COMMENTARII MATHEMATICI
UNIVERSITATIS SANCTI PAULI
Vol. 33, No. 2, 1984

ed. RIKKYO UNIV/MATH
IKEBUKURO TOKYO
171 JAPAN

A Note on the Distribution of Zeros of Solutions of Linear Differential Equations*

by

Steven B. BANK and Günter FRANK

(Received April 5, 1983)

ABSTRACT. In this paper, it is shown that for any linear differential equation of the form $y^{(k)}+A(z)y=0$, where $k\geq 2$, and $A(z)$ is a nonconstant polynomial of degree n, the following property holds: Any fundamental set of solutions contains at most one solution whose zero-sequence has exponent of convergence less than $(n+k)/k$.

1. Introduction

This paper is concerned with the distribution of zeros of solutions of equations of the form,

$$y^{(k)}+A(z)y=0\,, \tag{1}$$

where $k\geq 2$, and $A(z)$ is a nonconstant polynomial. More specifically, we are interested in the exponent of convergence (which we will denote by $N(f)$) of the zero-sequence of solutions $f\not\equiv 0$ of Eq. (1).

In a recent paper [2], a more general class of equations was considered, and the following theorem summarizes some of the results in [2]:

THEOREM A. *Given the linear differential equation,*

$$y^{(k)}+a_{k-1}(z)y^{(k-1)}+\cdots+a_0(z)y=0\,, \tag{2}$$

where $k\geq 2$, and the coefficients $a_j(z)$ are polynomials satisfying the following conditions: (i) *$a_0(z)$ is nonconstant;* (ii) *if $a_j\not\equiv 0$, say $a_j(z)=K_jz^{\alpha_j}+\cdots$, then the degree α_j satisfies $\alpha_j\leq(k-j)\alpha_0/k$ for each $j=1, 2, \cdots, k-1$;* (iii) *all roots of the polynomial $t^k+\sum K_jt^j$ (where the sum is over all j for which $a_j\not\equiv 0$ and $\alpha_j=(k-j)\alpha_0/k$) are simple. Then the following hold:*

(a) *Every solution $f\not\equiv 0$ of (2) is an entire function of order of growth $(\alpha_0+k)/k$.*

(b) *If a solution $f\not\equiv 0$ of (2) has the property that $N(f)<(\alpha_0+k)/k$, then f has only finitely many zeros.*

(c) *If $f_1, \cdots, f_k$ is a fundamental set for (2), then $N(f_j)=(\alpha_0+k)/k$ for at least one j in the set $\{1, 2, \cdots, k\}$.*

---

* This research was supported in part by the National Science Foundation (MCS-8200497).
AMS (MOS) Subject Classification Numbers: 34A20, 34C10, 30D15.

(Briefly, the proof is as follows: Part (a) follows from the Wiman-Valiron theory [6; Chapt. 4] or [7; Chapt. 1]. For Part (b), if a solution $f \not\equiv 0$ has the form ge^Q, where $Q(z) = cz^{(\alpha_0+k)/k}$, and g is of order $< (\alpha_0 + k)/k$, then applying Pöschl's theorem [5; Satz 2] to the equation obtained from (2) by the transformation $y = ue^Q$, we immediately obtain the conclusion that g has only finitely many zeros. Finally, Part (c) immediately follows from Part (b) by Frank's theorem [3; Satz 1].)

When the number $(\alpha_0 + k)/k$ is not a positive integer, then from Part (a) and the Hadamard factorization theorem, every solution $f \not\equiv 0$ of Eq. (2) satisfies $N(f) = (\alpha_0 + k)/k$. In the case where $(\alpha_0 + k)/k$ is a positive integer, the result in Part (c) raises a natural question, namely for how many members of a fundamental set $f_1, \cdots, f_k$ can the inequality $N(f_j) < (\alpha_0 + k)/k$ hold? In this paper, we answer this question for the special equation (1). Our main result is the following:

THEOREM B. *Let $A(z)$ be a nonconstant polynomial of degree n, and let k be an integer greater than* 1. *Assume that Eq.* (1) *possesses a solution $f(z)$ such that $N(f) < (n+k)/k$. Then, if $g(z)$ is any solution of* (1) *which is not a constant multiple of f, we have $N(g) = (n+k)/k$.*

The proof is based on first reducing the order of (1) using the solution f. Assuming that the conclusion fails, we then obtain a linear differential equation of order $k-1$, with rational functions for coefficients, which must possess a solution which is a nontrivial rational function. However, a method of factoring the zero-order coefficient in this equation, shows that no such rational function solution can exist by considering the Laurent expansion around ∞.

