta(n,\ x)|<\infty \vdash f(x)=\Sigma\theta(n,\ x))\,.$$

These relations imply

$$\Sigma J_2^1(|\theta(n)|)=\Sigma J_1(|\theta(n)|)<\infty\,.$$

By this and Proposition 4.3, we obtain $\mathrm{ae}\,(x,\ \Sigma|\theta(n,x)|<\infty,\ E^*,\ \chi^*)$ for some E^* and χ^*, where ae is understood to be "almost everywhere" with regards to $\mathscr{B}_2$, and hence

$$\mathrm{ae}\,(x,\ \Sigma\theta(n,\ x)=f(x),\ E^*,\ \chi^*)$$

in $\mathscr{B}_2$. We can thus conclude $L_2^1(f,\ \theta,\ E^*,\ \chi^*)$ and

$$\begin{aligned}
J_2^1(f,\ \theta,\ E^*,\ \chi^*)&=\Sigma J_2^1(\theta(n),\ \Theta(n),\ A(n),\ \Xi(n))\\
&=\Sigma J_1(\theta(n))\\
&=J_1^1(f,\ \Phi,\ E,\ \chi)\,.
\end{aligned}$$

COROLLARY. *Assume $\mathscr{A}$, $\mathscr{B}_1$, $\mathscr{B}_2$, $\mathscr{C}$, $L_1\subset L_2^1$, $L_2\subset L_1^1$, $\forall\phi\in L_1(J_1(\phi)=J_2^1(\phi))$, $\forall\psi\in L_2(J_2(\psi)=J_1^1(\psi))$. Then $L_1^1=L_2^1$ and $f\in L_1^1=L_2^1\to J_1^1(f)=J_2^1(f)$.*

THEOREM 5.2. *The coefficient set R (the set of reals) can be replaced by Q (the set of rationals) in the axioms of $\mathscr{B}$ (Definition 1.3). That is, in 1° and 3° of $\mathscr{B}$, $\forall a \in R \forall \phi \in L\ (a\phi \in L)$ and $\forall a \in R \forall \phi \in L(J(a\phi) = aJ(\phi))$ respectively can be replaced by $\forall r \in Q \forall \phi \in L\ (r\phi \in L)$ and $\forall r \in Q A\phi \in L(J(r\phi) = rJ(\phi))$.*

This means that one can develop the definable theory of integration with the modified axioms. Let us denote the modified $\mathscr{B}$ by $\mathscr{B}'$.

Proof. First note that we can redo the material in the preceding sections with the coefficient set Q instead of R.

Suppose $a \in R$ and $\phi \in L$.

$$a\phi = \lim \{r_n\phi;\ a-(1/n) < r_n < a+(1/n)\}\ ,$$

where r_n is a rational number satisfying the inequalities above, which can be definably specified by a and n. From this follows that $a\phi \in L^1$ and $J^1(a\phi) = \lim J(r_n\phi) = aJ(\phi)$. Let L_1 and J_1 respectively denote the L and the J in $\mathscr{B}$, and let L_2 and J_2 denote the corresponding notions in $\mathscr{B}'$. Then $L_1 \subset L_2^1$ and $J_1 = J_2^1$ on L_1. So, by Theorem 5.1, $L_1^1 \subset L_2^1$. The opposite inclusion is trivial, and thus $L_1^1 = L_2^1$.

§6. Measurability

DEFINITION 6.1. 5°. $\forall \phi(L(\phi) \vdash L(1 \wedge \phi))$, where 1 denotes $\{x\}\{t\}(t<1)$ here.

$$\mathscr{B}_0 : \mathscr{B} + \{5°\}$$

From now on (to Section 10) we work in the theory $\mathscr{J}$ with $\mathscr{B}_0$ instead of $\mathscr{B}$.

DEFINITION 6.2. 1) $\text{mbl}\,(f, \Theta, \Lambda, \Xi)$:

$$\forall \phi \in L(+)\ \text{itg}\,(\text{mid}(-\phi, f, \phi), \Theta(\phi), \Lambda(\phi), \Xi(\phi))$$

(f is measurable with respect to the parameters Θ, Λ and Ξ; the parameters may be abbreviated to a single letter W, or even omitted altogether.)

Note. mbl is not a definable notion.

2) $\text{mbl}'\,(f, \Theta', \Lambda', \Xi')$:

$$\forall \phi \in L(+)(\text{itg}\,(f^+ \wedge \phi, \Theta'(1), \Lambda'(1), \Xi'(1))$$
$$\wedge\ \text{itg}\,(f^- \wedge \phi, \Theta'(2), \Lambda'(2), \Xi'(2))$$

PROPOSITION 6.1. mbl *and* mbl′ *are mutually definably interpretable.*

Proof. This follows from the facts $f^+ \wedge \phi = (\text{mid}\,(-\phi, f, \phi))^+$ and $f^- \wedge \phi = (\text{mid}\,(-\phi, f, \phi))^+$ and Proposition 4.1.

PROPOSITION 6.2. 1) *There is a finite sequence W^* of appropriate objects such that* $\text{itg}\,(f, \Phi, E, \chi) \rightarrow \text{mbl}\,(f, W^*)$.

2) $\text{mbl}\,(1, W^*)$ *for some* W^*.

3) $a \in R$, mbl $(f) \to$ "*af, f^+, f^- and $|f|$ are measurable.*"

In fact we claim a more refined result. Consider af as an example. There are definable Φ^* and W^* (with appropriate parameters) such that

$$\phi \in L(+) \to \forall n(\Phi^*(n) \in L(+))$$

and

$$\phi \in L(+), \qquad \forall n \,\mathrm{itg}\,(\mathrm{mid}\,(-\Phi^*(n), f, \Phi^*(n)), W(\Phi^*(n)))$$
$$\to \mathrm{itg}\,(\mathrm{mid}\,(-\phi, af, \phi), W^*(\phi)).$$

From these and by the definable comprehension rule, we obtain

$$\mathrm{mbl}\,(f, W) \to \mathrm{mgl}\,(af, W^*).$$

4) mbl *is closed under the finite sum*, max *and* min. *See* 3) *above for the precise form of our claim.*

