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# Covering Spaces of $P^2$ Branched along Two Non-singular Curves with Normal Crossings

by

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Let $S_1$ and $S_2$ be non-singular curves in $P^2$ with normal crossings. In this paper, we shall study some of $n$-sheeted coverings of $P^2$ branched along $S_1 \cup S_2$. Our main result is the following theorem.

THEOREM. *A 3-sheeted covering space of $P^2$ branched along $S_1 \cup S_2$ is either a normal surface whose singularities are all rational double points (in this case we have*

$$p_g = g(S_1) + g(S_2) - \frac{1}{9}(S_1 - 2S_2)(2S_1 - S_2))$$

*or a normal surface whose singularities are all rational triple points (in this case we have*

$$p_g = g(S_1) + g(S_2) - \frac{2}{9}(S_1 - S_2)^2).$$

*Here $p_g$ is the geometric genus of the non-singular model of the covering surface and $g(S_i)$ is the genus of $S_i$.*

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## §1. Ramification curves and contractible curves

Let $S = \bigcup_{i=1}^{m} S_i$ be a curve on a non-singular algebraic surface $X$ with normal crossings such that each irreducible component $S_i$ of $S$ is non-singular. In this section we consider a surjective holomorphic map $\varphi: \tilde{X} \to X$ between non-singular algebraic surfaces which satisfies the following conditions:

(i) The restriction map of $\varphi$ to $X - \varphi^{-1}(S)$ is an unramified $n$-sheeted covering over $X - S$.

(ii) $\varphi^{-1}(S) = \bigcup_{\alpha=1}^{k} A_\alpha$ is a curve on $X$ with normal crossings such that each irreducible component $A_\alpha$ is non-singular and $\varphi(A_\alpha)$ is $S_i$ or a point of $S_i \cap S_j$ $(i \neq j)$.

(iii) If $\varphi(A_\alpha) = S_i$, then for an arbitrary point $\tilde{p} \in A_\alpha$, there are local coordinate systems $(t_1, t_2)$ and $(z_1, z_2)$ around $\tilde{p}$ and $p = \varphi(\tilde{p})$ respectively, such that the defining equation of $A_\alpha$ is $t_1 = 0$, the defining equation of $S_i$ is $z_1 = 0$ and $\varphi$ is expressed as

$$(z_1, z_2) = (t_1{}^a t_2{}^b, t_2{}^d), \tag{1.1}$$

where $a$, $b$, $d$ are non-negative integers with $ad \neq 0$.

(iv) If $\varphi(A_\alpha) = p$, where $p \in S_i \cap S_j$ $(i \neq j)$, then for an arbitrary point $\tilde{p} \in A_\alpha$, there are local coordinate systems $(t_1, t_2)$ and $(z_1, z_2)$ around $\tilde{p}$ and $p$ respectively, such that the defining equation of $A_\alpha$ is $t_1 = 0$, the defining equation of $S_i$ is $z_1 = 0$, the defining equation of $S_j$ is $z_2 = 0$ and $\varphi$ is expressed as

$$(z_1, z_2) = (t_1{}^a t_2{}^b, t_1{}^c t_2{}^d), \tag{1.2}$$

where $a$, $b$, $c$, $d$ are non-negative integers with $ad - bc \neq 0$.

It is easily checked that the integer $a$ in (1.1) and the pair of integers $(a, c)$ in (1.2) do not depend on the choice of local coordinate systems $(t_1, t_2)$ and $(z_1, z_2)$ such as in (iii) and (iv) respectively. Since $A_\alpha$ is connected, it follows from this that the above integer $a$ (or the pair of integers $(a, c)$) is uniquely determined by $A_\alpha$.

We call $A_\alpha$ a ramification curve of type $(a, S_i)$ (or a contractible curve of type $(a, c, S_i, S_j)$ respectively).

We denote by $R_\varphi$ the ramification divisor of $\varphi$. The proof of the following proposition is easy.

PROPOSITION 1. *Let $\varphi: \tilde{X} \to X$ be a surjective holomorphic map of non-singular algebraic surfaces which satisfies the conditions* (i), (ii), (iii) *and* (iv). *We suppose that the map $\varphi$ has ramification curves $\tilde{S}_i$ $(i = 1, \cdots, n)$ of types $(\eta_i, S_i)$ and contractible curves $A_\alpha$ $(\alpha = 1, \cdots, \lambda)$ of types $(p_\alpha, q_\alpha, S_i, S_j)$. Then we have*

$$R_\varphi = \sum_{i=1}^{n} (\eta_i - 1)\, \tilde{S}_i + \sum_{\alpha=1}^{\lambda} (p_\alpha + q_\alpha - 1) A_\alpha .$$

## § 2. Torus embeddings

We put

$$X = C^2 \quad \text{with coordinates} \quad (t_1, t_2),$$

$$Y = C^2 \quad \text{with coordinates} \quad (z_1, z_2),$$

$$A = \{(t_1, t_2) \in C^2 \mid t_1 t_2 = 0\},$$

$$B=\{(z_1, z_2)\in C^2 \mid z_1z_2=0\}$$

and let $g: X-A\to Y-B$ be a holomorphic map which is given by

$$(z_1, z_2)=(t_1{}^{\alpha}t_2{}^{n_1}, t_1{}^{\beta}t_2{}^{n_2}),$$

where $\alpha$, $\beta$, $n_1$, $n_2$ are integers with $\alpha n_2-\beta n_1>0$. Then the map $g$ is continued uniquely to a finite holomorphic map $g': X'\to Y$.

We can construct $g': X'\to Y$ by the theory of torus embeddings as follows. Let $N=Z^2$ and $N'$ be a submodule of $N$ generated by $(\alpha, \beta)$ and $(n_1, n_2)$. We denote by $i_R: N'_R\to N_R$ the linear map induced by an inclusion $i: N'\to N$, where $N'_R=N'\otimes R$ and $N_R=N\otimes R$. If we put $e_1=(1, 0)$, $e_2=(0, 1)$ and $e'_1=(\alpha, \beta)$, $e'_2=(n_1, n_2)$ and we take $\{e_1, e_2\}$ as a basis of $N_R$ and $\{e'_1, e'_2\}$ as a basis of $N'_R$, then the linear map $i_R$ is given by

$$(i_R(e'_1), i_R(e'_2))=(e_1, e_2)\begin{pmatrix}\alpha & n_1\\ \beta & n_2\end{pmatrix}$$

Let $\sigma$ be the cone in $N_R$ defined to be $\sigma=R_0e_1+R_0e_2$, where $R_0$ is the set of non-negative elements in $R$. We put

$$\sigma'=i_R^{-1}(\sigma).$$

It is easily checked that $\sigma'=R_0(n_2e'_1-\beta e'_2)+R_0(-n_1e'_1+\alpha e'_2)$. We denote by $\Delta'$ and $\Delta$ the r.p.p. decompositions of $\sigma'$ and $\sigma$ respectively. Then an inclusion $i: N'\to N$ induces a map $h': (N', \Delta')\to(N, \Delta)$ of r.p.p decompositions. Therefore we have the map of torus embeddings $f: T_{N'}\text{ emb }(\Delta')\to T_N\text{ emb }(\Delta)$ corresponding to $h'$ (see Oda [7]). We see easily that $f$ is a finite morphism and equivalent to $g': X'\to Y$.

Let $d_1(d_2)$ be the greatest common factor of $\alpha$ and $n_1$ ($\beta$ and $n_2$, respectively). We put

$$f_1=\frac{n_2}{d_2}e'_1+\left(\frac{-\beta}{d_2}\right)e'_2, \qquad f_2=\left(\frac{-n_1}{d_1}\right)e'_1+\frac{\alpha}{d_1}e'_2,$$

$$\tau'_i=R_0f_i \quad\text{and}\quad \tau_i=R_0e_i \quad (i=1, 2).$$

Then we have

$$\sigma'=R_0f_1+R_0f_2.$$

(I) First we assume that $\sigma'$ is not a non-singular cone. By Mumford et al. [6], the minimal resolution of $T_{N'}\text{ emb }(\Delta')$ is $T_{N'}\text{ emb }(\tilde{\Delta})$, where $(N', \tilde{\Delta})$ is a subdivision of $(N', \Delta')$ defined as follows.

(i) Let $\Sigma=$convex hull of $\sigma'\cap N'-\{0\}$.

(ii) Let $f_1, v_1, \cdots, v_k, f_2$ be the points of $N'$ on $\partial\Sigma$ between $f_1$ and $f_2$.

(iii) Subdivide $\sigma'$ by the set of rays $R_0v_i$ and denote the sectors between them by

$$\sigma_1=R_0f_1+R_0v_1,$$
$$\sigma_i=R_0v_{i-1}+R_0v_i \qquad (i=2, \cdots, k),$$
$$\sigma_{k+1}=R_0v_k+R_0f_2.$$

(iv) Let

$$\tilde{\Delta}=\{\sigma_i(i=1,\cdots,k+1),\ \chi_i\ (i=1,\cdots,k),\ \tau'_1,\ \tau'_2,\ \{0\}\},$$

where $\chi_i=R_0v_i$.

Let $\pi: T_{N'}\,\mathrm{emb}\,(\tilde{\Delta})\to T_N\,\mathrm{emb}\,(\Delta)$ be the map of torus embeddings corresponding to a map $\tilde{h}$: $(N',\tilde{\Delta})\to(N,\Delta)$ and put

$$A_i=\overline{\mathrm{orb}\,(\chi_i)}\qquad(i=1,\cdots,k),$$

$$\tilde{S}_i=\overline{\mathrm{orb}\,(\tau'_i)}\qquad(i=1,2),$$

$$S_i=\overline{\mathrm{orb}\,(\tau_i)}\qquad(i=1,2)\quad\text{and}\quad S=S_1\cup S_2.$$

Then it is elementary that there exist integers $a_i$ $(i=1,\cdots,k)$ such that (i) $a_i\leqq-2$, (ii) $f_1+v_2+a_1v_1=0$, $v_i+v_{i+2}+a_{i+1}v_{i+1}=0$ $(i=1,\cdots,k-2)$ and $v_{k-1}+f_2+a_kv_k=0$. It holds that $A_i^2=a_i$ (cf. Proposition 6.7 in Oda [7]). We see that $A_i$ is isomorphic to a non-singular rational curve $P^1$, $\pi(\tilde{S}_i)=S_i$ and $\pi(A_i)=0$, where $0=\mathrm{orb}\,(\sigma)$.

We put

$$v_i=k_ie_1+l_ie_2\qquad(i=1,\cdots,k),$$

and we denote by $m_i$ $(i=1,2)$ the degree of the restriction map of $\pi$ to $\tilde{S}_i$.

THEOREM 1. *A map* $\pi$: $T_{N'}\,\mathrm{emb}\,(\tilde{\Delta})\to T_N\,\mathrm{emb}\,(\Delta)$ *satisfies the conditions* (i), (ii), (iii), (iv) *of* § 1. *Furthermore we have the following.*

(i) $\pi^{-1}(S)=\tilde{S}_1\cup\tilde{S}_2\cup A_1\cup\cdots\cup A_k$ *and they intersect transversely and no three intersect at a point.*

(ii) $\tilde{S}_1$ *is a ramification curve of type* $(n/d_2, S_1)$ *and* $\tilde{S}_2$ *is a ramification curve of type* $(n/d_1, S_2)$.

(iii) $A_i$ *is a contractible curve of type* $(k_i\alpha+l_in_1,\ k_i\beta+l_in_2,\ S_1,\ S_2)$.

(iv) $$\pi^*S_1=\frac{n}{d_2}\tilde{S}_1+\sum_{i=1}^{k}(k_i\alpha+l_in_1)A_i,\qquad \pi^*S_2=\frac{n}{d_1}\tilde{S}_2+\sum_{i=1}^{k}(k_i\beta+l_in_2)A_i.$$

(v) $m_1=k_1\beta+l_1n_2$ *and* $m_2=k_k\alpha+l_kn_1$.

