COMMENTARII MATHEMATICI
UNIVERSITATIS SANCTI PAULI
Vol. 33, No. 2, 1984

ed. RIKKYO UNIV/MATH
IKEBUKURO TOKYO
171 JAPAN

A Definable Interpretation of Metric Spaces

by

Mariko YASUGI

(Received April 5, 1983)

Introduction

We have been investigating the "definability problems" in analysis. It is a program to develop certain mathematical theories in some formal systems which are modest extensions of Peano arithmetic, thus establishing the soundness of these theories relative to the given mathematical structures. The formal systems we employed in our forgoing researches were those based on many sorted logics in which the definable theory of the reals can be formalized and which have the principle of DDI (definitions by definable induction). For the detailed discussion of our standpoint, see Yasugi [8] and [10].

A recent work along this line concerns the elementary theory of topology (Yasugi [10]). Here as a sequel to it we take up the abstract theory of metric spaces. It turns out that the mathematical notions and theorems in this area admit the "definable interpretations". For example, the separability of a space is formulated in terms of a parameter; we say that a metric space is separable by E when E is supposed to be the separating system of the space. In the presence of the metric separability, various notions of compactness are mutually definably interpretable (§ 9). "A complete subset C of a metric space is closed" if there is a definable predicate by which each point of the closure of C is assigned a sequence from C converging to it. The uniform closure of an algebra of continuous functions can be defined in terms of a definable sequence of functions (§ 11).

[Convention] Acquaintance with [10] is assumed throughout. In as much as various definitions and theorems in it are valid here also, we reckon that it is better to avoid repetitions, and hence relevant items in [10] will be quoted with the asterisk affixed whenever it is sufficient to do so. Thus, for example, "Definition 1.1*" will stand for "Definition 1.1 in Yasugi [10]".

Mathematically we follow Royden [5], occasionally resorting to Kelley [3] and Loomis [4].

A preliminary report of this paper has appeared in [9]. [11] and [12] are included in the references, since they were worked on parallel to the present article and are closely related to it.

§ 1. Symbols and axioms

DEFINITION 1.1 1) Atomic types are three sorted; one for the rationals and two for the elements of two spaces. (For the sake of simplicity, we assume two spaces.) For the compound types, see Definition 1.1*, namely Definition 1.1 in [10].

2) For the basic language, see Definition 1.2*.

3) Symbols of metric spaces and of DDI are as follows.

$$X,\ Y,\ \mathrm{eq}\,(X;\ ,\ ),\ \mathrm{eq}\,(Y;\ ,\ ),\ \rho,\ \sigma,\ x_0,\ y_0,\ I_0,\ I_1,\ I_2,\ \cdots .$$

The intended interpretations and the types of these symbols will become clear later.

4) As for the notions of definability, terms, formulas, abstracts, **min**, sequents and substitution, see Definitions 1.3* and 1.4*.

5) Φ, Ψ, ψ, ξ, E, F, G, $\cdots$ will be used for the parameters of higher types. See also Section 1 of [8] for other notational conventions.

DEFINITION 1.2. 1) The logical system $\mathscr{L}$ is defined as in Definition 1.5*.

2) For the axiom sets $\mathscr{A}$ and $\mathscr{C}$, see Definition 1.6*. $\mathscr{C}$ consists of definitions by definable induction, which is abbreviated to DDI.

3) $\mathscr{B}$ consists of the axioms on (X, ρ) and (Y, σ). We shall employ conventional mathematical notations as much as possible. In particular, $\mathrm{eq}\,(X; x, u)$ will be denoted by $x=u$ and $\{t\}\rho(x, u, t)$ will be abbreviated to $\rho(x, u)$.

$\forall x(x\in X)$, $x_0\in X$

The equivalence relations on $=$

$\forall x\ \forall u R(\rho(x, u))$, where $R(a)$, or $a\in R$, expresses that a is a real number.

$\forall x\ \forall y\ \forall u\ \forall v(x=u\wedge y=v\vdash\rho(x, y)=\rho(u, v))$.

The metric property of ρ

Similarly for (Y, σ).

4) A sequent $\Gamma\to\Delta$ of our language is said to be a theorem of $\mathscr{M}_0$ if

$$\mathscr{A},\ \mathscr{B},\ \mathscr{C},\ \Gamma\to\Delta$$

is provable in the system $\mathscr{L}$. $\mathscr{M}_0$ will be called the theory of metric spaces.