5) mbl *is closed under the limit.*

Proof. 3) Consider af as an example. Define

$$g(n) \equiv \mathrm{mid}\,(-\phi, a\,\mathrm{mid}\,(-n\phi, f, n\phi), \phi)$$

and $\Phi^*(n) \equiv n\phi$. Then $\phi \in L(+) \to \Phi^*(n) \in L(+)$.

$$\mathrm{itg}\,(\mathrm{mid}\,(-\Phi^*(n), f, \Phi^*(n)), W(\Phi^*(n)))$$

implies itg $(g(n), U^*(n))$ for some U^*. $|g(n)| \leq \phi$, presuming that $\phi \in L(+)$. $\lim g(n) = \mathrm{mid}\,(-\phi, af, \phi)$ everywhere. So, by the Lebesgue's dominated convergence theorem,

$$\mathrm{itg}\,(\mathrm{mid}\,(-\phi, af, \phi), W^*(\phi)).$$

4) Consider first the finite sum, $\Sigma[f(i); i \leq m]$; suppose $\phi \in L(+)$. Define $\Phi^*(n) \equiv n\phi$ and

$$g(n) \equiv \mathrm{mid}\,(-\phi, \Sigma[\mathrm{mid}\,(-n\phi, f(i), n\phi); i \leq m], \phi).$$

Then as above

$$\lim g(n) = \mathrm{mid}\,(-\phi, \Sigma[f(i); i \leq m], \phi) \quad \text{and} \quad |g(n)| \leq \phi,$$

and hence we obtain $\forall n(\Phi^*(n) \in L(+))$ and

$$\forall i \leq m \forall n \,\mathrm{itg}\,(\mathrm{mid}\,(-\Phi^*(n), f(i), \Phi^*(n)), W(\Phi^*(n)))$$
$$\to \mathrm{itg}\,(\mathrm{mid}\,(-\phi, \Sigma[f(i); i \leq m], \phi), W^*(\phi)).$$

To deal with the finite maximum, max $[f(i); i \leq m]$, first recall that

$$\max(f, g) = f \vee g = 1/2(f+g) + 1/2\,|f-g|.$$

Using the results in 3), we obtain

(1)
$$\phi\in L(+)\to\forall n(\Phi_1(n)\in L(+)\wedge\Phi_2(n)\in L(+))$$
$$\wedge[\forall n\ \mathrm{itg}\,(\mathrm{mid}\,(-\Phi_1(n), f, \Phi_1(n)),\ W(\Phi_1(n)))$$
$$\wedge\forall n\ \mathrm{itg}\,(\mathrm{mid}\,(-\Phi_2(n), g, \Phi_2(n)),\ W(\Phi_2(n)))$$
$$\vdash\ \mathrm{itg}\,(\mathrm{mid}\,(-\phi, f\vee g, \phi),\ W^*(\phi))],$$

for some definable Φ_1, Φ_2 and W^*. Notice that (1) consists of definable formulas with parameters f and g (among others). By an application of DDI with (1) as the defining formula, we can construct Θ^* and U^* so that

$$\phi\in L(+),\qquad j\leq m\to\forall n\forall i\leq j(\Theta^*(i, n)\in L(+))$$
$$\wedge[\forall i\leq j\forall n\ \mathrm{itg}\,(\mathrm{mid}\,(-\Theta^*(i, n),\ f(i),\ \Theta^*(i, n)),\ W(\Theta^*(i, n)))$$
$$\vdash\mathrm{itg}\,(\mathrm{mid}\,(-\phi,\ \max\,[f(i);\ i\leq j],\ \phi),\ U^*(\phi))].$$

The proof is by induction on $j\ (\leq m)$ applied to the formula in the succedent, which is definable.

DEFINITION 6.3. $\mathrm{mbl}''\,(f, \Theta'', \Lambda'', \Xi'')$:
$$\forall g\forall\Phi\forall E\,\forall\chi(\mathrm{itg}\,(g, \Phi, E, \chi)\wedge g\geq 0$$
$$\vdash\mathrm{itg}\,(\mathrm{mid}\,(-g, f, g),\ \Theta''(W),\ \Lambda''(W),\ \Xi''(W)))$$
where W stands for the sequence of relevant parameters.

PROPOSITION 6.3. mbl *and* mbl$''$ *are mutually definably interpretable.*

PROPOSITION 6.4. *If different classes of elementary functions yield the same classes of integrable functions, then the corresponding classes of measurable functions are also identical.*

This is a corollary of Proposition 6.3 above; namely, mbl$''$ is expressed in terms of integrable functions (the g there).

From Proposition 6.3 follows also

PROPOSITION 6.5. $\mathrm{mbl}\,(f, \Theta, \Lambda, \Xi),\ g\geq 0,\ \mathrm{itg}\,(g, \Phi, E, \chi)$,
$$\mathrm{ae}\,(x,\ |f(x)|\leq g(x),\ E,\ \chi)\to\mathrm{itg}\,(f, \Phi^*, E^*, \chi^*)$$
for some Φ^*, E^* *and* χ^*.

PROPOSITION 6.6. mbl *forms an algebra; that is, measurability is closed under the scalar product, the finite sum and the finite product.*

Proof. We have only to consider the product. (See Proposition 6.2.) As in the usual mathematical proof, notice first that

$$\exp\,(a, 2)=\lim\,\{\max\,\{2r_k a-\exp\,(r_k, 2);\ 1\leq k\leq n\};\ n=1, 2, \cdots\},$$

where $\{r_k\}$ is a definable enumeration of the rationals, and

$$ab = [\exp(a+b, 2) - \exp(a-b, 2)]/4 .$$

Using these facts and the results in Proposition 6.2, the assertion for the finite product can be established by an application of DDI in a manner similar to the proof in 4) there.

DEFINITION 6.4. $C(\rho)$ will stand for the following condition:

$$\rho \subset L(+) \wedge \forall x \forall r > 0 \exists m \forall n \geq m (\rho(n, x) > r) .$$

PROPOSITION 6.7. *Under the assumption of $C(\rho)$, every measurable function is the limit of a sequence of integrable functions.*

Proof. Let f be measurable. Then $\mathrm{mid}\,(-\rho(n), f, \rho(n))$ is also, and

$$f \equiv \lim \mathrm{mid}\,(-\rho(n), f, \rho(n)) .$$

We shall henceforth assume the condition $C(\rho)$, since in most examples of measurable functions such a ρ can be found.