*Proof.* By Mumford et al. [6], we have (i). Let $M'$ and $M$ be the duals of $N'$ and $N$, respectively. Let $\{m'_1, m'_2\}$ be the basis of $M'$ dual to $\{e'_1, e'_2\}$ and $(m_1, m_2\}$ be the basis of $M$ dual to $\{e_1, e_2\}$. We denote by $i^*$: $M\to M'$ the injection induced by an inclusion homomorphism $i$: $N'\to N$. Then it is easily checked that

$$i^*(m_1)=\alpha m'_1+n_1m'_2,$$
(2.1)
$$i^*(m_2)=\beta m'_1+n_2m'_2.$$

Let $\sigma_i=R_0v_{i-1}+R_0v_i$ $(i=2,\cdots,k)$ be the cone in $N'_R$ and denote by $\check{\sigma}_i$ its dual in $M'_R=M'\otimes R$. Then by elementary computations we see that $\check{\sigma}_i\cap M'=Z_0v'_{i-1}+Z_0v'_i$, where

(2.2) $$v'_{i-1}=l_im'_1-k_im'_2\quad\text{and}\quad v'_i=-l_{i-1}m'_1+k_{i-1}m'_2$$

and $Z_0$ is the set of non-negative elements in $Z$. From (2.2) we have

(2.3) $$m'_1=k_{i-1}v'_{i-1}+k_iv'_i \quad \text{and} \quad m'_2=l_{i-1}v'_{i-1}+l_iv'_i .$$

Substituting (2.3) in the relations (2.1), we have

(2.4) $$i^*(m_1)=(k_{i-1}\alpha+l_{i-1}n_1)v'_{i-1}+(k_i\alpha+l_in_1)v'_i ,$$
$$i^*(m_2)=(k_{i-1}\beta+l_{i-1}n_2)v'_{i-1}+(k_i\beta+l_in_2)v'_i .$$

Let $\pi_i$ be the restriction map of $\pi$ to an affine open subset $U(\sigma_i)=\mathrm{Hom}_{\text{unit-semi gr}}(\check{\sigma}_i\cap M', C)$. The map $\pi_i: U(\sigma_i)\to T_N\,\mathrm{emb}\,(\Delta)=\mathrm{Hom}_{\text{unit-semi gr}}(\check{\sigma}\cap M, C)$ is induced by the map $h$: $T_{N'}=\mathrm{Hom}\,(M', C^*)\to T_N=\mathrm{Hom}\,(M, C^*)$. Since $h$ is defined by $h(\mu')(m)=\mu'(i^*(m))$, if we put $\mu=\pi_i(\mu_i)$, $z_j=\mu(m_j)$ $(j=1, 1)$, $x_i=\mu_i(v'_{i-1})$ and $y_i=\mu_i(v'_i)$, then $\pi_i$ is given by

(2.5) $$(z_1, z_2)=(x_i^{k_{i-1}\alpha+l_{i-1}n_1}y_i^{k_i\alpha+l_in_1},\ x_i^{k_{i-1}\beta+l_{i-1}n_2}y_i^{k_i\beta+l_in_2}) .$$

On the other hand, we see easily that $A_{i-1}=\overline{\mathrm{orb}\,(\chi_{i-1})}$ and $A_i=\overline{\mathrm{orb}\,(\chi_i)}$ are defined by the equations $x_i=0$ and $y_i=0$ respectively in $U(\sigma_i)$ and $S_i$ $(i=1, 2)$ is defined by the equation $z_i=0$. Therefore $\pi$ satisfies the condition (iv) in § 1 and $A_{i-1}$ is a contractible curve of type $(k_{i-1}\alpha+l_{i-1}n_1, k_{i-1}\beta+l_{i-1}n_2, S_1, S_2)$ and $A_i$ is a contractible curve of type $(k_i\alpha+l_in_1, k_i\beta+l_in_2, S_1, S_2)$.

Similary, we see that the restriction maps of $\pi$ to $U(\sigma_1)=\mathrm{Hom}_{\text{unit-semi gr}}(\check{\sigma}_1\cap M', C)$ and $U(\sigma_{k+1})=\mathrm{Hom}_{\text{unit-semi gr}}(\check{\sigma}_{k+1}\cap M', C)$ are given by

(2.6) $$(z_1, z_2)=(x^{n/d_2}y_1^{k_1\alpha+l_1n_1},\ y_1^{k_1\beta+l_1n_2}) ,$$

(2.7) $$(z_1, z_2)=(x_{k+1}^{k_k\alpha+l_kn_1},\ x_{k+1}^{k_k\beta+l_kn_2}y_{k+1}^{n/d_1}) ,$$

respectively, where

$$x_1=\mu_1(f'_1) \qquad (f'_1=l_1m'_1-k_1m'_2) ,$$
$$y_1=\mu_1(v'_1) \qquad \left(v'_1=\frac{\beta}{d_2}m'_1+\frac{n_2}{d_2}m'_2\right),$$
$$x_{k+1}=\mu_{k+1}(v'_k) \qquad \left(v'_k=\frac{\alpha}{d_1}m'_1+\frac{n_1}{d_1}m'_2\right),$$
$$y_{k+1}=\mu_{k+1}(f'_2) \qquad (f'_2=-l_km'_1+k_km'_2) ,$$

and $z_j=\mu(m_j)$ $(j=1, 2)$.

Since $S_1$ and $S_2$ are defined by the equations $x_1=0$ and $y_{k+1}=0$ respectively, we see that $\pi$ satisfies the condition (iii) in § 1 and $S_1$ is a ramification curve of type $(n/d_2, S_1)$ and $S_2$ is a ramification curve of type $(n/d_1, S_2)$.

From (2.5), (2.6) and (2.7), it is easy to check (iv) and (v).

(II) Next we assume that $\sigma'$ is a non-singular cone. In this case, $T_{N'}\,\mathrm{emb}\,(\Delta')$ is non-singular and by the same arguments as in Theorem 1, we see that a map $f$: $T_{N'}\,\mathrm{emb}(\Delta')\to T_N\,\mathrm{emb}(\Delta)$ satisfies the conditions (i), (ii), (iii) of § 1 and we have the following.

(i) $f^{-1}(S)=\tilde{S}_1\cup\tilde{S}_2$ and they intersect transversely.

(ii) $\tilde{S}_1$ is a ramification curve of type $(n/d_2, S_1)$ and $\tilde{S}_2$ is a ramification curve of type $(n/d_1, S_2)$.

(iii) $f^*S_1=\dfrac{n}{d_2}\tilde{S}_1$ and $f^*S_2=\dfrac{n}{d_1}\tilde{S}_2$.

(iv) $m_1=\dfrac{n}{d_1}$ and $m_2=\dfrac{n}{d_2}$.

Here $\tilde{S}_i=\overline{\mathrm{orb}\,(\tilde{\tau}_i)}$, $S_i=\overline{\mathrm{orb}\,(\tau_i)}$ and $m_i$ is the degree of the restriction map of $f$ to $\tilde{S}_i$ $(i=1, 2)$.

## §3. Coverings of $P^2$ branched along $S_1 \cup S_2$

Let $S_1$ and $S_2$ be non-singular curves in $P^2$ with normal crossings. By Oka [8], the fundamental group of $P^2-S_1 \cup S_2$ is abelian and $\pi_1(P^2-S_1 \cup S_2)=Z\gamma_1+Z\gamma_2/(v_1\gamma_1+v_2\gamma_2)$, where $\gamma_i$ is a small loop about $S_i$ and $v_i$ is the degree of $S_i$. We denote $\pi_1(P^2-S_1 \cup S_2)$ by $\bar{G}$. Let $\bar{H}$ be a subgroup of $\bar{G}$ whose index in $\bar{G}$ is $n$ $(n\geqq 2)$. Then we have a unique covering $\psi: X\to P^2$ associated to $\bar{H}$ (see Kawai [5]). $\psi: X\to P^2$ is an $n$-sheeted covering of $P^2$ whose branch locus is $S_i$ $(i=1, 2)$ or $S_1 \cup S_2$.

The following proposition is easily proved by virtue of Proposition in Kawai [5].

PROPOSITION 2. *Let $\psi: X\to P^2$ be the covering associated to $\bar{H}$ which is determined by the subgroup of $Z\gamma_1+Z\gamma_2$ generated by $\alpha\gamma_1+\beta\gamma_2$, $n_1\gamma_1+n_2\gamma_2$. Then for an arbitrary point $p\in S_1\cap S_2$, we have a local coordinate neighborhood $U$ of $p$ and a local coordinate system $(z_1, z_2)$ on $U$ such that*

(i) *$S_i$ is defined by $z_i=0$ $(i=1, 2)$,*

(ii) *$\psi^{-1}(U-S_1\cup S_2)$ is connected and an unramified covering of $U-S_1\cup S_2$ associated to the subgroup of $\pi_1(U-S_1\cup S_2)$ generated by $\alpha a_1+\beta a_2$, $n_1a_1+n_2a_2$, where $a_i$ $(i=1, 2)$ is a small loop about $S_i$ in $U-S_1\cup S_2$.*

Let $f: T_{N'}\,\mathrm{emb}\,(\Delta')\to T_N\,\mathrm{emb}\,(\Delta)$ be the same as in §2. Then we have $T_N\,\mathrm{emb}\,(\Delta)=C^2$. Put

$$V=\{(z_1, z_2)\in C^2\,|\,|z_i|<\varepsilon_i\}\,,$$

$$V'=\{(z_1, z_2)\in C^2\,|\,0<|z_i|<\varepsilon_i\}\,,$$

where $\varepsilon_i$ $(i=1, 2)$ is a small positive number. We denote by $f_V: f^{-1}(V)\to V$ the restriction map of $f$ to $f^{-1}(V)$.

THEOREM 2. *Let $\psi: X\to P^2$ be a covering which is the same as in Proposition 2. Then for an arbitrary point $p\in S_1\cap S_2$, we have a local coordinate neighborhood $U$ of $p$ such that the restriction map of $\psi$ to $\psi^{-1}(U)$ is equivalent to $f_V: f^{-1}(V)\to V$.*

*Proof.* Let $N$ be the fundamental group of $V'$. Then we may assume that $N=Zb_1+Zb_2$, where

$$b_1: (z_1, z_2)=\left(\frac{\varepsilon_1}{2}, \frac{\varepsilon_2}{2} e^{i\theta}\right) \qquad 0\leqq\theta\leqq 2\pi ,$$

$$b_2: (z_1, z_2)=\left(\frac{\varepsilon_1}{2} e^{i\theta}, \frac{\varepsilon_2}{2}\right) \qquad 0\leqq\theta\leqq 2\pi .$$

Let $N'$ be a subgroup of $N$ generated by $\alpha b_1+\beta b_2$, $n_1 b_1+n_2 b_2$. Since $f$ is equivalent to $g': X'\to Y=C^2$ in § 2, it is easy to see that $f_V$ is a covering associated to $N'$. On the other hand, by Proposition 2, we can take a local coordinate neighborhood $U$ of $p$ such that $U$ is isomorphic to $V$ and the restriction map of $\psi$ to $\psi^{-1}(U-S_1\cup S_2)$ is an unramified covering of $U-S_1\cup S_2$ associated to the subgroup of $\pi_1(U-S_1\cup S_2)$ generated by $\alpha a_1+\beta a_2$, $n_1 a_1+n_2 a_2$. Hence the restriction map of $\psi$ to $\psi^{-1}(U)$ is equivalent to $f_V: f^{-1}(V)\to V$.