We remark that it is tempting to try to extend Theorem B to the more general class of Eq. (2) subject to the conditions (i)–(iii) listed in Theorem A. However, our method does not seem to be applicable in this more general case for the following reason: The coefficients of the middle terms will appear in the zero-order coefficient of the equation described earlier, and will not permit the factorization technique to be used. The net effect is that the dominant terms around ∞ in the zero-order coefficient become unclear, and the method fails. Hence the possible extension of Theorem B to Eq. (2) remains as an open question.

2. Notation

If $R(z) \not\equiv 0$ is a rational function whose Laurent expansion around ∞ has the form,

$$R(z) = c_d z^d + c_{d-1} z^{d-1} + \cdots, \quad \text{where} \quad c_d \neq 0, \tag{3}$$

then we will denote d by $\delta(R)$. (If $R \not\equiv 0$, we will set $\delta(R) = -\infty$.) As is customary, we will also use the notation $\binom{n}{j}$ for the binomial coefficient $n!/(j!)((n-j)!)$.

LEMMA 1. *Let $P(z)$ be a polynomial of degree $m+1$, where m is a positive integer. Let $\Delta(z)$ be a polynomial of degree q, and assume $\Delta(z) \not\equiv 0$. Set $h = e^P$ and $f =$*

Δe^P. *Then the following are true*:

(a) *The Laurent expansion of* Δ'/Δ *around* ∞ *is of the form*,

(4) $$\Delta'(z)/\Delta(z)=qz^{-1}+\sigma_1 z^{-2}+\cdots,$$

where the σ_j *are constants.*

(b) *For* $n=2, 3, \cdots$, *the Laurent expansion of* $\Delta^{(n)}/\Delta$ *around* ∞ *is of the form*,

(5) $$\Delta^{(n)}(z)/\Delta(z)=\sigma_{nn}z^{-n}+\sigma_{n,n+1}z^{-(n+1)}+\cdots,$$

where the σ_{nj} *are constants* (*and* σ_{nn} *may be* 0).

(c) *For* $n=0, 1, 2, \cdots$, *we have*,

(6) $$h^{(n)}/h=(P')^n+(n(n-1)/2)(P')^{n-2}P''+V_n,$$

where V_n *is a polynomial of degree less than* $m(n-1)-1$.

(d) *For* $n=0, 1, 2, \cdots$, *we have*,

(7) $$f^{(n)}/f=(P')^n+(n(n-1)/2)(P')^{n-2}P''+n(\Delta'/\Delta)(P')^{n-1}+E_n,$$

where E_n *is a rational function satisfying* $\delta(E_n)<m(n-1)-1$.

Proof. Part (a) is obvious, and Part (b) follows from Part (a) by differentiation. Part (c) follows immediately from [4; Lemma 3.5, p. 73]. For Part (d), we have by Leibniz's rule,

(8) $$f^{(n)}/f=(h^{(n)}/h)+n(\Delta'/\Delta)(h^{(n-1)}/h)+\sum_{j=2}^{n}\binom{n}{j}(\Delta^{(j)}/\Delta)(h^{(n-j)}/h).$$

It is now obvious that (7) follows easily from (4), (5) and (6).

Proof of Theorem B. We are given that $A(z)$ is a polynomial of degree $n\geq 1$, and that Eq. (1) (where $k\geq 2$) possesses a solution $f(z)$ such that $N(f)<(n+k)/k$. We may assume that $k\geq 3$ since the case $k=2$ is covered by [1; Theorem 1]. By Theorem A, Parts (a) and (b), we can write $f=\Delta e^P$, where $\Delta(z)\not\equiv 0$ is a polynomial, and where $P(z)$ is a polynomial of degree $(n+k)/k$. Set $m=n/k$ so m is a positive integer (since $n\geq 1$). Under the change of variable,

(9) $$v=(y/f)',$$

Eq. (1) is transformed by the method of reduction of order into the equation,

(10) $$\sum_{j=1}^{k}\binom{k}{j}(f^{(k-j)}/f)v^{(j-1)}=0.$$

We now assume that the conclusion of Theorem B fails to hold, so that (1) possesses a solution $g(z)$ which is not a constant multiple of $f(z)$, and has the property that $N(g)<(n+k)/k$. In view of Parts (a) and (b) of Theorem A, we may write,