DEFINITION 6.5. $\mu(f, W)$: $\limsup J^1(\mathrm{mid}\,(-\rho(n), f, \rho(n)), W(\rho))$, where W is a sequence of appropriate parameters.

PROPOSITION 6.8. 1) $\mathrm{mbl}\,(f, W) \rightarrow$ "*$\mu(f, W)$ is uniquely determined by f and $\mu(f, W) \in R \cup \{\infty\}$.*"

2) $\mathrm{itg}\,(f, \Phi, E, \chi) \rightarrow \mu(f, W^*) = J^1(f, \Phi, E, \chi)$.

3) $\mathrm{mbl}\,(f, W_1), \mathrm{mbl}\,(f, W_2), f \leq g \rightarrow \mu(f, W_1) \leq \mu(g, W_1)$.

4) $\forall i\, \mathrm{mbl}\,(F(i), W(i)) \rightarrow \mu(\Sigma F(i), W^*) = \Sigma \mu(F(i), W(i))$ *for some* W^*.

5) $\mathrm{mbl}\,(f, W), a \in R \rightarrow \mu(af, W^*) = a\mu(f, W)$.

Proof of 2).

$$f = \lim \mathrm{mid}\,(-\rho(n), f, \rho(n))$$

by $C(\rho)$. If f is integrable, then so is $\phi(n) \equiv \mathrm{mid}\,(-\rho(n), f, \rho(n))$. Since $|\phi(n)| \leq f$, the Lebesgue's dominated convergence theorem applies and

$$J^1(f) = \lim J^1(\phi(n)) = \mu(f) .$$

§7. Measurable sets

DEFINITION 7.1. $P(D, x, t)$:

$$(x \in D \wedge t < 1) \vee (x \notin D \wedge t < 0)$$

$$\chi_D: \{x\}\{t\} P(D, x, t)$$

$$\mathrm{itgs}\,(D, \Phi, E, \chi);\ \mathrm{itg}\,(\chi_D, \Phi, E, \chi)$$

$$\text{mbls}\,(D, \Theta, \Lambda, \Xi): \text{mbl}\,(\chi_D, \Theta, \Lambda, \Xi)$$

$$\mu s(D, \Theta, \Lambda, \Xi): \mu(\chi_D, \Theta, \Lambda, \Xi)$$

We shall abbreviate the parameters to U, W, $\cdots$, or even omit them in the expressions above whenever possible. We also assume $\text{ss}\,(X, D)$, $\text{ss}\,(X, D_i)$ etc. throughout.

PROPOSITION 7.1. *The properties* itgs *and* mbls, *and the function* μs *are uniquely determined by D, independent of the parameters. They respectively express "D is an integrable set," "D is a measurable set" and "the measure of D."*

The mathematical proofs go through, now that the basic properties of the integration and the measurability of functions have been established in our theory.

PROPOSITION 7.2. 1) *The following* (1) *to* (5) *are all mutually definably interpretable.*

(1) $\text{mbl}\,(f)$
(2) $\forall r\, \text{mbls}\,(\{x;\, f(x) \geq r\})$
(3) $\forall r\, \text{mbls}\,(\{x;\, f(x) > r\})$
(4) $\forall r\, \text{mbls}\,(\{x;\, f(x) \leq r\})$
(5) $\forall r\, \text{mbls}\,(\{x;\, f(x) < r\})$

We have omitted the parameters.

2) *Any of* (1) *to* (5) *implies*

(6) $\forall r \in Q \cup \{-\infty, \infty\}\, \text{mbls}\,(\{x;\, f(x) = r\})$.

The mathematical proof goes through since the objects which are used there are uniformly definable.

PROPOSITION 7.3 (Egoroff). *There are definable M^* and W^* such that*

$$\text{itgs}\,(D, U),\ \forall x \notin D(f(x)=0 \wedge \forall n F(n, x)=0)\,,$$

$$\text{mbl}\,(f, W),\ \forall n\, \text{mbl}\,(F(n), W),\ \text{nls}\,(C, \theta)\,, \qquad C \subset D\,,$$

$$\forall x \in D - C(f(x) = \lim F(n, x))\,, \qquad \varepsilon > 0$$

$$\to \text{mbls}\,(M^*(\varepsilon), W^*(\varepsilon)) \wedge \mu s(M^*(\varepsilon), W^*(\varepsilon)) \leq \varepsilon$$

$$\wedge \text{“}\lim F(n) = f \text{ uniformly on } D - M^*(\varepsilon).\text{”}$$

Proof. Define

$$A(n, k) \equiv \{x; \in D \wedge |f(x) - F(k, x)| \geq (1/n)\}$$

and

$$B(n, m) \equiv \bigcup \{A(n, k);\, k \leq m\}\,.$$

Then

$$\forall n(\{B(n, m);\, m \geq 1\} \text{ is decreasing})$$

and

$$\forall n(\bigcap\{B(n, m); m \geq 1\} \subset C).$$

So, nls $(\bigcap\{B(n, m); m \geq 1\}, \theta)$, and hence

$$\lim \{\mu s(B(n, m), V^*); m \geq 1\} = 0$$

for some V^*. Thus,

$$\nu(n, \varepsilon) = \mathbf{min}\ (m, \mu s(B(n, m), V^*) \leq \varepsilon \exp(2, -n))$$

is well-defined. Put

$$M^*(\varepsilon) \equiv \bigcup\{B(n, \nu(n, \varepsilon)); n \geq 1\}.$$

W^* can be induced from M^*.