Definitions 1.8* to 1.10* are valid as they are. Those are concerned with the definable instantiations and the systems $\mathscr{M}$ and $\mathscr{P}$.

Note. 1) Given a formula $A(\phi, \psi)$ with free variables ϕ and ψ, we may write $A(\psi)$ for the abstract $\{\phi\}A(\phi, \psi)$. Similarly for the formulas with more free variables.

2) We shall use κ for an arithmetically definable enumeration of $N\times N$, the pairs of natural numbers.

§ 2. Relative soundness

Regarding the soundness of the theory under consideration, Section 2* nearly goes through.

THEOREM 1. *Let $\Gamma \to \Delta$ be a sequent which expresses an elementary theorem of the reals or of the metric spaces. Then it is a theorem of $\mathscr{M}_0$; that is,*

$$\mathscr{A}, \mathscr{B}, \mathscr{C}, \Gamma \to \Delta$$

is provable in $\mathscr{L}$, hence without cuts.

The major content of this article consists in the proof of this theorem.

THEOREM 2 (Relative soundness). *The elementary theory $\mathscr{M}_0$ of the metric spaces is sound relative to the axioms of the spaces concerned.*

See Theorem 3* for the proof.

§ 3. Topology

Note. Here and in all the sections that follow, the propositions are meant to be the theorems of $\mathscr{M}_0$. Also, we assume the definable theory of the reals throughout. (See Takeuti [6].) Since the technicalities of the proofs are much the same as those in our previous works, we shall limit ourselves to explicitly presenting the definable objects.

We use the set-theoretical notations in Section 3*.

THEOREM 4*. *The definability property and the subset property are both preserved under the basic set-theoretical operations.*

DEFINITION 3.1. 1) Q^+ will denote the set of the positive rationals, and r, s, t, ε, δ, $\cdots$ will stand for the rationals.

2) Λ will denote the set $(X \times Q^+) \cup \{\lambda_0\}$, where λ_0 is a new designated symbol.

3) If $\lambda = (x, r) \in X \times Q^+$, then $B(x; \lambda)$, or $B(\lambda)$ for short, abbreviates $\{u\}(\rho(x, u)) < r$. Thus $u \in B(\lambda)$ will mean $\rho(x, u) < r$. $B(\lambda_0)$ is defined to be the whole space X.

PROPOSITION 3.1.

$$\begin{aligned} & r_1 > 0\,, \qquad r_2 > 0\,, \qquad u \in B(x_1, r_1) \cap B(x_2, r_2) \\ & \to \exists r > 0 \;\; (r < \min\,(r_1 - \rho(u, x_1)\,,\; r_2 - \rho(u, x_2)) \\ & \wedge u \in B(u, r) \subset B(x_1, r_1) \cap B(x_2, r_2))\,. \end{aligned}$$

PROPOSITION 3.2. *$\{B(\lambda); \lambda \in \Lambda\}$ satisfies the axioms of the base of topology with the index set Λ. See* 2) *of Definition* 1.6*.

THEOREM 3. *The elementary theory of topology which is induced by* ρ *as in Proposition* 3.2 *above is sound relative to the given metric space.*

Proof. Proposition 3.2 is established in $\mathscr{M}_0$, and hence, by virtue of Theorem 2, is sound relative to the axioms (X, ρ). On the other hand, Theorem 3* claims that the elementary theory of topology is sound relative to a given base.

We do not repeat the definitions of various objects in topology and the relevant propositions. The reader should refer to Section 3*. Let us show just one example.

$$\text{opn}\,(A)\text{: ss}\,(X; A) \wedge \forall x \in A \exists y \exists r > 0 (x \in B(y, r) \subset A)\,,$$

where ss $(X; A)$ is read "A is a subset of X" and opn(A) is read "A is open".

PROPOSITION 3.3. 1) opn $(A) \leftrightarrow \forall x \in A \exists r > 0 (B(x, r) \subset A)$.

2) $r > 0 \rightarrow \text{cl}\,(B(x, r) \equiv \{y\}(\rho(x, y) \leq r)$,
where cl (A) *is read "the closure of A".*

3) $0 < r < s \rightarrow \text{cl}\,(B(s, r)) \subset B(x, s)$.

§ 4. Separability, countability and normality

DEFINITION 4.1. MS (X, E) (X is metric separable by E.):

$$\text{sq}\,(E) \wedge \forall x \forall r > 0 \exists n \exists y (E(n, y) \wedge \rho(y, x) < r)\,,$$

where sq (E) *is read "E is a sequence from X." See Definition* 3.4*.