(11) $$g=He^W,$$

where $H(z)\not\equiv 0$ is a polynomial, and where W is a polynomial of degree $m+1$. In view of (9), the function $v=(g/f)'$ is a nontrivial solution of (10). Clearly, we may write $v=Re^Q$ where Q is a polynomial, and where the rational function R is given by,

$$R=(H/\Delta)'+(H/\Delta)Q' . \tag{12}$$

Now in view of Part (d) of Lemma 1, the coefficient of $v^{(j-1)}$ in Eq. (10) is a rational function satisfying,

$$\delta\left(\binom{k}{j}(f^{(k-j)}/f)\right)=m(k-j) , \tag{13}$$

for each $j=1, 2, \cdots, k$, and clearly the number on the right side of (13) is maximum when $j=1$. It easily follows that (10) can have no nontrivial rational solutions, for if we consider the Laurent expansion of a rational function $v\not\equiv 0$ around ∞, then by (13) we would obtain,

$$\delta\left(\binom{k}{j}(f^{(k-j)}/f)v^{(j-1)}\right)<\delta\left(\binom{k}{1}(f^{(k-1)}/f)v\right) \tag{14}$$

for $j=2, \cdots, k$. In view of (13), it now follows from the Wiman-Valiron theory that if Eq. (10) possesses a solution of the form $v=z^nF(z)$ where n is an integer and F is entire, then the order of F is $m+1$. Since $v=Re^Q$, we see that the degree of the polynomial Q must be $m+1$.

We now set $w=e^Q$ so that $v=Rw$, and we compute the derivatives of v by Leibniz's rule. Substituting into Eq. (10), rearranging terms, and dividing by w, we see that the rational function $R(z)$ satisfies the equation,

$$\sum_{t=0}^{k-1} A_t R^{(t)}=0 , \tag{15}$$

where

$$A_t=\sum_{j=t+1}^{k}\binom{k}{j}\binom{j-1}{t}(f^{(k-j)}/f)(w^{(j-1-t)}/w) . \tag{16}$$

We now analyze the zero-order coefficient which is given by the formula,

$$A_0=\sum_{j=1}^{k}\binom{k}{j}(f^{(k-j)}/f)(w^{(j-1)}/w) . \tag{17}$$

Using (7) to compute the terms $f^{(k-j)}/f$ (since $f=\Delta e^P$), and using (6) to compute the terms $w^{(j-1)}/w$ (since $w=e^Q$ where Q is of degree $m+1$) we can write,

$$A_0=F_1+F_2+F^*+F_3 \tag{18}$$

where

$$F_1=\sum_{j=1}^{k}\binom{k}{j}(P')^{k-j}(Q')^{j-1} , \tag{19}$$

(20) $$F_2=\sum_{j=1}^{k}\binom{k}{j}\left(\binom{j-1}{2}(P')^{k-j}(Q')^{j-3}Q''+\binom{k-j}{2}(Q')^{j-1}(P')^{k-j-2}P''\right),$$

(21) $$F^*=\sum_{j=1}^{k}\binom{k}{j}((k-j)(\Delta'/\Delta)(P')^{k-j-1}(Q')^{j-1}),$$

and where F_3 is a rational function satisfying,

(22) $$\delta(F_3)<m(k-2)-1.$$

In order to sum the expressions in (19)–(21), we introduce the function $\varphi(s)$ defined by,

(23) $$\varphi(s)=((s+1)^k-1)/s,$$

and so by the binomial theorem,

(24) $$\varphi(s)=\sum_{j=1}^{k}\binom{k}{j}s^{j-1}.$$

Let $b_1, \cdots, b_{k-1}$ denote the roots of $\varphi(s)$ so that,

(25) $$\varphi(s)=(s-b_1)\cdots(s-b_{k-1}).$$

From (23) it is obvious that $b_1, \cdots, b_{k-1}$ are distinct and nonzero. Furthermore, if some b_j is real, then it must equal -2 (and this can happen only if k is even.) From (19), (24), and (25) it is easy to see that,

(26) $$F_1=(Q'-b_1P')\cdots(Q'-b_{k-1}P').$$

We observe that by (16) and Lemma 1, it easily follows that

(27) $$\delta(A_t)\le m(k-1-t) \quad \text{for} \quad t=0, 1, \cdots, k-1.$$

We now divide the proof into two cases.