§ 8. Axiomatic measure theory

DEFINITION 8.1. 1) Atomic symbols: X, $\mathscr{M}$, σ

2) The axiom set $\mathscr{D}$: the axioms on the measure space $(X, \mathscr{M}, \sigma)$

THEOREM 8.1. *The axiom set $\mathscr{B}$ in Definition 1.3 and the $\mathscr{D}$ above are mutually definably interpretable.*

Proof. The interpretation of $\mathscr{D}$ in terms of $\mathscr{B}$ has been carried out in Section 7. For the converse, let $H(\phi, n)$ denote

$$n \simeq (r_1, \cdots, r_k) \wedge \forall i \leq k(\{x; \phi(x) = r_i\} \in \mathscr{M}$$
$$\wedge\ \sigma(\{x; \phi(x) = r_i\}) < \infty$$
$$\wedge\ \phi = \Sigma[r_i \chi_{A(i)}; i \leq k],$$

where $n \simeq (r_1, \cdots, r_k)$ expresses that n represents a k-tuple of distinct rationals and $A(i) \equiv \{x; \phi(x) = r_i\}$.

$$\nu(\phi) = \mathbf{min}\ (n, H(\phi, n))$$

is well-defined, presuming that

$$\mathrm{mp}\ (\phi, X, R) \wedge \exists n H(\phi, n).$$

Now let $L(\phi)$ denote $\mathrm{mp}\ (\phi, X, R) \wedge \exists n H(\phi, n)$, and define $J(\phi)$ to be $\Sigma\{r_i \sigma(A(i)); i \leq k\}$ when $\nu(\phi) \simeq (r_1, \cdots, r_k)$.

The mathematical proof for the fact that 1° to 5° (Definitions 1.3 and 6.1) are satisfied by these L and J can be formalized in the theory with $\mathscr{D}$.

§ 9. Simple functions

In this section, we consider L, J, itg and J^1 in the context of Theorem 5.2.

DEFINITION 9.1. We shall use $\boldsymbol{n}$ to denote a natural number in a specific context.

sqn ($\boldsymbol{n}$): "$\boldsymbol{n}$ is a finite sequence of distinct rationals, say $(r_1, \cdots, r_l)$, arranged in the natural, increasing order."

lg ($\boldsymbol{n}$): the length of $\boldsymbol{n}$; that is, the l above.

$\boldsymbol{n}(k)$: r_k if $1 \le k \le l$.

$\mathscr{S}_0(\alpha, \boldsymbol{n}, \Phi, E, \chi)$: $\forall x \exists ! k \le \lg(\boldsymbol{n})(\alpha(x) = r_k) \wedge \mathrm{itg}(\alpha, \Phi, E, \chi)$

(α is a simple function with respect to $\boldsymbol{n}$, Φ, E, χ.)

$K(\alpha, \boldsymbol{n}, k)$: χ_D, where $D \equiv \{x;\ \alpha(x) = r_k\}$. See Definition 7.1 for χ_D.

$J_0(\alpha, \boldsymbol{n}, \Phi, E, \chi)$: $\Sigma\{r_k J^1(K(\alpha, \boldsymbol{n}, k), \phi, E, \chi);\ k \le \lg(\boldsymbol{n})\}$

We may omit or abbreviate Φ, E, χ, and even $\boldsymbol{n}$ when the circumstances allow us to.

PROPOSITION 9.1. 1) *The functions and the predicates defined above are definable. In particular,* J_0 *is arithmetically definable.*

2) $\mathscr{S}_0(\alpha, \boldsymbol{m}, W_1), \mathscr{S}_0(\alpha, \boldsymbol{n}, W_2) \rightarrow \boldsymbol{m} = \boldsymbol{n}$.

3) $\mathscr{S}_0(\alpha, \boldsymbol{n}, W) \rightarrow \forall k \le l \mathscr{S}_0(K(\alpha, \boldsymbol{n}, k), U(k))$ *for some* U.

4) $J_0(\alpha, \boldsymbol{n}, W)$ *is uniquely determined by* α, *presuming that* $\mathscr{S}_0(\alpha, \boldsymbol{n}, W)$ *holds, and then* $J_0(\alpha) = J^1(\alpha)$.

5) $\mathscr{S}_0$ *satisfies the axioms in* 1° *to* 5° *for the rational coefficients; that is, the axioms in* $\mathscr{B}'$ *are satisfied by* $\mathscr{S}_0$. *See Theorem* 5.2 *for* $\mathscr{B}'$.

Proof. 4) If $\mathscr{S}_0(\alpha, \boldsymbol{n}, W)$, then $\alpha = \Sigma[r_k K(\alpha, \boldsymbol{n}, k);\ k \le l]$. Since $\mathscr{S}_0(K(\alpha, \boldsymbol{n}, k))$ by 3), itg $(K(\alpha, \boldsymbol{n}, k))$, and hence

$$J_0(\alpha) = \Sigma\{r_k J^1(K(\alpha, \boldsymbol{n}, k));\ k \le l\}$$

is well-defined. $J_0(\alpha) = J^1(\alpha)$ follows from Proposition 4.1.

5) Let us first take up the finite maximum as an example. Suppose $\mathscr{S}_0(\alpha, \boldsymbol{m}, W_1)$ and $\mathscr{S}_0(\beta, \boldsymbol{n}, W_2)$ hold. Then itg $(\alpha \vee \beta, W^*)$ for some W^* (Proposition 4.1). Let $\boldsymbol{j}$ denote the finite sequence of rationals which consists of all the distinct rationals among those in $\boldsymbol{m}$ and $\boldsymbol{n}$ and which is arranged in the increasing order. ($\boldsymbol{j}$ is definable in $\boldsymbol{m}$ and $\boldsymbol{n}$.) Then $\mathscr{S}_0(\alpha \vee \beta, \boldsymbol{j}, W^*)$. The general case can be dealt with similarly.

The linearity of J_0 easily follows from 4) and Proposition 4.1. The continuity of J_0 is also a trivial consequence of 4) and Proposition 4.1.

$$1 \wedge \alpha = \Sigma[\min(1, r_k) K(\alpha, \boldsymbol{n}, k);\ k \le l].$$

Let $\boldsymbol{j}$ be the sequence of distinct numbers among $\{\min(1, r_k);\ k \le l\}$ arranged in the increasing order. Let $C(k)$ be $\{x;\ \alpha(x) = r_k\}$ and let $D(i)$ be $\bigcup\{D(k);\ \min(1, r_k) =$

$j(i)\}$, $i \leq \lg(j)$. Then itg $(\chi_{D(i)})$ and $1 \wedge \alpha$ is a linear combination of $\chi_{D(i)}$, $i \leq \lg(j)$, with the rational coefficients. Thus $\mathscr{S}_0(1 \wedge \alpha, j)$.