We write $E(n)$ for the y satisfying $E(n, y)$.

PROPOSITION 4.1.

$$\text{MS}\,(X, E),\ \text{opn}\,(A) \rightarrow$$
$$\forall x \in A \exists n \exists r > 0 (x \in B(E(n), r) \subset A)\,.$$

PROPOSITION 4.2 (Equivalence of the metric separability and the second countability). 1) MS $(X, E) \rightarrow$
"$\{n, m\}B(E(n), 1/m)$ *forms a countable base for the topology defined in Proposition* 3.2 *and* $\forall n \forall m (E(n) \in B(E(n), 1/m))$".

2)
$$\text{sq}\,(E),\ \text{opnsq}\,(\Phi),\ \forall l \exists n (E(n) \in \Phi(l)),$$
$$\forall x \forall r > 0 \exists l (x \in \Phi(l) \subset B(x, r)) \rightarrow \text{MS}\,(X, E)\,,$$

where opnsq (Φ) *is read "Φ is a sequence of open sets." See Definition* 6.2*.

DEFINITION 4.2.

$$\rho(x, F)\text{: } \{t\}(\forall u \in F \rho(x, u, t)),\ \textit{or}\ \inf\{\rho(x, u);\ u \in F\}$$

Notice that $\rho(x, F)$ is definable in our language.

COROLLARY. $\text{ss}(X, F) \rightarrow R(\rho(x, F))$.

PROPOSITION 4.3. *$\mathbf{T}(4; \theta, \eta)$ holds with definable θ and η; see Definition 5.1*.*

Proof. $\theta(F, G)$: $\{x\}(\rho(x, F)<\rho(x, G))$ and $\eta(F, G)$: $\{x\}(\rho(x, G)<\rho(x, F))$ will do.

§ 5. Sequences and convergence

The notions of sequences, convergence, cluster points, subsequences, etc. are defined as in Definitions 3.4* and 3.5*. Those can be equivalently formulated in terms of the metric. Let us give one example.

PROPOSITION 5.1. *When* sq (S) *is assumed,*

$$\text{clost}\,(S, x) \leftrightarrow \forall\varepsilon>0\ \forall n \exists m \geq n(\rho(x, S(m))<\varepsilon)\,,$$

where clst (S, x) *is read "x is a cluster point of S."* $\rho(x, S(m))<\varepsilon$ *abbreviates* $\forall y(S(m, y) \vdash \rho(x, y)<\varepsilon)$.

DEFINITION 5.1. 1) Csq (ξ) (ξ is a Cauchy sequence.):

$$\text{sq}\,(\xi) \wedge \forall\varepsilon>0 \exists l\ \forall m \geq l\ \forall n \geq l(\rho(\xi(m),\xi(n))<\varepsilon)$$

2) cmpl (A)(A is a complete subset of X.):

$$\text{ss}(X, A) \wedge \forall\xi(\text{sq}\,(A, \xi) \wedge \text{Csq}\,(\xi) \vdash \exists x\, \text{cnv}\,(\xi, x))\,,$$

where sq $(a.\ \xi)$ is read "ξ is a sequence from A" and cnv (ξ, x) is read "ξ converges to x".

3) Let S be a sequence of reals. limsup S, liminf S and lim S (when it exists) are defined as in [6].

4) CS (X) (Cauchy space): $\{\xi\}$ Csq (ξ)

$\xi=\eta$: $\forall n\ \forall x\ \forall u(x=u \vdash (\xi(n, x) \dashv\vdash \eta(n, u)))$

$\rho^*(\xi, \eta)$: limsup $\{\rho(n), \eta(n)); n=1, 2, \cdots\}$

5) sq*(Ξ): $\forall n$ Csq $(\{i, x\}\, \Xi(m, i, x))$

Csq*(Ξ): sq*$(\Xi) \wedge \forall\varepsilon>0 \exists l\ \forall m \geq l\ \forall n \geq l(\rho^*(\Xi(m), \Xi(n))<\varepsilon)$

$\langle x\rangle$: $\{n, z\}$ $(x=z)$

Note. Let $\Sigma(\Xi, n, \xi)$ denote $\forall i\ \forall x(\xi(i, x) \dashv\vdash \Xi(n, i, x))$. Under the assumption of sq*(Ξ), $\{n, \xi\}\Sigma(\Xi, n, \xi)$ is a sequence from CS (X) in the sense that $\forall n \exists \xi\ \forall \eta(\xi=\eta \dashv\vdash \Sigma(\Xi, n, \eta))$ is provable (with $\{i, x\}\, \Xi(n, i, x)$ as the ξ). Due to the specific form of Σ, we may regard Ξ itself as a sequence from CS (X).