## § 4. The main theorem

Let $\psi: X\to P^2$ be an $n$-sheeted covering of $P^2$ branched along $S_1\cup S_2$. If $X$ has singularities, then we consider the desingularization of $X$ obtained by the procedure of reducing singularities such as in § 2 and we denote it by $\sigma: \tilde{X}\to X$. We denote the composition of $\sigma$ and $\psi$ by $\varphi: \tilde{X}\to P^2$. Let $p$ be an arbitrary point of $S_1\cap S_2$. By Theorem 2, we can take a local coordinate neighborhood $U$ of $p$ such that the restriction map of $\psi$ to $\psi^{-1}(U)$ is equivalent to the map $f_V: f^{-1}(V)\to V$. Hence if $\psi^{-1}(U)$ has a singularity, then the restriction map of $\varphi$ to $\varphi^{-1}(U)$ is equivalent to the map $\pi_V: \pi^{-1}(V)\to V$, where $V=\{(z_1, z_2)\in C^2 \mid |z_i|<\varepsilon_i\}$ is an open subset of $T_N$ emb $(\Delta)=C^2$ and $f_V$ and $\pi_V$ are the restriction maps of $f: T_{N'}$ emb $(\Delta')\to T_N$ emb $(\Delta)$ to $f^{-1}(V)$ and $\pi: T_{N'}$ emb $(\tilde{\Delta})\to T_N$ emb $(\Delta)$ to $\pi^{-1}(V)$, respectively. Our main theorem is the following.

THEOREM 3. *Let $\psi: X\to P^2$ be the covering which satisfies the following properties*:

(i) *$\psi$ is an n-sheeted covering of $P^2$ branched along $S_1\cup S_2$.*

(ii) *For an each point $p\in S_1\cap S_2$, there is a local coordinate neighborhood $U$ of $p$ such that the restriction map of $\psi$ to $\psi^{-1}(U)$ is equivalent to the map $f_V$.*

(I) *If $\sigma'$ is not a non-singular cone and a subdivision $(N', \tilde{\Delta})$ of $(N', \Delta')$ is defined to be*

$$\tilde{\Delta}=\{\sigma_i\ (i=1, \cdots, k+1),\ \chi_i\ (i=1, \cdots, k),\ \tau'_1,\ \tau'_2,\ \{0\}\}$$

*as in § 2, then the first Chern class $c_1(\tilde{X})$ and the second Chern class $c_2(\tilde{X})$ are given by the following formulas*:

$$c_1(\tilde{X})=\varphi^* c_1(P^2)-\left(\frac{n}{d_2}-1\right)\tilde{S}_1-\left(\frac{n}{d_1}-1\right)\tilde{S}_2$$
$$-\sum_{p\in S_1\cap S_2}\sum_{i=1}^{k}\{k_i(\alpha+\beta)+l_i(n_1+n_2)-1\}A_{ip},$$

$$c_2(\tilde{X})=\{2(m_1+m_2)-n\}+2(n-m_1)g(S_1)$$
$$+2(n-m_2)g(S_2)+(k+n+1-m_1-m_2)S_1S_2 .$$

*Here* $\tilde{S}_i$ *is the proper transform of* $S_i$ *by* $\varphi$, $A_{ip}=\overline{\mathrm{orb}\,(\chi_i)}$ and $n$, $m_i$, $d_i$ *are the same as in* §2. $g(S_i)$ *is the genus of the non-singular curve* $S_i$.

(II) *If* $\sigma'$ *is a non-singular cone, then* $X$ *is non-singular and the Chern classes* $c_i(X)$ $(i=1, 2)$ *are given by*

$$c_1(X)=\psi^*c_1(P^2)-\left(\frac{n}{d_2}-1\right)\tilde{S}_1-\left(\frac{n}{d_1}-1\right)\tilde{S}_2,$$
$$c_2(X)=\{2(m_1+m_2)-n\}+2(n-m_1)g(S_1)$$
$$+2(n-m_2)g(S_2)+(n+1-m_1-m_2)S_1S_2 ,$$

*where* $\tilde{S}_i$ *is the proper transform of* $S_i$ *by* $\psi$.

*Proof.* We shall prove (I). By the property (ii), we may assume that the restriction map of $\varphi$ to $\varphi^{-1}(U)$ is $\pi: T_{N'}\,\mathrm{emb}\,(\tilde{\Delta})\to T_N\,\mathrm{emb}\,(\Delta)$. Then, by the same consideration as in the proof of Theorem 1 of §2, we see that $\varphi: \tilde{X}\to P^2$ is a proper surjective holomorphic map satisfying the conditions (i), (ii), (iii), (iv) of §1. By the same theorem, it is easy to check that the map $\varphi$ has a ramification curve $\tilde{S}_1$ of type $(n/d_2, S_1)$, a ramification curve $\tilde{S}_2$ of type $(n/d_1, S_2)$ and contractible curves $A_{ip}$ $(i=1, \cdots, k, p\in S_1\cap S_2)$ of types $(k_i\alpha+l_in_1, k_i\beta+l_in_2, S_1, S_2)$. Therefore it follows from Proposition 1 that the ramification divisor of $\varphi$ is given by

$$R_\varphi=\left(\frac{n}{d_2}-1\right)\tilde{S}_1+\left(\frac{n}{d_1}-1\right)\tilde{S}_2+\sum_{p\in S_1\cap S_2}\sum_{i=1}^{k}\{k_i(\alpha+\beta)+l_i(n_1+n_2)-1\}A_{ip} .$$

On the other hand, for a surjective holomorphic map $\varphi: \tilde{X}\to P^2$, the divisor $R_\varphi-\varphi^*c_1(P^2)$ is a canonical divisor on $\tilde{X}$. Therefore we have

$$c_1(\tilde{X})=\varphi^*c_1(P^2)-R_\varphi .$$

Hence we have the formula:

$$c_1(\tilde{X})=\varphi^*c_1(P^2)-\left(\frac{n}{d_2}-1\right)\tilde{S}_1-\left(\frac{n}{d_1}-1\right)\tilde{S}_2$$
$$-\sum_{p\in S_1\cap S_2}\sum_{i=1}^{k}\{k_i(\alpha+\beta)+l_i(n_1+n_2)-1\}A_{ip} .$$

To obtain the formula for the second Chern class, we shall calculate the Euler characteristic $\chi(\tilde{X})$ of $\tilde{X}$. First, taking a triangulation of $\tilde{X}$ in which $\varphi^{-1}(S_1\cup S_2)$ appears as a subcomplex, we have

$$\chi(\tilde{X})=\chi(\tilde{X}-\varphi^{-1}(S_1\cup S_2))+\chi(\varphi^{-1}(S_1\cup S_2)) . \tag{4.1}$$

Next, taking a triangulation of $\varphi^{-1}(S_1\cup S_2)$ in which $\varphi^{-1}(S_1\cap S_2)$ appears as a subcomplex, we have

(4.2) $$\chi(\varphi^{-1}(S_1 \cup S_2) = \chi(\varphi^{-1}(S_1 \cup S_2) - \varphi^{-1}(S_1 \cap S_2)) + \chi(\varphi^{-1}(S_1 \cap S_2)) .$$

On the other hand the restriction map of $\varphi$ to $\tilde{X} - \varphi^{-1}(S_1 \cup S_2)$ is an unramified $n$-sheeted covering over $P^2 - S_1 \cup S_2$. Thus we have

(4.3) $$\chi(\tilde{X} - \varphi^{-1}(S_1 \cup S_2)) = n\chi(P^2 - S_1 \cup S_2) .$$

Since

$$\varphi^{-1}(S_1 \cup S_2) - \varphi^{-1}(S_2 \cap S_2) = \bigcup_{i=1,2} \{\tilde{S}_i - \tilde{S}_i \cap \varphi^{-1}(S_1 \cap S_2)\}$$

and the restriction map of $\varphi$ to $\tilde{S}_i - S_i \cap \varphi^{-1}(S_1 \cap S_2)$ is an unramified $m_i$-sheeted covering over $S_i - S_1 \cap S_2$, where $m_i$ is the degree of the restriction map of $\varphi$ to $\tilde{S}_i - \tilde{S}_i \cap \varphi^{-1}(S_1 \cap S_2)$, if we take a triangulation of $\tilde{S}_i$ in which $\tilde{S}_i \cap \varphi^{-1}(S_1 \cap S_2)$ appears as vertices, then we have

(4.4) $$\chi(\varphi^{-1}(S_1 \cup S_2) - \varphi^{-1}(S_1 \cap S_2)) = m_1\chi(S_1 - S_1 \cap S_2) + m_2\chi(S_2 - S_1 \cap S_2) .$$

Moreover, taking a triangulation of $P^2$ in which $S_1 \cup S_2$ appears as a subcomplex, we have

(4.5) $$\chi(P^2 - S_1 \cup S_2) = \chi(P^2) - \chi(S_1 \cup S_2) .$$

It is easily checked that

(4.6) $$\chi(S_i - S_1 \cap S_2) = \chi(S_i) - S_1 S_2 ,$$

(4.7) $$\chi(S_1 \cup S_2) = \chi(S_1) + \chi(S_2) - S_1 S_2 ,$$

(4.8) $$\begin{aligned}\chi(\varphi^{-1}(S_1 \cap S_2)) &= \sum_{p \in S_1 \cap S_2} \chi\left(\bigcup_{i=1,\cdots,k} A_{ip}\right) \\ &= \left\{\sum_{i=1}^{k} \chi(P^1) - (k-1)\right\} S_1 S_2 \\ &= (k+1) S_1 S_2 .\end{aligned}$$

Hence we infer from (4.1), $\cdots$, (4.8) that

$$\begin{aligned}\chi(\tilde{X}) &= n\chi(P^2 - S_1 \cup S_2) + \chi(\varphi^{-1}(S_1 \cup S_2) - \varphi^{-1}(S_1 \cap S_2)) \\ &\quad + \chi(\varphi^{-1}(S_1 \cap S_2)) \\ &= n\{\chi(P^2) - \chi(S_1 \cup S_2)\} + m_1\chi(S_1 - S_1 \cap S_2) \\ &\quad + m_2\chi(S_2 - S_1 \cap S_2) + (k+1) S_1 S_2 \\ &= n\{\chi(P^2) - \chi(S_1) - \chi(S_2) + S_1 S_2\} + \{m_1\chi(S_1) \\ &\quad + m_2\chi(S_2) - (m_1 + m_2) S_1 S_2\} + (k+1) S_1 S_2 \\ &= n\chi(P^2) - (n - m_1)\chi(S_1) - (n - m_2)\chi(S_2) \\ &\quad + \{n - (m_1 + m_2) + k + 1\} S_1 S_2 .\end{aligned}$$

Noting $\chi(S_i)=2-2g(S_i)$ and $\chi(P^2)=3$, we have

$$\chi(\tilde{X})=\{2(m_1+m_2)-n\}+2(n-m_1)g(S_1)$$
$$+2(n-m_2)g(S_2)+(n+k+1-m_1-m_2)S_1S_2.$$

The second assersion is proved in the same manner as (I).

## §5. 2 and 3-sheeted coverings of $P^2$

Let $\psi: X\to P^2$ be the $n$-sheeted covering of $P^2$ associated to a subgroup $\bar{H}$ of $\bar{G}$ which is generated by $\alpha\gamma_1+\beta\gamma_2$, $n_1\gamma_1+n_2\gamma_2$ as in §3. We have $n=\alpha n_2-\beta n_1$. By Theorem 2, we can take a local coordinate neighborhood $U$ of $p$ ($p\in S_1\cap S_2$) such that the restriction map of $\psi$ to $\psi^{-1}(U)$ is equivalent to the map $f_V$. $f_V$ is the restriction map of $f: T_{N'}$ emb $(\Delta')\to T_N$ emb $(\Delta)$ to $f^{-1}(V)$, where $V=\{(z_1, z_2)\in C^2 \mid |z_i|<\varepsilon_i\}$ is a small open subset of $T_N$ emb$(\Delta)=C^2$. The above $(N', \Delta')$ is the r.p.p decomposition of $\sigma'=R_0(n_2, -\beta)+R_0(-n_1, \alpha)$ in $N'_R$. We put

$$o=(0, 0),$$
$$p_1=(n_2, -\beta),$$
$$p_2=(-n_1, \alpha)$$

and we denote by $\Delta(o, p_1, p_2)$ the triangle with $o$, $p_1$, $p_2$ as vertices. Then since $\alpha n_2-\beta n_1=n$, we see that the area of $\Delta(o, p_1, p_2)$ is $n/2$. We put

$$f_1=\left(\frac{n_2}{d_2}, -\frac{\beta}{d_2}\right),$$
$$f_2=\left(-\frac{n_1}{d_1}, \frac{\alpha}{d_1}\right),$$

where $d_i$ is the same as in §2. Then $\sigma'=R_0f_1+R_0f_2$. We see that $\sigma'$ is a non-singular cone if and only if the area of $\Delta(o, p_1, p_2)$ is equal to 1/2. If $\sigma'$ is not so, then we consider the subdivision $(N', \tilde{\Delta})$ of $(N', \Delta')$ consisting of non-singular cones $\sigma_i$ $(i=1, \cdots, k+1)$ such as in §2, where

$$\sigma_1=R_0f_1+R_0v_1,$$
$$\sigma_i=R_0v_{i-1}+R_0v_i \qquad (i=2, \cdots, k),$$
$$\sigma_{k+1}=R_0v_k+R_0f_2.$$

Here $v_i=(k_i, l_i)$. Then, since each $\sigma_i$ is a non-singular cone, the areas of triangles $\Delta(o, f_1, v_1)$, $\Delta(o, v_1, v_2)$, $\cdots$, $\Delta(o, v_k, f_2)$ are all equal to 1/2.