PROPOSITION 9.2. *Let $\mathscr{S}'_0$ denote the class of "integrable" functions based on $(\mathscr{S}_0, J_0)$, and let J'_0 be the integral of the functions in $\mathscr{S}'_0$. (See Definition 1.3 and Theorem 5.2.) Then $\mathscr{S}'_0 \subset$ itg and $J'_0 = J^1$ on $\mathscr{S}'_0$.*

This follows immediately from Proposition 9.1 and a modified version (in the context of Theorem 5.2) of Theorem 5.1.

Let us give a more precise presentation of Proposition 9.2. First define nls_0, ae_0 and $\mathscr{S}'_0$. U and V will represent finite sequences of parameters.

$$\begin{aligned} &\mathrm{nls}_0(E, \chi, U)\colon \mathrm{ss}(X, E) \wedge \forall j \mathscr{S}_0(\chi(j), U) \\ &\quad \wedge \forall j(\chi(j) \leq \chi(j+1)) \\ &\quad \wedge \forall x \in E \forall r > 0 \exists j(\chi(j, x) > r) \\ &\quad \wedge \lim J_0(\chi(j), U) \in R \\ &\mathrm{ae}_0(x, P, E, \chi, U)\colon \mathrm{nls}_0(E, \chi, U) \wedge \forall x \notin E\, P(x) \\ &\mathscr{S}'_0(f, \Phi, E, \chi, U, \Psi, V)\colon \forall i \mathscr{S}_0(\Psi(i), V(i)) \\ &\quad \wedge \mathrm{ae}_0(x, f(x) = \Sigma \Psi(i, x), E, \chi, U) \\ &\quad \wedge \Sigma J_0(|\Psi(i)|, V(i)) \in R \end{aligned}$$

The proposition then claims that there is a finite sequence of definable parameters V^* defined from $f, \Phi, E, \chi, U, \Psi, V$ such that

$$\mathscr{S}'_0(f, \Phi, E, \chi, U, \Psi, V) \rightarrow \mathrm{itg}(f, V^*)$$

and $J'_0(f, \Phi, E, \chi, U, \Psi, V) = J^1(f, V^*)$.

PROPOSITION 9.3. *$L \subset \mathscr{S}'_0$ and $J = J'_0$ on L.*

Proof. Suppose $\phi \in L$. We can construct Ψ and v so that, for each j,

$$\mathrm{itg}(\Psi(n)) \wedge \mathscr{S}_0(\Psi(n), v(n)) \wedge J_0(\Psi(n)) \leq J(\phi),$$

and that $\{\Psi(n)\}_n$ converges monotonically to ϕ from below. Then, by the monotone convergence theorem, $\mathscr{S}'_0(\phi)$ and

$$J'_0(\phi) = \lim J_0(\Psi(n)) = \lim J^1(\Psi(n)) = J^1(\phi) = J(\phi).$$

We present the explicit construction of Ψ and v, although it is nearly a copy of the usual mathematical proof. First define $E(k, n, i)$, $k = 1, 2, 3, 4$.

$$E(1, n, i) \equiv \mathrm{inv}(\phi, [(i-1)/\exp(2, n), i/\exp(2, n))), \qquad 1 \leq i \leq n \exp(2, n)$$

where $\mathrm{inv}(f, I) = \{x; f(x) \in I\}$.

$$E(2, n, i) \equiv \mathrm{inv}\,(\phi, [-i/\exp(2, n), -(i-1)/\exp(2, n))), \qquad 2 \leq i \leq n \exp(2, n).$$

$$E(3, n, 0) = \mathrm{inv}\,(\phi, [n, \infty)).$$

$$E(4, n, 0) = \mathrm{inv}\,(\phi, (-\infty, -n)).$$

For every $x \in X$, there is a unique (k, n, i) satisfying a certain condition such that $x \in E(k, n, i)$. Now define

$$r(k, n, i) = \begin{cases} (i-1)/\exp(2, n) & \text{if } k=1, \\ -i/\exp(2, n) & \text{if } k=2, \\ n & \text{if } k=3, \\ -(n+1) & \text{if } k=4. \end{cases}$$

For each n, re-arrange $\{(k, i)\}$, so that $\{r(k, n, i)\}$ becomes an increasing sequence, say $s(n, 1) < \cdots < s(n, p(n))$. Then define

$$\Psi(n) \equiv \Sigma[s(n, j)\chi_{E(j)};\ 1 \leq j \leq p(n)],$$

where $E(j) = E(k, n, i)$ for appropriate k and i. $v(n)$ is defined to be $(s(n, 1), \cdots, s(n, p(n)))$.

THEOREM 9.1. *$\mathscr{S}'_0 = \mathrm{itg}$ and $J'_0 = J^1$.*

This follows from Propositions 9.2 and 9.3, and Theorem 5.1.

Conclusion. We may assume $\mathscr{S}_0$ as the class of elementary functions in developing the theory of integration in our definable system.

§ 10. Integration of functions of two variables

DEFINITION 10.1. 1) Let $\mathscr{J}_2$ be the theory $\mathscr{J}$ with the axiom sets of two integration spaces $\mathscr{X} = (X, L_1, J_1)$ and $\mathscr{Y} = (Y, L_2, J_2)$ in the place of $\mathscr{B}$. (See Definition 1.3 and Theorem in Section 1.) The properties which are claimed in this section are the theorems of $\mathscr{J}_2$.

2) $Z \equiv X \times Y = \{(x, y); x \in X, y \in Y\}$

$$(x, y) = (u, v) \colon x = u \wedge y = v$$

3) $\mathscr{S}_1$: the $\mathscr{S}_0$ for L_1 in $\mathscr{X}$
$\mathscr{S}_2$: the $\mathscr{S}_0$ for L_2 in $\mathscr{Y}$
I_1: the J_0 for $\mathscr{S}_1$
I_2: the J_0 for $\mathscr{S}_2$

(See Definition 9.1.)

4) $\mathscr{S}(m, \alpha, \xi, \beta, \eta) \colon \forall k \leq m(\mathscr{S}_1(\alpha(k), \xi(k)) \wedge \mathscr{S}_2(\beta(k), \eta(k)))$, where $\xi(k)$ and $\eta(k)$ each stands for four parameters.