Case 1: $Q'\not\equiv b_jP'$ for each $j=1, \cdots, k-1$. Since Q' and P' are of degree m, and since the b_j are distinct, it is clear in Case 1 that $Q'-b_jP'$ is of degree m for all j with the possible exception of one value of j, and if the exceptional value of j exists, the factor $Q'-b_jP'$ is not identically zero. Hence from (26), we see that,

(28) $$\delta(F_1)\ge m(k-2).$$

But from (4), (20), and (21) we see that $\delta(F_2)$ and $\delta(F^*)$ are at most $m(k-2)-1$, and so in view of (18) and (22), we have

(29) $$\delta(A_0)\ge m(k-2).$$

It now follows from (27) and (29) that Eq. (15) cannot possess a rational solution $R\not\equiv 0$ since we easily see that

(30) $$\delta(A_tR^{(t)})<\delta(A_0R) \quad \text{for} \quad t=1, \cdots, k-1,$$

for any rational function $R\not\equiv 0$. This shows that Case 1 is not possible under our assumption. Hence we must have the following case:

Case 2: There exists an r in $\{1, 2, \cdots, k-1\}$ such that,

$$Q'\equiv b_r P', \quad \text{and thus} \quad F_1\equiv 0 . \tag{31}$$

In this case, we proceed to sum the expression in (20) for F_2, and the expression for F^* in (21). In view of (20) and (31), we can write,

$$F_2=K_1(P')^{k-3}P'', \quad \text{where} \quad K_1=K_2+K_3 , \tag{32}$$

and where,

$$K_2=\sum_{j=1}^{k}\binom{k}{j}\binom{j-1}{2}b_r^{j-2}, \tag{33}$$

and

$$K_3=\sum_{j=1}^{k}\binom{k}{j}\binom{k-j}{2}b_r^{j-1}. \tag{34}$$

In view of (24), it is easy to see that,

$$K_2=b_r\varphi''(b_r)/2 . \tag{35}$$

To find K_3, we now set,

$$\psi(s)=s^{k-1}\varphi(1/s)=\sum_{j=1}^{k}\binom{k}{j}s^{k-j}, \tag{36}$$

and so from (34) we obtain,

$$K_3=b_r^{k-3}\psi''(1/b_r)/2 . \tag{37}$$

Computing ψ'' from (36), and noting that $\varphi(b_r)=0$ we obtain,

$$K_3=(b_r^2\varphi''(b_r)-(2k-4)b_r\varphi'(b_r))/2 . \tag{38}$$

Using the definition $\varphi(s)=((s+1)^k-1)/s$ to compute φ' and φ'', and using the fact that $(b_r+1)^k=1$ (since $\varphi(b_r)=0$), we obtain from (32), (35), and (38) that,

$$K_1=(-k/b_r)+(1/2)k(3-k)(b_r+1)^{k-1} . \tag{39}$$

Since $(b_r+1)^{k-1}=1/(b_r+1)$, we thus have,

$$K_1=(-k/b_r)+(1/2)k(3-k)(1/(b_r+1)) . \tag{40}$$

We now compute F^* using (21) and (31). We see that,

$$F^*=K^*(\Delta'/\Delta)(P')^{k-2} \tag{41}$$

where

(42) $$K^* = \sum_{j=1}^{k} \binom{k}{j} (k-j) b_r^{j-1} .$$

Using the function $\psi(s)$ defined in (36), we easily see that,

(43) $$K^* = b_r^{k-2} \psi'(1/b_r) .$$

Using the definition of $\psi(s)$ in (36), we see that $K^* = -b_r \varphi'(b_r)$. Computing φ' from (23) (and noting that $(b_r+1)^k = 1$), we find,

(44) $$K^* = -k/(b_r+1) .$$

We now compute the coefficient A_1 which by (16) is given by the formula,

(45) $$A_1 = \sum_{j=2}^{k} \binom{k}{j} (j-1)(f^{(k-j)}/f)(w^{(j-2)}/w) .$$

Recalling that $f = \Delta e^P$ and $w = e^Q$, it now follows from Lemma 1 that,

(46) $$A_1 = \sum_{j=2}^{k} \binom{k}{j} (j-1)(P')^{k-j}(Q')^{j-2} + G ,$$

where G is a rational function satisfying,

(47) $$\delta(G) \leq m(k-2) - m - 1 .$$

In view of (31), we have,

(48) $$A_1 = K_4 (P')^{k-2} + G ,$$

where

(49) $$K_4 = \sum_{j=2}^{k} \binom{k}{j} (j-1) b_r^{j-2} .$$

In view of (24), clearly $K_4 = \varphi'(b_r)$. Computing φ' from (23) (and using the fact that $(b_r+1)^k = 1$), we find that,

(50) $$K_4 = k/b_r(b_r+1) .$$

Now from our assumption that the conclusion fails to hold, Eq. (15) must possess a rational solution $R(z) \not\equiv 0$. Let the Laurent expansion of R around ∞ be,

(51) $$R(z) = c_d z^d + c_{d-1} z^{d-1} + \cdots , \quad \text{where} \quad c_d \neq 0 .$$