If $X$ has singularities, then we denote by $\varphi: \tilde{X}\to P^2$ the composition of the desingularization $\sigma$ of $X$ and $\psi$. We denote by $q$ and $p_g$ respectively the irregularity and geometric genus of $X$ or the non-singular model $\tilde{X}$ of $X$. By Kawai [5], we see that

$q=0$.

(I) First, we shall examine the case in which $n=3$. Since the area of $\Delta(o, p_1, p_2)$ is equal to 3/2, we see easily that $(N', \Delta')$ is equal to one of the following two r.p.p decompositions $\Delta_1'$ and $\Delta_2'$

(1) $\Delta_1' = \{\sigma' = R_0 f_1 + R_0 f_2$ and its faces$\}$, where

$$f_1 = (n_2, -\beta) \quad \text{and} \quad f_2 = \left(\frac{-n_1}{3}, \frac{\alpha}{3}\right),$$

(2) $\Delta_2' = \{\sigma' = R_0 f_1 + R_0 f_2$ and its faces$\}$, where

$$f_1 = \left(\frac{n_2}{3}, \frac{-\beta}{3}\right) \quad \text{and} \quad f_2 = (-n_1, \alpha),$$

or $(N', \Delta')$ has one of the following two subdivisions $\tilde{\Delta}_3$ and $\tilde{\Delta}_4$

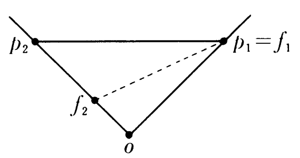
(3) $\tilde{\Delta}_3 = \{\sigma_1 = R_0 f_1 + R_0 v_1,\ \sigma_2 = R_0 v_1 + R_0 v_2,\ \sigma_3 = R_0 v_2 + R_0 f_2$ and their faces$\}$, where

$$f_1 = (n_2, -\beta), \qquad v_1 = \left(\frac{-n_1 + 2n_2}{3}, \frac{\alpha - 2\beta}{3}\right),$$

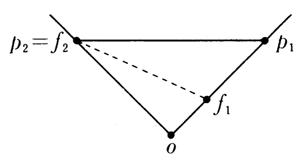
$$v_2 = \left(\frac{-2n_1 + n_2}{3}, \frac{2\alpha - \beta}{3}\right) \quad \text{and} \quad f_2 = (-n_1, \alpha),$$

(4) $\tilde{\Delta}_4 = \{\sigma_1 = R_0 f_1 + R_0 v_1,\ \sigma_2 = R_0 v_1 + R_0 f_2$ and their faces$\}$, where

$$f_1 = (n_2, -\beta), \qquad v_1 = \left(\frac{-n_1 + n_2}{3}, \frac{\alpha - \beta}{3}\right) \quad \text{and} \quad f_2 = (-n_1, \alpha).$$




(1)

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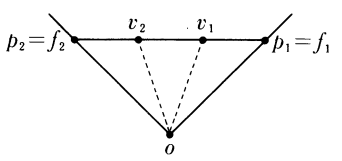


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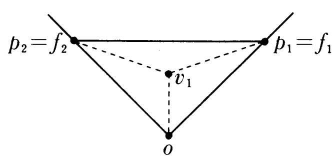



(4)

THEOREM 4. *A 3-sheeted covering space of $P^2$ branched along $S_1 \cup S_2$ is either a normal surface whose singularities are all rational double points (in this case we have*

$$p_g = g(S_1) + g(S_2) - \frac{1}{9}(S_1 - 2S_2)(2S_1 - S_2)),$$

*or a normal surface whose singularities are all rational triple points* (*in this case we have*

$$p_g=g(S_1)+g(S_2)-\frac{2}{9}(S_1-S_2)^2).$$

*Proof.* Let $p$ be an arbitrary point of $S_1 \cap S_2$. By an appropriate choice of the local coordinate neighborhood $U$ of $p$, we may assume that the restriction map of $\psi$ to $\psi^{-1}(U)$ is $f_V$. If $(N', \Delta')$ is $\Delta_1$ or $\Delta_2$, then one of $S_i$ is not a branch locus. If $(N', \Delta')$ has the subdivision $(N', \tilde{\Delta}_3)$, then $\psi^{-1}(U)$ has a singularity and its minimal desingularization is given by $T_{N'}\,\mathrm{emb}\,(\tilde{\Delta}_3)$. Then it is easily checked that $A_i^2=-2$ $(i=1, 2)$, where $A_i=\overline{\mathrm{orb}\,(\chi_i)}$ and $\chi_i=R_0 v_i$. By Artin [1], we see that the singularity of $\psi^{-1}(U)$ is a rational double point. If $(N', \Delta')$ has the subdivision $(N', \tilde{\Delta}_4)$, then the minimal desingularization of $\psi^{-1}(U)$ is given by $T_{N'}\,\mathrm{emb}\,(\tilde{\Delta}_4)$. Then we have $A_1^2=-3$, where $A_1=\overline{\mathrm{orb}\,(\chi_1)}$ and $\chi_1=R_0 v_1$. By Artin [1], we see that the singularity of $\psi^{-1}(U)$ is a rational triple point.

We shall calculate the geometric genus $p_g$.

(i) The case in which $(N', \Delta')$ has the subdivision $(N', \tilde{\Delta}_3)$. The minimal desingularization of $\psi^{-1}(U)$ is given by $T_{N'}\,\mathrm{emb}\,(\tilde{\Delta}_3)$. Then since we have $d_1=d_2=1$, by Theorem 3, we see that

$$c_1(\tilde{X})=\varphi^*c_1(P^2)-2\left\{\tilde{S}_1+\tilde{S}_2+\sum_{p\in S_1\cap S_2}(A_{1p}+A_{2p})\right\}$$

and since, by Theorem 1, we have $m_1=m_2=1$, we see that

$$c_2(\tilde{X})=1+4\{g(S_1)+g(S_2)\}+4S_1S_2\,.$$

On the other hand, by Theorem 1, we have

$$\varphi^*(S_1)=3\tilde{S}_1+\sum_{p\in S_1\cap S_2}(2A_{1p}+A_{2p}),$$

$$\varphi^*(S_2)=3\tilde{S}_2+\sum_{p\in S_1\cap S_2}(A_{1p}+2A_{2p}),$$

where $A_{ip}=\overline{\mathrm{orb}\,(\chi_i)}$. Therefore we have

$$(\deg\varphi)(c_1(P^2),\, S_1)=\left(\varphi^*c_1(P^2),\quad 3\tilde{S}_1+\sum_{p\in S_1\cap S_2}2A_{1p}+A_{2p}\right).$$

Hence we have

$$(\varphi^*c_1(P^2),\, \tilde{S}_1)=3\nu_1\,.$$

Similary we have

$$(\varphi^*c_1(P^2),\, \tilde{S}_2)=3\nu_2\,.$$

Since we have

$$(\tilde{S}_i,\, A_{jp})=\delta_{ij},\ A_{jp}{}^2=-2 \quad\text{and}\quad (A_{1p},\, A_{2p})=1\,,$$

we see that

$$\tilde{S}_1^{\,2}=\frac{1}{3}(S_1^{\,2}-2S_1S_2),$$

$$\tilde{S}_2^{\,2}=\frac{1}{3}(S_2^{\,2}-2S_1S_2).$$

Then by easy computations we have

$$c_1(\tilde{X})^2=11+8\{g(S_1)+g(S_2)\}-\frac{8}{3}(S_1^{\,2}+S_2^{\,2})+\frac{8}{3}S_1S_2.$$

Since $q=0$, it follows from Neother's formula that

$$p_g(\tilde{X})=g(S_1)+g(S_2)-\frac{1}{9}(S_1-2S_2)(2S_1-S_2).$$

(ii) The case in which $(N', \Delta')$ has the subdivision $(N', \tilde{\Delta}_4)$. We infer in the same manner as in (i) that

$$c_1(\tilde{X})=\varphi^*c_1(P^2)-2(\tilde{S}_1+S_2)-\sum_{p\in S_1\cap S_2}A_{1p},$$

$$c_2(\tilde{X})=1+4\{g(S_1)+g(S_2)\}+3S_1S_2.$$

And moreover following equations are proved in a similar manner:

$$(\varphi^*c_1(P^2), \tilde{S}_i)=3v_i \qquad (i=1, 2),$$

$$(A_{1p}, \tilde{S}_i)=1 \qquad (i=1, 2),$$

$$A_{1p}^{\,2}=-3,$$

$$\tilde{S}_1^{\,2}=\frac{1}{3}(S_1^{\,2}-S_1S_2),$$

$$\tilde{S}_2^{\,2}=\frac{1}{3}(S_2^{\,2}-S_1S_2).$$

Then by easy computations we have

$$c_1(\tilde{X})^2=11+8\{g(S_1)+g(S_2)\}-\frac{8}{3}(S_1^{\,2}+S_2^{\,2})+\frac{7}{3}S_1S_2.$$

Since $q=0$, we have

$$p_g(\tilde{X})=g(S_1)+g(S_2)-\frac{2}{9}(S_1-S_2)^2.$$

(II) Next, we shall examine the case in which $n=2$. Since the area of $\Delta(o, p_1, p_2)$ is equal to 1, we see easily that $(N', \Delta')$ is equal to one of the following two r.p.p decompositions $\Delta_1$ and $\Delta_2$

(1) $\Delta_1=\{\sigma'=R_0f_1+R_0f_2$ and its faces$\}$, where

$$f_1=(n_2, -\beta) \quad \text{and} \quad f_2=\left(\frac{-n_1}{2}, \frac{\alpha}{2}\right),$$

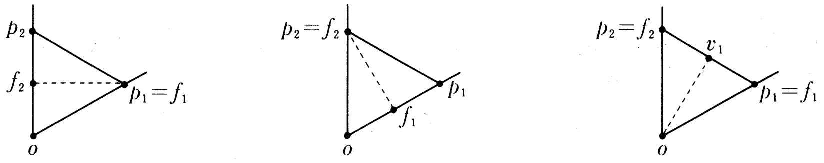
(2) $\Delta_2=\{\sigma'=R_0f_1+R_0f_2$ and its faces$\}$, where

$$f_1=\left(\frac{n_2}{2}, \frac{-\beta}{2}\right) \quad \text{and} \quad f_2=(-n_1, \alpha),$$

or $(N', \Delta')$ has the following subdivision $\tilde{\Delta}_3$

(3) $\tilde{\Delta}_3=\{\sigma_1=R_0f_1+R_0v_1,\ \sigma_2=R_0v_1+R_0f_2$ and their faces$\}$, where

$$f_1=(n_2, -\beta), \qquad v_1=\left(\frac{n_2-n_1}{2}, \frac{\alpha-\beta}{2}\right) \quad \text{and} \quad f_2=(-n_1, \alpha).$$




(1) (2) (3)