5) $\pi(m, \alpha, \beta, z) \colon \Sigma\{\alpha(k, x)\beta(k, y);\ k \leq m\}$,

where $z=(x, y)$.

PROPOSITION 10.1. 1) $\mathscr{S}(m, \alpha, \xi, \beta, \eta) \to \mathrm{mp}\,(\pi(m, \alpha, \beta), Z, Q)$.

2) *$\mathscr{S}$ is a linear space with the rational coefficients. The meaning of this assertion will become clear in the proof.*

3) *$\mathscr{S}$ is closed with respect to the absolute value.*

Proof. 2) Assume, for example, $\mathscr{S}(m, \alpha, \xi, \beta, \eta)$ and $\mathscr{S}(n, \gamma, \zeta, \delta, \theta)$. Define $l=m+n$, $\alpha^*(k)\equiv\alpha(k)$ if $1\le k\le m$, and $\alpha^*(m+k)\equiv\gamma(k)$ if $1\le k\le n$. ξ^*, β^* and η^* are defined similarly. Then $\mathscr{S}(l, \alpha^*, \xi^*, \beta^*, \eta^*)$ and

$$\pi(m, \alpha, \beta)+\pi(n, \gamma, \delta)=\pi(l, \alpha^*, \beta^*)\,.$$

3) What must be shown is that there are n, α^*, ξ^*, β^* and η^* for which holds

$$\mathscr{S}(m, \alpha, \xi, \beta, \eta) \to \mathscr{S}(n, \alpha^*, \xi^*, \beta^*, \eta^*) \wedge |\pi(m, \alpha, \beta)|=\pi(n, \alpha^*, \beta^*)\,.$$

Suppose the first entry of ξ is v. Let

$$\{\lambda(k, i);\ k\le m \text{ and } i\le n \text{ for some } n\}$$

exhaust all the distinct m-tuples $(v(1, j_1), \cdots, v(m, j_m))$, $j_k\le \lg\,(v(k))$, $k\le m$, which are different from the origin, and define

$$\omega(i, y)=\Sigma\{\lambda(k, i)\beta(k, y)\,;\ k\le m\}\,.$$

Then $\omega(i)\in\mathscr{S}_2$, and hence $|\omega(i)|\in\mathscr{S}_2$ (for some definable parameters) if $i\le n$. (See Proposition 9.1.) Define

$$D(i)\equiv\{x\,;\ \Sigma\{\exp\,(\alpha(k, x)-\lambda(k, i), 2)\,;\ k\le m\}=0\}\,.$$

$\chi_{D(i)}\in\mathscr{S}_1$. Since

$$|\pi(m, \alpha, \beta, z)|=\Sigma\{\chi_{D(i)}(x)\,|\omega(i, y)|\,i\le n\}\,,$$

we can take $\{i\}\chi_{D(i)}$ as α^* and $\{i\}\,|\omega(i)|$ as β^*.

PROPOSITION 10.2. *Put $\tau_x\equiv\{v\}\pi(m, \alpha, \beta, x, v)$ and $\tau_y\equiv\{u\}\pi(m, \alpha, \beta, u, y)$. Then*

$$\begin{aligned}
&\mathscr{S}(m, \alpha, \xi, \beta, \eta),\ x\in X,\ y\in Y\\
&\to\mathscr{S}_2(\tau_x)\wedge\mathscr{S}_1(\tau_y)\wedge\mathscr{S}_2(\{y\}I_1(\tau_y))\wedge\mathscr{S}_1(\{x\}I_2(\tau_x))\\
&\quad\wedge[I_2^1(\{y\}I_1(\tau_y))\\
&=I_1^1(\{x\}I_2(\tau_x))\\
&=\Sigma\{I_1(\alpha(k))I_2(\beta(k))\,;\ k\le m\}]\,.
\end{aligned}$$

(We have omitted the parameters, which are definable objects.)

DEFINITION 10.2. $I(m, \alpha, \beta)\colon I_2^1(\{y\}I_1(\tau_y))$

PROPOSITION 10.3. $\mathscr{S}(m, \alpha, \xi, \beta, \eta)$

$$\to I(m, \alpha, \beta) = I_1^1(\{x\}I_2(\tau_x))$$
$$= \Sigma\{I_1(\alpha(k))I_2(\beta(k))\,;\ k \le m\} \in R\,.$$

PROPOSITION 10.4. *$(\mathscr{S}, I)$ satisfies* 1° *to* 5° *(over the coefficient set Q).*

Proof. The elementary properties can be derived from Proposition 10.1 by using DDI and induction (applied to definable formulas). For the continuity, suppose $\Phi \subset \mathscr{S}$ and that Φ decreases to 0. Precisely, let ν, Ψ_1, Ψ_2, Ξ_1 and Ξ_2 be parameters of appropriate types. Suppose $\forall n \mathscr{S}(\nu(n),\ \Psi_1(n),\ \Xi_1(n),\ \Psi_2(n),\ \Xi_2(n)),\ \Phi(n) \equiv \pi(\nu(n), \Psi_1(n), \Psi_2(n))$ and that $\{\Phi(n, z)\}_n$ decreases to 0 for each $z \in Z$. Let $\tau(n; x)$ denote the τ_x in Proposition 10.2, where $m = \nu(n)$, $\alpha \equiv \Psi_1(n)$ and $\beta \equiv \Psi_2(n)$. Similarly for $\tau(n; y)$. Then

$$\forall y(\mathscr{S}_1(\tau(n; y)) \wedge \text{“}\tau(n; y) \text{ decreases to } 0\text{”} \wedge \mathscr{S}_2(I_1(\tau(n; y)))$$

by Proposition 10.2. This and 4° for $\mathscr{X}$ imply $\lim I_1\ (\tau(n; y)) = 0$; this and 4° for $\mathscr{Y}$ imply $\lim I_2(\{y\}I_1(\tau(n; y))) = 0$, or $I(\Phi(n))$ tends to 0.

To prove 5°, suppose $\mathscr{S}(m, \alpha, \xi, \beta, \eta)$. Then

$$\pi(m, \alpha, \beta, z) = \Sigma\{\chi_{D(i)}(x)\omega(i, y)\,;\ i \le n\}\,.$$

(See the proof of Proposition 10.1.)

$$(\pi(m, \alpha, \beta) \wedge 1)\ (z) = \Sigma\{\chi_{D(i)}(x)(\omega(i, y) \wedge 1);\ i \le n\}\,,$$

and $\mathscr{S}_2(\{y\}(\omega(i, y) \wedge 1))$ by 5° for $\mathscr{Y}$. This proves $\pi(m, \alpha, \beta) \wedge 1 \in \mathscr{S}$.