Since P' is a polynomial of degree m, we can write,

(52) $$P' = c z^m + \cdots , \quad \text{where} \quad c \neq 0 .$$

We can write (15) in the form,

(53) $$A_1 R' + A_0 R = -\sum_{t=2}^{k-1} A_t R^{(t)} .$$

Now from (18), (31), (32), and (41), we have

$$A_0=K_1(P')^{k-3}P''+K^*(\Delta'/\Delta)(P')^{k-2}+F_3\,. \tag{54}$$

Letting q denote the degree of Δ, we see from (4), (22), and (54) that the Laurent expansion around ∞ of A_0 is of the form,

$$A_0=c^{k-2}K_5z^{m(k-2)-1}+F_4 \tag{55}$$

where,

$$K_5=K_1m+K^*q\,, \quad \text{and} \quad \delta(F_4)<m(k-2)-1\,. \tag{56}$$

We assert that $K_5\neq 0$. If we assume the contrary, then using (40) and (44) we would obtain,

$$b_r(-q+(m(1-k)/2))=m\,. \tag{57}$$

Since $q\geq 0$, $m\geq 1$, and $k\geq 3$, the relation (57) would show that b_r is real, and hence we would have $b_r=-2$ as previously noted. But then (57) would yield $2q=m(2-k)$ which would imply $q<0$. Hence $K_5\neq 0$.

Now, from (47), (48), and (52), we see that the Laurent expansion of A_1 around ∞ is of the form,

$$A_1=c^{k-2}K_4z^{m(k-2)}+G_1\,, \tag{58}$$

where $\delta(G_1)<m(k-2)$. (We observe that $K_4\neq 0$ by (50).) Finally, we observe that from (27) and (51), we have,

$$\delta(A_tR^{(t)})<m(k-2)+d-1 \tag{59}$$

for $t=2, \cdots, k-1$. Thus from (51), (53), (55), and (58), we see that,

$$c_dc^{k-2}(K_5+K_4d)z^{m(k-2)+d-1}=G_2 \tag{60}$$

where $\delta(G_2)<m(k-2)+d-1$, and thus we must have,

$$K_5+K_4d=0 \tag{61}$$

since $c_dc\neq 0$. (We remark that since $K_5\neq 0$, we must have $d\neq 0$.) Using the formulas for K_5, K_1, K^*, and K_4 developed in (40), (44), (50), and (56), it follows from (61) after elementary manipulation that,

$$b_r(-q+(m(1-k)/2))=m-d \tag{62}$$

which as in (57) shows that b_r is real (since the coefficient of b_r is not zero). Thus $b_r=-2$ and (62) then yields,

$$d=m(2-k)-2q\,. \tag{63}$$

We now show (63) is impossible. Let σ denote the degree of the polynomial H in the representation (11) for g. Noting that Δ is of degree q, and Q' is of degree m, it follows from (12) and (51) that

(64) $$d=\delta(R)=\sigma-q+m\,.$$

Together with (63), this yields,

(65) $$\sigma=m(1-k)-q\,.$$

Since $m\geq 1$, $k\geq 3$, and $q\geq 0$, relation (65) shows $\sigma<0$ which is absurd since σ is the degree of the polynomial H. This contradiction shows that Case 2 is also impossible under our assumption, and the proof of Theorem B is complete.

References

[1] BANK, S. and LAINE, I.; On the oscillation theory of $f''+Af=0$ where A is entire, *Bull. Amer. Math. Soc.*, **6** (1982), 95–98.

[2] BANK, S. FRANK, G. and LAINE, I.; Über die Nullstellen von Lösungen linearer Differentialgleichungen, *Math. Zeit.*, **183** (1983), 355–364.

[3] FRANK, G.; Picardsche Ausnahmewerte bei Lösungen linearer Differentialgleichungen, *Manuscripta Math.*, **2** (1970), 181–190.

[4] HAYMAN, W.; *Meromorphic Functions*, Clarendon Press, Oxford, 1964.

[5] PÖSCHL, K.; Über Anwachsen und Nullstellenverteilung der ganzen transcendenten Lösungen linearer Differentialgleichungen I, *J. reine angew. Math.*, **199** (1958), 121–138.

[6] VALIRON, G.; *Lectures on the General Theory of Integral Functions*, Chelsea, New York, 1949.

[7] WITTICH, H.; *Neure Untersuchungen über eindeutige analytische Funktionen*, Ergebnisse der Math. Heft 8, Springer-Verlag, Berlin, 1955.

Department of Mathematics
1409 West Green Street
University of Illinois
Urbana, Illinois 61801
U.S.A.

Abteilung Mathematik
Universität Dortmund
Postfach 50 05 00
4600 Dortmund 50
West Germany