From this, by the same arguments as in above, we see easily that a 2-sheeted covering space of $P^2$ branched along $S_1 \cup S_2$ is a normal surface whose singularities are all rational double points (see Perrson [9]). In this case, for the non-singular model $\tilde{X}$ of $X$ such as in §4, we have

$$c_1(\tilde{X})^2=10+4\{g(S_1)+g(S_2)\}+S_1S_2-\frac{3}{2}(S_1{}^2+S_2{}^2),$$

$$c_2(\tilde{X})=2+2\{g(S_1)+g(S_2)\}+2S_1S_2,$$

$$p_g(\tilde{X})=\frac{1}{2}\{g(S_1)+g(S_2)\}-\frac{1}{8}(S_1-S_2)^2.$$

*Remark.* Let $\psi: X \to P^2$ be a $p$-sheeted covering of $P^2$ branched along $S_1 \cup S_2$, where $p$ is a prime number. In this case, we see that $\psi^{-1}(U)$ has a rational double point if and only if $(N', \Delta')$ has the following subdivision $\tilde{\Delta}$

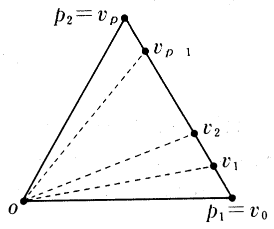
$$\tilde{\Delta}=\{\sigma_i=R_0v_{i-1}+R_0v_i\ (i=1, \cdots, p) \text{ and their faces}\},$$

where

$$v_0=p_1=(n_2, -\beta),$$

$$v_i=\left(\frac{-in_1+(p-i)n_2}{p}, \frac{i\alpha-(p-i)\beta}{p}\right),$$

$$v_p=p_2=(-n_1, \alpha).$$



From this, by the same arguments as in above, we see that if $X$ is a $p$-sheeted covering surface of $P^2$ branched along $S_1 \cup S_2$ and has only rational double points as singularities, then for the non-singular model $\tilde{X}$ of $X$ such as in § 4, we have

$$c_1(\tilde{X})^2 = (p+8) + 4(p-1)\{g(S_1)+g(S_2)\} + \frac{1-p^2}{p}(S_1{}^2+S_2{}^2) + \frac{2(p-1)^2}{p} S_1S_2 ,$$

$$c_2(\tilde{X}) = (4-p) + 2(p-1)\{g(S_1)+g(S_2)\} + 2(p-1)S_1S_2 ,$$

$$p_g(\tilde{X}) = \frac{(p-1)}{2}\{g(S_1)+g(S_2)\} - \frac{(p-1)}{12p}\{(p+1)(S_1{}^2+S_2{}^2) - 2(2p-1)S_1S_2\} .$$

## § 6. The difference of tangent bundles

Let $\varphi: \tilde{X} \to X$ be the same as in § 1. We denote by $\mathscr{T}_{\tilde{X}}$ and $\mathscr{T}_X$ respectively the tangent bundles of $\tilde{X}$ and $X$. Then we may assume that for an arbitrary point $\tilde{p} \in \tilde{X}$, there are local coordinate systems $(t_1, t_2)$ and $(z_1, z_2)$ around $\tilde{p}$ and $p = \varphi(\tilde{p})$ respectively such that $\varphi$ is expressed as

$$(z_1, z_2) = (t_1^a t_2^b,\ t_1^c t_2^d) ,$$

where $a$, $b$, $c$, $d$ are non-negative integers with the property

$$(6.1) \qquad ad - bc \neq 0 .$$

Then we have a sheaf homomorphism $\phi: \mathscr{T}_{\tilde{X}} \to \varphi^*\mathscr{T}_X$ defined by

$$\phi\left(\frac{\partial}{\partial t_1}\right) = \frac{\partial z_1}{\partial t_1}\frac{\partial}{\partial z_1} + \frac{\partial z_2}{\partial t_1}\frac{\partial}{\partial z_2} ,$$

$$\phi\left(\frac{\partial}{\partial t_2}\right) = \frac{\partial z_1}{\partial t_2}\frac{\partial}{\partial z_1} + \frac{\partial z_2}{\partial t_2}\frac{\partial}{\partial z_2} ,$$

where $\partial/\partial t_i$ and $\partial/\partial z_i$ are tangent vectors.

By (6.1), we see easily that $\phi$ is a monomorphism.

For each point $\tilde{p} \in \tilde{X}$, we denote by $\phi_{\tilde{p}}: (\mathscr{T}_{\tilde{X}})_{\tilde{p}} \to (\varphi^*\mathscr{T}_X)_{\tilde{p}}$ the homomorphism of stalks induced by $\phi$. We shall prove the following lemma.

LEMMA. *Let*

$$\rho = g_1(t_1, t_2)\frac{\partial}{\partial z_1} + g_2(t_1, t_2)\frac{\partial}{\partial z_2}$$

*be a section of* $\varphi^*\mathscr{T}_X$ *over* $\tilde{X}$ *and we denote by* $\rho_{\tilde{p}}$ *its germ at a point* $\tilde{p}$ *of* $\tilde{X}$. *We put* $\tilde{p} =$ $(t_1, t_2)$. *Then we have*

(I) *Under the conditions* $t_1 \neq 0$ *and* $t_2 \neq 0$, $\rho_{\tilde{p}}$ *always belongs to the image of* $\phi_{\tilde{p}}$.

(II) *Under the conditions* $t_1 = 0$ *and* $t_2 \neq 0$, $\rho_{\tilde{p}}$ *belongs to the image of* $\phi_{\tilde{p}}$ *if and only if*, (1) *if* $a, c \geqq 1$, *then* $g_1$ *and* $g_2$ *can be written in the forms*

$$g_1(t_1, t_2) = t_1^{a-1}\{\alpha_1(t_2) + t_1\beta_1(t_1, t_2)\} ,$$

$$g_2(t_1, t_2)=t_1^{c-1}\{\alpha_2(t_2)+t_1\beta_2(t_1, t_2)\}\,,$$

*where* $\alpha_i$, $\beta_i$ *are holomorphic functions at* $\tilde{p}$ *and* $\alpha_1(t_2)$ *and* $\alpha_2(t_2)$ *satisfy the relation*

$$ct_2^d\alpha_1(t_2)-at_2^b\alpha_2(t_2)=0\,,$$

*and* (2) *if* $c=0$ *and* $a,\ d\geqq 1$, *then* $g_1$ *can be written in the form*

$$g_1(t_1, t_2)=t_1^{a-1}\varepsilon(t_1, t_2)\,,$$

*where* $\varepsilon$ *is a holomorphic function at* $\tilde{p}$.

(III) *Under the conditions* $t_1=0$ *and* $t_2=0$, $\rho_{\tilde{p}}$ *belongs to the image of* $\phi_{\tilde{p}}$ *if and only if,* (1) *if* $a,\ b,\ c,\ d\geqq 1$, *then* $g_1$ *and* $g_2$ *can be written in the forms*

$$g_1(t_1, t_2)=t_1^{a-1}t_2^{b-1}h_1(t_1, t_2)\,,$$
$$g_2(t_1, t_2)=t_1^{c-1}t_2^{d-1}h_2(t_1, t_2),$$

*where* $h_i$ *is a holomorphic function at* $\tilde{p}$ *and moreover* $dh_1-bh_2$ *and* $-ch_1+ah_2$ *can be expressed in the forms*

$$dh_1(t_1, t_2)-bh_2(t_1, t_2)=t_2\varepsilon_1(t_1, t_2)\,,$$
$$-ch_1(t_1, t_2)+ah_2(t_1, t_2)=t_1\varepsilon_2(t_1, t_2)\,,$$

*and* (2) *if* $c=0$ *and* $a,\ b,\ d\geqq 1$, *then* $g_1$ *and* $g_2$ *can be written in the forms*

$$g_1(t_1, t_2)=t_1^{a-1}t_2^{b-1}h_1(t_1, t_2)\,,$$
$$g_2(t_1, t_2)=\qquad t_2^{d-1}h_2(t_1, t_2)\,,$$

*where* $h_i$ *is a holomorphic function at* $\tilde{p}$ *and moreover* $dh_1-bt_1h_2$ *can be expressed in the form*

$$dh_1(t_1, t_2)-bt_1h_2(t_1, t_2)=t_2\varepsilon(t_1, t_2)\,.$$

*Here* $\varepsilon_i$ *and* $\varepsilon$ *are holomorphic functions at* $\tilde{p}$.

*Proof.* First we assume that $t_1=0$, $t_2=0$ and $a,\ b,\ c,\ d\geqq 1$. We suppose that $g_1$ and $g_2$ satisfy the conditions in (1) of (III). Then we put

$$f_1(t_1, t_2)=\frac{1}{ad-bc}\,\varepsilon_1(t_1, t_2)\,,\qquad f_2(t_1, t_2)=\frac{1}{ad-bc}\,\varepsilon_2(t_1, t_2)\,.$$

By the definition of $\phi$, we have

$$\phi\left(\frac{\partial}{\partial t_1}\right)=at_1^{a-1}t_2^b\frac{\partial}{\partial z_1}+ct_1^{c-1}t_2^d\frac{\partial}{\partial z_2}\,,$$
$$\phi\left(\frac{\partial}{\partial t_2}\right)=bt_1^at_2^{b-1}\frac{\partial}{\partial z_1}+dt_1^ct_2^{d-1}\frac{\partial}{\partial z_2}\,. \tag{6.2}$$

Then it follows from (6.2) that

$$\begin{aligned}\phi\left(f_1\frac{\partial}{\partial t_1}+f_2\frac{\partial}{\partial t_2}\right)&=\frac{1}{ad-bc}\,\varepsilon_1\left(at_1^{a-1}t_2^b\frac{\partial}{\partial z_1}+ct_1^{c-1}t_2^d\frac{\partial}{\partial z_2}\right)\\&\quad+\frac{1}{ad-bc}\,\varepsilon_2\left(bt_1^at_2^{b-1}\frac{\partial}{\partial z_1}+dt_1^ct_2^{d-1}\frac{\partial}{\partial z_2}\right)\\&=\frac{1}{ad-bc}\left\{t_1^{a-1}t_2^{b-1}(at_2\varepsilon_1+bt_1\varepsilon_2)\frac{\partial}{\partial z_1}\right.\\&\quad\left.+t_1^{c-1}t_2^{d-1}(ct_2\varepsilon_1+dt_1\varepsilon_2)\frac{\partial}{\partial z_2}\right\}\\&=\frac{1}{ad-bc}\left[t_1^{a-1}t_2^{b-1}\{a(dh_1-bh_2)+b(-ch_1+ah_2)\}\frac{\partial}{\partial z_1}\right.\\&\quad\left.+t_1^{c-1}t_2^{d-1}\{c(dh_1-bh_2)+d(-ch_1+ah_2)\}\frac{\partial}{\partial z_2}\right]\\&=t_1^{a-1}t_2^{b-1}h_1\frac{\partial}{\partial z_1}+t_1^{c-1}t_2^{d-1}h_2\frac{\partial}{\partial z_2}\\&=g_1\frac{\partial}{\partial z_1}+g_2\frac{\partial}{\partial z_2}\,.\end{aligned}$$

Conversely we suppose that there is a section

$$f_1(t_1, t_2)\frac{\partial}{\partial t_1}+f_2(t_1, t_2)\frac{\partial}{\partial t_2}$$

of $\mathscr{T}_{\tilde{X}}$ over a neighborhood of $\tilde{p}$ such that

$$g_1\frac{\partial}{\partial z_1}+g_2\frac{\partial}{\partial z_2}=\phi\left(f_1\frac{\partial}{\partial t_1}+f_2\frac{\partial}{\partial t_2}\right).$$

Then we infer from (6.2) that

$$\begin{aligned}g_1(t_1, t_2)&=t_1^{a-1}t_2^{b-1}\{at_2f_1(t_1, t_2)+bt_1f_2(t_1, t_2)\}\,,\\g_2(t_1, t_2)&=t_1^{c-1}t_2^{d-1}\{ct_2f_1(t_1, t_2)+dt_1f_2(t_1, t_2)\}\,.\end{aligned}$$

We put

$$\begin{aligned}h_1(t_1, t_2)&=at_2f_1(t_1, t_2)+bt_1f_2(t_1, t_2)\,,\\h_2(t_1, t_2)&=ct_2f_1(t_1, t_2)+dt_1f_2(t_1, t_2)\,.\end{aligned}$$

It is easily chekced that

$$\begin{aligned}dh_1(t_1, t_2)-bh_2(t_1, t_2)&=(ad-bc)t_2f_1(t_1, t_2)\,,\\-ch_1(t_1, t_2)+ah_2(t_1, t_2)&=(ad-bc)t_1f_2(t_1, t_2)\,.\end{aligned}$$

This implies that $g_1$ and $g_2$ satisfy the desired conditions. Hence we can prove (1) of (III).