DEFINITION 10.3. f_y: $\{x\}f(x, y)$

$I[1, 2, f]$: $I_2^1\{y\}(I_1^1(f_y))$

$I[2, 1, f]$: $I_1^1(\{x\}I_2^1(f_x))$

$\mathscr{S}^1(f, \Phi, E, \chi)$: “$f$ is integrable with respect to Φ, E and χ in the theory of $(Z, \mathscr{S}, I)$.

$I^1(f, \Phi, E, \chi)$: $\Sigma\{I(\Phi(n))\,;\ n = 1, 2, \cdots\}$.

PROPOSITION 10.5. *If* $\mathrm{ae}\,(y,\ \mathscr{S}_1(f_y),\ E,\ \chi)$ *(in $\mathscr{Y}$), then* $\mathrm{ae}\,(y,\ \mathscr{S}_2(I_1(f_y)),\ E^*,\ \chi^*)$ *(in $\mathscr{Y}$) and $I[1, 2, f] \in R$ for some E^* and χ^*.*

PROPOSITION 10.6 (Fubini).

$$\mathscr{S}^1(f, \Phi, E, \chi) \to I[1, 2, f] = I[2, 1, f] = I^1(f, \Phi, E, \chi) \in R\,,$$

where the unwritten parameters are definable in Φ, E, χ.

Proof. Assume $\mathscr{S}^1(f, \Phi, E, \chi)$, or by 5) of Proposition 3.3, assume

$$\Psi \subset \mathscr{S} \wedge \Sigma I(|\Psi(m)|) < \infty$$
$$\wedge \forall z(\Sigma |\Psi(m, z)| < \infty \vdash f(z) = \Sigma \Psi(m, z))\,.$$

(See the proof of Proposition 10.4 for the notation.)

$$\Psi(m, z)=\Sigma\{\chi_{D(m,i)}(x)\omega(m, i, y); i\leq\nu(m)\}$$

by the proof of Proposition 10.1, where $D(m, i)$ is the $D(i)$, $\omega(m, i)$ is the $\omega(i)$ and $\nu(m)$ is the n there corresponding to m. By a variation of Fatou's lemma in $\mathscr{Y}$, we have

$$\text{ae}\,(y, \Sigma I_1(|\tau(m; y)|<\infty, E^*, \chi^*),$$

where $\tau(m; y)$ denotes the τ_y for $\Psi(m)$ (Proposition 10.2).

$$y\notin E^*\rightarrow\text{ae}\,(x, \Sigma|\psi(m, x, y)|<\infty, E^1, \chi^1)$$

for some E^1 and χ^1. So $y\notin E^*$ implies

$$\text{ae}\,(x, f_y(x)=\Sigma\tau(m; y)(x), E^1, \chi^1).$$

Thus "f_y is integrable in $\mathscr{X}$," and

$$I_1^1(f_y)=\Sigma I_1(\tau(m; y)).$$

Define $\theta(m, y)=I_1(\tau(m; y))$. $\theta(m)$ is "integrable in $\mathscr{Y}$."

$$I_1(\tau(m; y))=\Sigma\{I_1(\chi_{D(m,i)})\omega(m, i, y); i\leq\nu(m)\}$$

and

$$I_1(|\tau(m; y)|)=\Sigma\{I_1(\chi_{D(m,i)}|\omega(m, i, y)|; i\leq\nu(m)\}.$$

Thus $|I_1(\tau(m; y))|\leq I_1(|\tau(m; y)|)$, and so

$$\Sigma I_1(|\tau(m; y)|)<\infty\rightarrow\Sigma|I_1(\tau(m; y))|<\infty,$$

hence $\Sigma I_1(\tau(m; y))$ is convergent.

$$\begin{aligned}\Sigma I[1, 2, |\theta(m)|]&=\Sigma I[1, 2, |I_1(\tau(m; y))|]\\ &\leq\Sigma I[1, 2, I_1(|\tau(m; y)|)]<\infty.\end{aligned}$$

So by Beppo-Levi theorem,

$$\text{ae}\,(y, \Sigma\theta(m, y)<\infty, E'', \chi'').$$

Since $I_1^1(f_y)=\Sigma\theta(m, y)$,

$$\begin{aligned}I[1, 2, f]&=I_2^1(\{y\}I_1^1(f_y))\\ &=\Sigma I_2^1(\theta(m))=\Sigma I_2^1(I_1(\tau(m; y)))\\ &=\Sigma I(\Psi(m))=I^1(f)\end{aligned}$$

(by Mikusiński).

The other half can be proved by symmetry.

PROPOSITION 10.7 (Tonelli). *Assume* mp (f, Z, R). *The following* (i) *and* (ii) *are mutually definably interpretable* (*in* Z).

(i) $\mathscr{S}^1(f, \Phi, E, \chi)$.

(ii) $\text{mbl}\,(f, \Theta, \Lambda, \Xi) \wedge \forall m \mathscr{S}^1(\chi_{A(m)}, \Gamma(m), B(m), \Delta(m))$

$$\wedge \forall z \notin \bigcup A\ (f(z)=0)$$

$$\wedge (I[1, 2, f] \in R \vee I[2, 1, f] \in R)\ .$$

Proof. Assume (i) and define $A(m)=\{z;\ |f(z)| \geq (1/m)\}$. Then f vanishes off $\bigcup A$, and $\chi_{A(m)} \in \mathscr{S}^1$ follows from Proposition 6.5.

$$I[1, 2, f]=I[2, 1, f]=I^1(f) \in R$$

by Fubini's theorem.

Assume (ii) and define $\Psi(n)=\chi_{A[n]}$, where $A[n]=\bigcup\{A(i);\ i \leq n\}$. Suppose $I[1, 2, f] \in R$. $\{(n\Psi(n)) \wedge |f|\}_n$ is an increasing sequence of integrable functions, and $|f|=\lim\,(n\Psi(n)) \wedge\ |f|$ everywhere. By Fubini's theorem,

$$I^1((n\Psi(n)) \wedge |f|)=I[1, 2, (n\Psi(n)) \wedge |f|)$$
$$\leq I[1, 2, |f|] \in R.$$

So, by the monotone convergence theorem in Z, $|f|$ is integrable. The integrability of f is then immediate.

§ 11. Signed integral

DEFINITION 11.1. Let $\mathscr{D}$ be the axiom set $\mathscr{B}$ (in Definition 1.3) modified as follows.

(a) The primitive symbols δ, J_0^+, J_0^- and $\imath$ are added.

(b) The condition $\forall \phi \in L(+)(J(\phi) \geq 0)$ is eliminated.

(c) The continuity property 4° is replaced by a stronger one:

4′. $$\forall \varepsilon > 0 \forall \phi \in L(\|\phi\| < \delta(\varepsilon) \vdash |J(\phi)| < \varepsilon)\ ,$$

where $\|\phi\|=\sup\{|\phi(x)|;\ x \in X\}$.

(d) The axioms on J_0^+, J_0^- and $\imath$ are added.

$$\forall \phi \in L(+);$$
$$J_0^+(\phi), \qquad J_0^-(\phi), \qquad \imath(\phi) \in R,$$
$$J_0^+(\phi)=\sup\{J(\psi);\ 0 \leq \psi \leq \phi,\ \psi \in L(+)\}\ ,$$
$$J_0^-(\phi)=-\inf\{J(\psi);\ 0 \leq \psi \leq \phi,\ \psi \in L(+)\}\ ,$$
$$\imath(\phi)=\sup\{J_0^+(\psi)-J_0^-(\psi);\ 0 \leq \psi \leq \phi,\ \psi \in L(+)\}\ .$$

(Here the sup in the right hand side is not a formal object, but the entire equation represents a relation which determines the property of $J_0^+(\phi)$ (or $\imath(\phi)$). Similarly for inf.)

Let $\mathcal{J}'$ be the theory obtained from $\mathcal{J}$ by replacing $\mathcal{B}$ by $\mathcal{D}$. $\mathcal{J}'$ will be called the theory of signed integral and J will be called a signed integral on L.

In this section we work in $\mathcal{J}'$, and all the functions are supposed to be in L.

PROPOSITION 11.1. 1) *4° is provable in* $\mathcal{J}'$.

2) *If I_1 and I_2 are integrals (in the original sense) on L, then $J=I_1-I_2$ satisfies $\mathcal{D}$ except* (d). (*This is provable in* $\mathcal{J}$.)

3) $\phi\in L(+)\to J_0^+(\phi)\geq J(\phi)\wedge J_0^-(\phi)\geq -J(\phi)$.

4) $J(0)=0$.

5) *J_0^+ and J_0^- are linear on $L(+)$ with respect to the non-negative coefficients.*

DEFINITION 11.2. $J^+(\phi)\colon J_0^+(\phi^+)-J_0^+(\phi^-)$; $J^-(\phi)\colon J_0^-(\phi^+)-J_0^-(\phi^-)$.

PROPOSITION 11.2. 1) $\phi\in L(+)\to J^+(\phi)=J_0^+(\phi)\wedge J^-(\phi)=J_0^-(\phi)$.

2) *J^+ and J^- are respectively unique extensions of J_0^+ and J_0^- to L.*

3) *J^+ and J^- are integrals (in the original sense) satisfying 4′ on $L(+)$.*

4) $J=J^+-J^-$.

Proof of 4). Suppose first $\phi\in L(+)$, and let $G(\psi,\varepsilon)$ denote

$$\psi\in L(+)\wedge\psi\leq\phi\wedge J^-(\phi)\leq -J(\psi)+\varepsilon\,.$$

$$\forall\varepsilon>0\exists\psi G(\psi,\varepsilon) \tag{1}$$

by the axiom on J^-; here $J^-=J_0^-$.

$$\begin{aligned}\varepsilon>0\,,\ G(\psi,\varepsilon)\to\phi-\psi\in L(+)\wedge\phi-\psi\leq\phi\wedge(J+J^-)(\phi)-\varepsilon\\ \leq J(\phi-\psi)\leq J^+(\phi)\,.\end{aligned} \tag{2}$$

By applications of the $\exists\psi$ in the antecedent and the cut applied to (1) and (2), we obtain

$$(J+J^-)(\phi)\leq J^+(\phi)\,.$$

Notice that, in deriving (1), no comprehension is involved. The opposite direction can be established in a similar manner. Thus $J^+=J+J^-$ on $L(+)$, or $J=J^+-J^-$ on $L(+)$. The general case follows from the fact that $\phi=\phi^+-\phi^-$, the linearity of J and the definitions of J^+ and J^- on L.

DEFINITION 11.3. 1) For any two integrals (in the original sense), we say I_1 and I_2 are compatible by K if

$$\forall\phi\in L(+)(K(\phi)\in R\wedge K(\phi)=\sup\{I_1(\psi)-I_2(\psi)\,;\,0\leq\psi\leq\phi,\ \psi\in L(+)\})\,.$$

In such a case, define $I_1\wedge I_2$ to be I_1-K (on $L(+)$).

2) If $I_1\wedge I_2\equiv 0$ on $L(+)$, we say I_1 and I_2 are mutually singular (with respect to K).

PROPOSITION 11.3. 1) *J^+ and J^- are compatible by* ι.

2) *J^+ and J^- are mutually singular.*

3) *$J=J^{+}-J^{-}$ is the unique decomposition of J by mutually singular integrals.*

Proof. 2) $\phi\in L(+)\rightarrow\iota(\phi)=J^{+}(\phi)$, and so $(J^{+}\wedge J^{-})(\phi)=0$.

3) Suppose there is another pair (I_1, I_2) satisfying the condition. Then for any $\phi\in L(+)$, $K(\phi)=J^{+}(\phi)$ is obvious, and hence $I_1\wedge I_2=I_1-J^{+}\equiv 0$, or $I_1\equiv J^{+}$ on $L(+)$. This also implies $I_2\equiv J^{-}$. The equations extend to L.

By virtue of the proposition above, (J^{+}, J^{-}) can be regarded as the Jordan decomposition of J.

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