Next we assume that $t_1=0$, $t_2\neq 0$ and $a, c\geqq 1$. We suppose that $g_1$ and $g_2$ satisfy the conditions in (1) of (II). We put

$$f_1(t_1, t_2)=\frac{1}{(ad-bc)t_2^{b+d-1}}\{(d\alpha_1(t_2)t_2^{d-1}-b\alpha_2(t_2)t_2^{b-1})$$
$$+t_1(d\beta_1(t_1, t_2)t_2^{d-1}-b\beta_2(t_1, t_2)t_2^{b-1})\},$$
$$f_2(t_1, t_2)=\frac{1}{(ad-bc)t_2^{b+d-1}}\{-c\beta_1(t_1, t_2)t_2^d+a\beta_2(t_1, t_2)t_2^b\}.$$

Then, by the same arguments as in (1) of (III), we have

$$\phi\left(f_1\frac{\partial}{\partial t_1}+f_2\frac{\partial}{\partial t_2}\right)=g_1\frac{\partial}{\partial z_1}+g_2\frac{\partial}{\partial z_2}.$$

Conversely we suppose that there is a section

$$f_1(t_1, t_2)\frac{\partial}{\partial t_1}+f_2(t_1, t_2)\frac{\partial}{\partial t_2}$$

of $\mathscr{T}_{\tilde{X}}$ over a neighborhood of $\tilde{p}$ such that

$$\phi\left(f_1\frac{\partial}{\partial t_1}+f_2\frac{\partial}{\partial t_2}\right)=g_1\frac{\partial}{\partial z_1}+g_2\frac{\partial}{\partial z_2}.$$

Then we infer from (6.2) that

$$g_1(t_1, t_2)=t_1^{a-1}\{t_2^{b-1}(at_2f_1(t_1, t_2)+bt_1f_2(t_1, t_2))\},$$
$$g_2(t_1, t_2)=t_1^{c-1}\{t_2^{d-1}(ct_2f_1(t_1, t_2)+dt_1f_2(t_1, t_2))\}.$$

Now we put

$$\text{(6.3)}\quad \begin{aligned} t_2^{b-1}(at_2f_1(t_1, t_2)+bt_1f_2(t_1, t_2))&=\alpha_1(t_2)+t_1\beta_1(t_1, t_2),\\ t_2^{d-1}(ct_2f_1(t_1, t_2)+dt_1f_2(t_1, t_2))&=\alpha_2(t_2)+t_1\beta_2(t_1, t_2),\end{aligned}$$

$$\text{(6.4)}\quad f_1(t_1, t_2)=\gamma(t_2)+t_1\delta(t_1, t_2).$$

Then, substituting (6.4) in (6.3), we have

$$\alpha_1(t_2)=at_2^b\gamma(t_2)\quad\text{and}\quad \alpha_2(t_2)=ct_2^d\gamma(t_2).$$

This implies that

$$ct_2^d\alpha_1(t_2)-at_2^b\alpha_2(t_2)=0.$$

Hence we can prove (1) of (II).

To prove (2) of (III), we assume that $t_1=0$, $t_2=0$ and $c=0$, $a, b, d\geqq 1$. We suppose that $g_1$ and $g_2$ satisfy the conditions in (2) of (III). Then we put

$$f_1(t_1, t_2)=\frac{\varepsilon(t_1, t_2)}{ad} \quad \text{and} \quad f_2(t_1, t_2)=\frac{h_2(t_1, t_2)}{d}.$$

In this case, by the definition of $\phi$, we have

$$\text{(6.5)} \quad \begin{aligned} \phi\left(\frac{\partial}{\partial t_1}\right)&=at_1^{a-1}t_2^b\frac{\partial}{\partial z_1}, \\ \phi\left(\frac{\partial}{\partial t_2}\right)&=bt_1^a t_2^{b-1}\frac{\partial}{\partial z_1}+dt_2^{d-1}\frac{\partial}{\partial z_2}. \end{aligned}$$

Then it follows from (6.5) that

$$\begin{aligned} \phi\left(f_1\frac{\partial}{\partial t_1}+f_2\frac{\partial}{\partial t_2}\right)&=\frac{\varepsilon}{ad}at_1^{a-1}t_2^b\frac{\partial}{\partial z_1}+\frac{h_2}{d}\left(bt_1^a t_2^{b-1}\frac{\partial}{\partial z_1}+dt_2^{d-1}\frac{\partial}{\partial z_2}\right) \\ &=\left\{\frac{1}{d}t_1^{a-1}t_2^{b-1}(dh_1-bt_1h_2)+\frac{1}{d}t_1^a t_2^{b-1}bh_2\right\}\frac{\partial}{\partial z_1}+t_2^{d-1}h_2\frac{\partial}{\partial z_2} \\ &=g_1\frac{\partial}{\partial z_1}+g_2\frac{\partial}{\partial z_2}. \end{aligned}$$

Conversely we suppose that there is a section

$$f_1(t_1, t_2)\frac{\partial}{\partial t_1}+f_2(t_1, t_2)\frac{\partial}{\partial t_2}$$

of $\mathscr{T}_{\tilde{X}}$ over a neighborhood of $\tilde{p}$ such that

$$g_1\frac{\partial}{\partial z_1}+g_2\frac{\partial}{\partial z_2}=\phi\left(f_1\frac{\partial}{\partial t_1}+f_2\frac{\partial}{\partial t_2}\right).$$

Then we infer from (6.5) that

$$\begin{aligned} g_1(t_1, t_2)&=t_1^{a-1}t_2^{b-1}\{at_2f_1(t_1, t_2)+bt_1f_2(t_1, t_2)\}, \\ g_2(t_1, t_2)&=t_2^{d-1}\{df_2(t_1, t_2)\}. \end{aligned}$$

We put

$$\begin{aligned} h_1(t_1, t_2)&=at_2f_1(t_1, t_2)+bt_1f_2(t_1, t_2), \\ h_2(t_1, t_2)&=df_2(t_1, t_2). \end{aligned}$$

It is easy to see that

$$dh_1(t_1, t_2)-bt_1h_2(t_1, t_2)=t_2\{adf_1(t_1, t_2)\}.$$

This implies that $g_1$ and $g_2$ satisfy the desired conditions. Hence we can prove (2) of (III).

As the cases in which (I) and (2) of (II) are easily proved, we omit them.

We may assume that $\mathscr{T}_{\tilde{X}}$ is a submodule of $\varphi^*\mathscr{T}_X$. We shall prove the following propositions by a similar argument as in Kawai [5].

PROPOSITION 3. *If* $\varphi(A_\alpha)\cap S_i\neq\emptyset$, *then we have the following sheaf homomorphism*

$$\psi^k_{A_\alpha,S_i}\colon \lambda^\alpha_*[\varphi^*_\alpha(\bar{\nu}(S_i))\otimes(\nu(A_\alpha)^*)^{\otimes k}]\to\varphi^*\mathscr{T}_X/\mathscr{T}_{\tilde{X}}$$

*for a non negative integer* $k$, *where* $\bar{\nu}(S_i)$ *is the restriction of normal bundle* $\nu(S_i)$ *of* $S_i$ *in* $X$ *to* $\varphi(A_\alpha)$, $\nu(A_\alpha)^*$ *is the dual bundle of the normal bundle* $\nu(A_\alpha)$ *of* $A_\alpha$ *in* $\tilde{X}$, $\varphi_\alpha$ *is the restriction map of* $\varphi$ *to* $A_\alpha$ *and* $\lambda^\alpha$ *is the injection of* $A_\alpha$ *into* $\tilde{X}$.

*Proof.* Let $\tilde{p}$ be an arbitrary point of $A_\alpha$. By the condition (iii) of § 1, we may assume that the defining equation of $A_\alpha$ is $t_1=0$, the defining equation of $S_i$ is $z_1=0$. With respect to these coordinate systems, we shall define the homomorphism $\psi^k_{A_\alpha,S_i}$ by

$$\psi^k_{A_\alpha,S_i}\left(g(t_2)\frac{\partial}{\partial z_1}\otimes dt_1^{\otimes k}\right)=t_1^k g(t_2)\frac{\partial}{\partial z_1},$$

where $\partial/\partial z_1$ is considered to be a normal vector on the left and a tangent vector on the right and $dt_1$ is a conormal vector.

To show that $\psi^k_{A_\alpha,S_i}$ determines the sheaf homomorphism, we take any other coordinate systems $(\bar{t}_1, \bar{t}_2)$ and $(\bar{z}_1, \bar{z}_2)$ such that the defining equation of $A_\alpha$ is $\bar{t}_1=0$, the defining equation of $S_i$ is $\bar{z}_1=0$ and $\varphi$ is expressed as

$$(\bar{z}_1, \bar{z}_2)=(\bar{t}_1^a\bar{t}_2^b, \bar{t}_1^c\bar{t}_2^d)\,.$$

Now, we may assume that $\psi^k_{A_\alpha,S_i}$ is given by

$$\psi^k_{A_\alpha,S_i}(\bar{g}(\bar{t}_2)\frac{\partial}{\partial\bar{z}_1}\otimes d\bar{t}_1^{\otimes k})=\bar{t}_1^k\bar{g}(\bar{t}_2)\frac{\partial}{\partial\bar{z}_1}\,.$$

Then it is sufficient to show that if

$$g(t_2)\frac{\partial}{\partial z_1}\otimes dt_1^{\otimes k}\quad\text{and}\quad \bar{g}(\bar{t}_2)\frac{\partial}{\partial\bar{z}_1}\otimes d\bar{t}_1^{\otimes k}$$

are the same holomorphic sections of $\lambda^\alpha_*[\varphi^*_\alpha(\bar{\nu}(S_i))\otimes(\nu(A_\alpha)^*)^{\otimes k}]$, then

$$t_1^k g(t_2)\frac{\partial}{\partial z_1}\quad\text{and}\quad \bar{t}_1^k\bar{g}(\bar{t}_2)\frac{\partial}{\partial\bar{z}_1}$$

are the same elements of $\varphi^*\mathscr{T}_X/\mathscr{T}_{\tilde{X}}$. Since we have

$$z_1=a_1(\bar{z}_1, \bar{z}_2)\bar{z}_1\quad\text{and}\quad t_1=b(\bar{t}_1, \bar{t}_2)\bar{t}_1\,,$$

where $a_1(\bar{z}_1, \bar{z}_2)$ and $b(\bar{t}_1, \bar{t}_2)$ are non-vanishing holomorphic functions, we have the following relations with respect to (co) normal vectors

$$\frac{\partial}{\partial\bar{z}_1}=a_1(\bar{z}_1, \bar{z}_2)\frac{\partial}{\partial z_1}\quad\text{and}\quad dt_1=b(\bar{t}_1, \bar{t}_2)d\bar{t}_1\,.$$

Hence we infer from the assumption that

$$g(t_2)\frac{\partial}{\partial z_1}\otimes dt_1^{\otimes k} \quad \text{and} \quad \bar{g}(\bar{t}_2)\frac{\partial}{\partial \bar{z}_1}\otimes d\bar{t}_1^{\otimes k}$$

are the same holomorphic sections that

$$g(t_2)b(\bar{t}_1, \bar{t}_2)^k = \bar{g}(\bar{t}_2)a_1(\bar{z}_1, \bar{z}_2)\,.$$

Therefore, by easy computations, we have the following equations with respect to elements of $\varphi^*\mathscr{T}_X$

(6.6)
$$gt_1^k\frac{\partial}{\partial z_1} - \bar{g}\bar{t}_1^k\frac{\partial}{\partial \bar{z}_1} = gt_1^k\frac{\partial}{\partial z_1} - \bar{g}\bar{t}_1^k\left(\frac{\partial z_1}{\partial \bar{z}_1}\frac{\partial}{\partial z_1} + \frac{\partial z_2}{\partial \bar{z}_1}\frac{\partial}{\partial z_2}\right)$$
$$= -\bar{g}\bar{t}_1^k\left(\frac{\partial a_1}{\partial \bar{z}_1}\bar{z}_1\frac{\partial}{\partial z_1} + \frac{\partial z_2}{\partial \bar{z}_1}\frac{\partial}{\partial z_2}\right).$$

Here we put

(6.7)
$$g_1(t_1, t_2) = -\bar{g}\bar{t}_1^k\frac{\partial a_1}{\partial \bar{z}_1}\bar{z}_1 = -\frac{1}{a_1}\bar{g}\bar{t}_1^k\frac{\partial a_1}{\partial \bar{z}_1}z_1\,,$$
$$g_2(t_1, t_2) = -\bar{g}\bar{t}_1^k\frac{\partial z_2}{\partial \bar{z}_1}.$$

First we consider the case in which $A_\alpha$ is a ramification curve of type $(\eta, S_i)$. If

$$\tilde{p}\notin \bigcup_{\beta\neq\alpha}(A_\alpha\cap A_\beta)\,,$$

then we may assume that $t_1=0$, $t_2\neq 0$ and $\varphi$ is expressed as

$$(z_1, z_2) = (t_1^\eta t_2^b, t_2^d) \qquad (\eta, d\geqq 1)\,.$$

Then it is easily checked that $g_1$ is written in the form

$$g_1(t_1, t_2) = t_1^\eta t_2^b h_1(t_1, t_2)\,,$$

where $h_1$ is a holomorphic function at $\tilde{p}$. Then, by (2) in (II) of Lemma, we see that the element (6.6) belongs to a germ of $(\mathscr{T}_{\tilde{X}})_{\tilde{p}}$. If

$$\tilde{p}\in \bigcup_{\beta\neq\alpha}(A_\alpha\cap A_\beta)$$

for some contractible curve $A_\beta$ $(\beta\neq\alpha)$, then we may assume that $t_1=0$, $t_2=0$ and $\varphi$ is expressed as

$$(z_1, z_2) = (t_1^\eta t_2^b, t_2^d) \qquad (\eta, b, d\geqq 1)\,.$$

Moreover we may assume that $p\in S_i\cap S_j$ for some index $j$ $(\neq i)$. Then, since $z_2=0$ and $\bar{z}_2=0$ are the defining equations of $S_j$, we have

$$z_2 = a_2(\bar{z}_1, \bar{z}_2)\bar{z}_2\,,$$

where $a_2$ is a non-vanishing holomorphic function at $p$.

Hence we have

$$\frac{\partial z_2}{\partial \bar{z}_1}=\frac{\partial a_2}{\partial \bar{z}_1}\bar{z}_2.$$

Substituting this in the right of (6.7), we have

$$g_2(t_1, t_2)=-\bar{g}\bar{t}_1^k\frac{\partial a_2}{\partial \bar{z}_1}\bar{z}_2=-\frac{1}{a_2}\bar{g}\bar{t}_1^k\frac{\partial a_2}{\partial \bar{z}_1}z_2\,.$$

This implies that $g_1$ and $g_2$ are written in the forms

$$g_1(t_1, t_2)=t_1^\eta t_2^b h_1(t_1, t_2)\,,$$

$$g_2(t_1, t_2)=\quad t_2^d h_2(t_1, t_2)\,,$$

where $h_i$ is a holomorphic function at $\tilde{p}$. Again, by (2) in (III) of Lemma, we see that the element (6.6) belongs to a germ of $(\mathscr{T}_{\tilde{X}})_{\tilde{p}}$.

Next we consider the case in which $A_\alpha$ is a contractible curve of type $(p, q, S_i, S_j)$. By the same arguments as in above, we obtain

$$g_1(t_1, t_2)=t_1^p t_2^b h_1(t_1, t_2)\,,$$

$$g_2(t_1, t_2)=t_1^q t_2^d h_2(t_1, t_2)\,,$$

where $h_i$ is a holomorphic function at $\tilde{p}$. If

$$\tilde{p}\notin\bigcup_{\beta\neq\alpha}(A_\alpha\cap A_\beta)\,,$$

then we may assume that $t_1=0$, $t_2\neq 0$ and $\varphi$ is expressed as

$$(z_1, z_2)=(t_1^p t_2^b,\ t_1^q t_2^d)\qquad (p, q\geqq 1)\,.$$

Hence, by (1) in (II) of Lemma, we see that the element (6.6) belongs to a germ of $(\mathscr{T}_{\tilde{X}})_{\tilde{p}}$. If $\tilde{p}\in A_\alpha\cap A_\beta$ for some contractible curve $A_\beta$, then we may assume that $t_1=0$, $t_2=0$ and $\varphi$ is expressed as

$$(z_1, z_2)=(t_1^p t_2^b,\ t_1^q t_2^d)\qquad (p, q, b, d\geqq 1)\,.$$

Then, by (1) in (III) of Lemma, we see that the element (6.6) belongs to a germ of $(\mathscr{T}_{\tilde{X}})_{\tilde{p}}$. As the other cases are easily checked, we omit them.

PROPOSITION 4. (1) *If $A_\alpha$ is a ramification curve of type $(\eta, S_i)$, then we have a homomorphism*

$$\psi_{A_\alpha}=\sum_{k=0}^{\eta-2}\psi^k_{A_\alpha,S_i}\colon \sum_{k=0}^{\eta-2}\lambda^\alpha_*[\varphi^*_\alpha(\bar{\nu}(S_i))\otimes(\nu(A_\alpha)^*)^{\otimes k}]\to\varphi^*\mathscr{T}_X/\mathscr{T}_{\tilde{X}}$$

*such that the restriction map of $\psi_{A_\alpha}$ to $\tilde{X}-\bigcup_{\beta\neq\alpha}(A_\alpha\cap A_\beta)$ is an isomorphism.*

(2) *If $A_\alpha$ is a contractible curve of type $(p, q, S_i, S_j)$, then we have a homomorphism*

$$\psi_{A_\alpha}=\sum_{k=0}^{p-2}\psi^k_{A_\alpha,S_i}+\sum_{k=0}^{q-1}\psi^k_{A_\alpha,S_i}$$

$$:\sum_{k=0}^{p-2}\lambda^\alpha_*[\varphi^*_\alpha(\bar{\nu}(S_i))\otimes(\nu(A_\alpha)^*)^{\otimes k}]+\sum_{k=0}^{q-1}\lambda^\alpha_*[\varphi^*_\alpha(\bar{\nu}(S_j))\otimes(\nu(A_\alpha)^*)^{\otimes k}]$$

$$\to\varphi^*\mathcal{T}_X/\mathcal{T}_{\tilde{X}}$$

*such that the restriction map of* $\psi_{A_\alpha}$ to $\tilde{X}-\bigcup_{\beta\neq\alpha}(A_\alpha\cap A_\beta)$ *is an isomorphism.*

*Proof.* We shall use the notations as the same as the proof of Proposition 3.

(1) By the definition, we have

$$\psi_{A_\alpha}\left(\sum_{k=0}^{\eta-2}g_k(t_2)\frac{\partial}{\partial z_2}\otimes dt_1^{\otimes k}\right)=\sum_{k=0}^{\eta-2}t_1^k g_k(t_2)\frac{\partial}{\partial z_1}.$$

Let $\tilde{p}$ be an arbitrary point of

$$A_\alpha-\bigcup_{\beta\neq\alpha}(A_\alpha\cap A_\beta)\quad\text{and}\quad g_1(t_1,t_2)\frac{\partial}{\partial z_1}+g_2(t_1,t_2)\frac{\partial}{\partial z_2}$$

be an arbitrary element of $(\varphi^*\mathcal{T}_X)_{\tilde{p}}$. Then, by (2) in (II) of Lemma, we see that

$$g_1(t_1,t_2)\frac{\partial}{\partial z_1}+g_2(t_1,t_2)\frac{\partial}{\partial z_2}\equiv\left(\sum_{k=0}^{\eta-2}t_1^k\alpha_k(t_2)\right)\frac{\partial}{\partial z_1}$$

mod $(\mathcal{T}_{\tilde{X}})_{\tilde{p}}$, where

$$g_1(t_1,t_2)=\sum_{k=0}^{\eta-2}\alpha_k(t_2)t_1^k+t_1^{\eta-1}\varepsilon_1(t_1,t_2).$$

This implies that $(\psi_{A_\alpha})_{\tilde{p}}$ is bijective.

(2) By the definition, we have

$$\psi_{A_\alpha}\left(\sum_{k=0}^{p-2}g_k(t_2)\frac{\partial}{\partial z_1}\otimes dt_1^{\otimes k}+\sum_{k=0}^{q-1}h_k(t_2)\frac{\partial}{\partial z_2}\otimes dt_1^{\otimes k}\right)$$
$$=\sum_{k=0}^{p-2}t_1^k g_k(t_2)\frac{\partial}{\partial z_1}+\sum_{k=0}^{q-1}t_1^k h_k(t_2)\frac{\partial}{\partial z_2}.$$

Let $\tilde{p}$ be an arbitrary point of

$$A_\alpha-\bigcup_{\beta\neq\alpha}(A_\alpha\cap A_\beta)\quad\text{and}\quad g(t_1,t_2)\frac{\partial}{\partial z_1}+h(t_1,t_2)\frac{\partial}{\partial z_2}$$

be an arbitrary element of $(\varphi^*\mathcal{T}_X)_{\tilde{p}}$. We put

$$g(t_1,t_2)=\sum_{k=0}^{p-2}t_1^k g_k(t_2)+t_1^{p-1}\{\alpha_1(t_2)+t_1\beta_1(t_1,t_2)\},$$

$$h(t_1,t_2)=\sum_{k=0}^{q-2}t_1^k h_k(t_2)+t_1^{q-1}\{\alpha_2(t_2)+t_1\beta_2(t_1,t_2)\}.$$

Then we can write $h(t_1, t_2)$ in the form

$$h(t_1, t_2)=\sum_{k=0}^{q-2} t_1^k h_k(t_2)+t_1^{q-1}\left\{\alpha_2(t_2)-\frac{q}{p}t_2^{b-d}\alpha_1(t_2)\right\}$$
$$+t_1^{q-1}\left\{\frac{q}{p}\, t_2^{b-d}\alpha_1(t_2)+t_1\beta_2(t_1, t_2)\right\}.$$

By (1) in (II) of Lemma, we infer that

$$g(t_1, t_2)\frac{\partial}{\partial z_1}+h(t_1, t_2)\frac{\partial}{\partial z_2} \quad \text{and} \quad \sum_{k=0}^{p-2} t_1^k g_k(t_2)\frac{\partial}{\partial z_1}+\sum_{k=0}^{q-1} t_1^k h_k(t_2)\frac{\partial}{\partial z_2}$$

are the same elements of $(\varphi^*\mathscr{T}_X/\mathscr{T}_{\tilde{X}})_{\tilde{p}}$, where

$$h_{q-1}(t_2)=\alpha_2(t_2)-\frac{q}{p}\, t_2^{b-d}\alpha_1(t_2).$$

This implies that $(\psi_{A_\alpha})_{\tilde{p}}$ is surjective. It is obvious from the same lemma that $(\psi_{A_\alpha})_{\tilde{p}}$ is injective.

PROPOSITION 5. (1) *If $A_\alpha$ is a ramification curve of type $(\nu, S_i)$ and $A_\beta$ is a ramification curve of type $(\mu, S_j)$, then for an arbitrary point $\tilde{p}\in A_\alpha\cap A_\beta$, the homomorphism $(\psi_{A_\alpha}+\psi_{A_\beta})_{\tilde{p}}$ is an isomorphism.*

(2) *Let $A_\alpha$ be a contractible curve of type $(p, q, S_i, S_j)$. If $A_\beta$ is a ramification curve of type $(\nu, S_i)$, then for an arbitrary point $\tilde{p}\in A_\alpha\cap A_\beta$, the kernel and cokernel of the homomorphism $(\psi_{A_\alpha}+\psi_{A_\beta})_{\tilde{p}}$ are given as follows:*

$$\mathrm{Ker}\,(\psi_{A_\alpha}+\psi_{A_\beta})_{\tilde{p}}\cong C^{(\nu-1)(p-1)},$$
$$\mathrm{Coker}\,(\psi_{A_\alpha}+\psi_{A_\beta})_{\tilde{p}}\cong C.$$

*and if $A_\beta$ is a ramification curve of type $(\mu, S_j)$, then for an arbitrary point $\tilde{p}\in A_\alpha\cap A_\beta$, we have*

$$\mathrm{Ker}\,(\psi_{A_\alpha}+\psi_{A_\beta})_{\tilde{p}}\cong C^{(\mu-1)q},$$
$$\mathrm{Coker}\,(\psi_{A_\alpha}+\psi_{A_\beta})_{\tilde{p}}\cong 0.$$

(3) *If $A_\alpha$ is a contractible curve of type $(p, q, S_i, S_j)$ and $A_\beta$ is a contractible curve of type $(s, t, S_i, S_j)$, then for an arbitrary point $\tilde{p}\in A_\alpha\cap A_\beta$, we have*

$$\mathrm{Ker}\,(\psi_{A_\alpha}+\psi_{A_\beta})_{\tilde{p}}\cong C^{(p-1)(s-1)+qt},$$
$$\mathrm{Coker}\,(\psi_{A_\alpha}+\psi_{A_\beta})_{\tilde{p}}\cong C.$$

*Proof.* We shall prove (3). We may assume that $\varphi$ is expressed as

$$(z_1, z_2)=(t_1^p t_2^s,\ t_1^q t_2^t)$$

and $A_\alpha$ is defined by $t_1=0$, $A_\beta$ is defined by $t_2=0$, $S_i$ is defined by $z_1=0$ and $S_j$ is defined by $z_2=0$. With respect to these coordinate systems, the homomorphism

$\psi_{A_\alpha}+\psi_{A_\beta}$ is given by

$$(\psi_{A_\alpha}+\psi_{A_\beta})\Big(\sum_{k=0}^{p-2}\alpha_k(t_2)\frac{\partial}{\partial z_1}\otimes dt_1^{\otimes k}+\sum_{k=0}^{q-1}\beta_k(t_2)\frac{\partial}{\partial z_2}\otimes dt_1^{\otimes k}$$
$$+\sum_{k=0}^{s-2}\gamma_k(t_1)\frac{\partial}{\partial z_1}\otimes dt_2^{\otimes k}+\sum_{k=0}^{t-1}\delta_k(t_1)\frac{\partial}{\partial z_2}\otimes dt_2^{\otimes k}\Big)$$
$$=g_1(t_1, t_2)\frac{\partial}{\partial z_1}+g_2(t_1, t_2)\frac{\partial}{\partial z_2},$$

where

$$g_1(t_1, t_2)=\sum_{k=0}^{p-2}\alpha_k(t_2)t_1^k+\sum_{k=0}^{s-2}\gamma_k(t_1)t_2^k,$$
$$g_2(t_1, t_2)=\sum_{k=0}^{q-1}\beta_k(t_2)t_1^k+\sum_{k=0}^{t-1}\delta_k(t_1)t_2^k.$$

We put

$$\text{(6.9)}\qquad \begin{aligned}\alpha_k(t_2)&=\sum_{i=0}^{\infty}a_{ki}t_2^i \qquad (k=0, \cdots, p-2),\\ \beta_k(t_2)&=\sum_{i=0}^{\infty}b_{ki}t_2^i \qquad (k=0, \cdots, q-1),\\ \gamma_k(t_1)&=\sum_{i=0}^{\infty}c_{ki}t_1^i \qquad (k=0, \cdots, s-2),\\ \delta_k(t_1)&=\sum_{i=0}^{\infty}d_{ki}t_1^i \qquad (k=0, \cdots, t-1).\end{aligned}$$

Substituting (6.9) in (6.8), we have

$$g_1(t_1, t_2)=\sum_{i=0}^{p-2}\sum_{k=0}^{s-2}(a_{ik}+c_{ki})t_1^i t_2^k+\sum_{i=0}^{p-2}\sum_{k=s-1}^{\infty}a_{ki}t_1^i t_2^k+\sum_{i=p-1}^{\infty}\sum_{k=0}^{s-2}c_{ik}t_1^i t_2^k,$$
$$g_2(t_1, t_2)=\sum_{i=0}^{q-1}\sum_{k=0}^{t-1}(b_{ik}+d_{ki})t_1^i t_2^k+\sum_{i=0}^{q-1}\sum_{k=t}^{\infty}b_{ik}t_1^i t_2^k+\sum_{i=q}^{\infty}\sum_{k=0}^{t-1}d_{ki}t_1^i t_2^k.$$

First we suppose that

$$g_1(t_1, t_2)\frac{\partial}{\partial z_1}+g_2(t_1, t_2)\frac{\partial}{\partial z_2}$$

is the image of $\phi$ at $\tilde{p}$. Then, by (III) of Lemma, monomial $t_1^{p-1}t_2^{q-1}$ divides $g_1(t_1, t_2)$ and $t_1^{s-1}t_2^{t-1}$ divides $g_2(t_1, t_2)$. Hence we have

$$(6.10)\qquad \begin{aligned} &a_{ik}+c_{ki}=0 && (i=0,\cdots,p-2,\ k=0,\cdots,s-2)\,,\\ &a_{ik}=0 && (i=0,\cdots,p-2,\ k\geqq s-1)\,,\\ &c_{ki}=0 && (i\geqq p-1,\ k=0,\cdots,s-2)\,,\\ &b_{ik}+d_{ki}=0 && (i=0,\cdots,q-1,\ k=0,\cdots,t-2)\\ & \qquad\qquad\text{or} && (i=0,\cdots,q-2,\ k=0,\cdots,t-1)\,,\\ &b_{ik}=0 && (i=0,\cdots,q-2,\ k\geqq t)\,,\\ &d_{ki}=0 && (i\geqq q,\ k=0,\cdots,t-2)\,. \end{aligned}$$

From (6.10) we have

$$g_1(t_1, t_2)=t_1^{p-1}t_2^{s-1}h_1(t_1, t_2)\,,$$
$$g_2(t_1, t_2)=t_1^{q-1}t_2^{t-1}h_2(t_1, t_2)\,,$$

where

$$h_1(t_1, t_2)=0\,,$$
$$h_2(t_1, t_2)=(b_{q-1\,t-1}+d_{t-1\,q-1})+\sum_{i=1}^{\infty} d_{t-1\,i+q-1}t_1^i+\sum_{k=1}^{\infty} b_{q-1\,k+t-1}t_2^k\,.$$

Then, since $t_i$ $(i=1, 2)$ divides $h_2(t_1, t_2)$, we see that

$$\begin{aligned} &b_{q-1\,t-1}+d_{t-1\,q-1}=0\,,\\ &b_{q-1\,k}=0 \qquad (k\geqq t)\,,\\ &d_{t-1\,i}=0 \qquad (i\geqq q)\,. \end{aligned}$$

Hence we have

$$(6.11)\qquad \begin{aligned} &b_{ik}+d_{ki}=0 && (i=0,\cdots,q-1,\ k=0,\cdots,t-1)\,,\\ &b_{ik}=0 && (i=0,\cdots,q-1,\ k\geqq t)\,,\\ &d_{ki}=0 && (i\geqq q,\ k=0,\cdots,t-1)\,. \end{aligned}$$

From (6.10) and (6.11) we see easily that

$$\begin{aligned} \operatorname{Ker}(\psi_{A_\alpha}+\psi_{A_\beta})_{\tilde p}=\Bigg\{&\sum_{k=0}^{p-2}\left(\sum_{i=0}^{s-2} a_{ki}t_2^i\right)\frac{\partial}{\partial z_1}\otimes dt_1^{\otimes k}\\ &+\sum_{k=0}^{q-1}\left(\sum_{i=0}^{t-1} b_{ki}t_2^i\right)\frac{\partial}{\partial z_2}\otimes dt_1^{\otimes k}+\sum_{k=0}^{s-2}\left(\sum_{i=0}^{p-2}(-a_{ik})t_1^i\right)\frac{\partial}{\partial z_1}\otimes dt_2^{\otimes k}\\ &+\sum_{k=0}^{t-1}\left(\sum_{i=0}^{q-1}(-b_{ik})t_1^i\right)\frac{\partial}{\partial z_2}\otimes dt_2^{\otimes k}\ \Bigg|\ a_{ik}, b_{ik}\in C\Bigg\}\,. \end{aligned}$$

Hence we have

$$\mathrm{Ker}\,(\psi_{A_\alpha}+\psi_{A\beta})_{\tilde{p}}\cong C^{(p-1)(s-1)+qt}.$$

Next let

$$g_1(t_1, t_2)\frac{\partial}{\partial z_1}+g_2(t_1, t_2)\frac{\partial}{\partial z_2}$$

be an arbitrary section of $\varphi^*\mathscr{T}_X$ at $\tilde{p}$. We put

$$\begin{aligned}
g_1(t_1, t_2)=&\sum_{k=0}^{p-2} a_1^k(t_2)t_1^k+\sum_{k=0}^{s-2} b_1^k(t_1)t_2^k\\
&+t_1^{p-1}t_2^{s-1}\{a_1+t_1b_1(t_1)+t_2c_1(t_2)+t_1t_2d_1(t_1, t_2)\},\\
g_2(t_1, t_2)=&\sum_{k=0}^{q-2} a_2^k(t_2)t_1^k+\sum_{k=0}^{t-2} b_2^k(t_1)t_2^k\\
&+t_1^{q-1}t_2^{t-1}\{b_2(t_1)+c_2(t_2)+t_1t_2d_2(t_1, t_2)\}.
\end{aligned}$$

If we set

$$\begin{aligned}
&\alpha_k(t_2)=a_1^k(t_2) \qquad (k=0, \cdots, p-2),\\
&\beta_k(t_2)=a_2^k(t_2) \qquad (k=0, \cdots, q-2),\\
&\beta_{q-1}(t_2)=t_2^{t-1}c_1(t_2)-\frac{q}{p}\,t_2^t c_1(t_2),\\
&\gamma_k(t_1)=b_1^k(t_1) \qquad (k=0, \cdots, s-2),\\
&\delta_k(t_1)=b_2^k(t_1) \qquad (k=0, \cdots\cdot, t-2),\\
&\delta_{t-1}(t_1)=t_1^{q-1}b_2(t_1)-\frac{t}{s}\,t_1^q b_1(t_1),
\end{aligned}$$

then, by (III) of Lemma, we see easily that

$$\begin{aligned}
&\left\{g_1(t_1, t_2)\frac{\partial}{\partial z_1}+g_2(t_1, t_2)\frac{\partial}{\partial z_2}\right\}-\left\{a_1t_1^{p-1}t_2^{s-1}\frac{\partial}{\partial z_1}\right\}\\
&\equiv\left(\sum_{k=0}^{p-2}\alpha_k(t_2)t_1^k+\sum_{k=0}^{s-2}\gamma_k(t_1)t_2^k\right)\frac{\partial}{\partial z_1}\\
&\quad+\left(\sum_{k=0}^{q-1}\beta_k(t_2)t_1^k+\sum_{k=0}^{t-1}\delta_k(t_1)t_2^k\right)\frac{\partial}{\partial z_2}
\end{aligned}$$

mod $(\mathscr{T}_{\tilde{X}})_{\tilde{p}}$. From this we have

$$\mathrm{Coker}\,(\psi_{A_\alpha}+\psi_{A\beta})_{\tilde{p}}=\left\{at_1^{p-1}t_2^{s-1}\frac{\partial}{\partial z_1}\,\middle|\, a\in C\right\}.$$

Hence we have

$$\mathrm{Coker}\,(\psi_{A_\alpha}+\psi_{A_\beta})_{\tilde{p}}\cong C\,.$$

The other assertions are proved in a similar manner.

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