PROPOSITION 5.2. 1) *(CS (X), $=$, ρ^*) is a pseudo-metric space.*

2) *$\langle x\rangle \in$ CS (X), and $\forall x\ \forall u(\rho^*(\langle x\rangle, \langle u\rangle)=\rho(x, u))$.*

3) *$\xi \in$ CS $(X) \rightarrow \forall\varepsilon>0 \exists x \in X(\rho^*(\xi, \langle x\rangle)<\varepsilon)$.*

4) *(CS (X), $=$, ρ^*) is definably complete; that is,*

$$\mathrm{Csq}^*(\Xi)\to\xi^*(\Xi)\in \mathrm{CS}\,(X)\wedge\ \forall\varepsilon>0\exists l\ \forall n\geq l(\rho^*(\xi^*(\Xi),\ \Xi(n))<\varepsilon)$$

with a definable ξ^*, *where* $\Xi(n)$ *abbreviates* $\{i, x\}\,\Xi(n, i, x)$.

Proof of 4). Assume $\mathrm{Csq}^*(\Xi)$ and define ν and ξ^* as follows.

$$\nu(n)=\mathbf{min}\,(k,\ \forall i\geq k\ \forall j\geq k\ \forall x\ \forall u(\Xi(n, i, x)\wedge \Xi(n, j, u)\vdash\rho(x, u)<1/n))\,;$$

$$\xi^*(\Xi, n, x):\ \Xi(n, \nu(n), x)\,.$$

§ 6. Continuous functions

Here we consider two metric spaces (X, ρ) and (Y, σ), and assume $x, u, z, \cdots\in X$, $y, v, \cdots\in Y$, $\lambda\in\Lambda$ and $\mu\in(Y\times Q^+)\cup\{\mu_0\}$. See Definition 3.1, Proposition 3.2 and Section 4*.

PROPOSITION 6.1. *Assume* mp (f, X, Y) (f *is a map from* X *to* Y). *Then*

$$\forall\mu\ \mathrm{opn}\,(\mathrm{inv}\,(f,\ B(Y;\ \mu)))\leftrightarrow$$

$$\forall\varepsilon>0\ \forall x\in W\exists\delta>0\ \forall u(\rho(x, u)<\delta\vdash\sigma(f(x),\ f(u))<\varepsilon)\,,$$

where $W=\mathrm{inv}\,(f,\ B(Y;\ \mu))$.

Proof. Given $\varepsilon>0$ and $x\in W$. Put $V\equiv B(Y;\ f(x),\ \varepsilon)$. Then

$$\mathrm{opn}\,(\mathrm{inv}\,(f,\ V))\to$$

$$\exists z\exists r>0(x\in B(X;\ z,\ r)\subset\mathrm{inv}\,(f,\ V))\,.$$

Put $a=\rho(x, z)$. Then $a\geq 0\wedge\exists\delta>0(\delta<r-a)$. Such a δ will do. Conversely, assume the right hand side and consider $U=B(Y;\ y,\ r)$ for $r>0$. Suppose $x\in U$ and consider $B(Y;\ f(x),\ r)$. Then $f(x)\in B(Y;\ y,\ r)\cap B(Y;\ f(x),\ r)$, and hence

$$f(x)\in B(Y;\ f(x),\ s)\subset B(Y;\ y,\ r)\cap B(Y;\ f(x),\ r)$$

for an s, $0<s<r$.

$$\exists\delta>0(u\in B(X;\ x,\ \delta)\vdash f(u)\in B(Y;\ f(x),\ s)\subset B(Y;\ y,\ r))\,,$$

so $\mathrm{opn}\,(\mathrm{inv}\,(f,\ B(Y;\ y,\ r)))$.

Various properties concerning the continuous functions and the homeomorphism can be stated and proved as in the general setting. See Section 4*.

DEFINITION 6.1. unfcnt (f, A, Y) (uniform continuity): mp $(f, A, Y)\wedge\ \forall\varepsilon>0\exists\delta>0\ \forall x\ \forall u(\rho(x, u)<\delta\vdash\sigma(f(x), f(u))<\varepsilon)$

PROPOSITION 6.2. unfcnt $(f)\wedge\mathrm{Csq}\,(X;\ S)\to\mathrm{Csq}\,(Y; f(S))$, *where* $\mathrm{Csq}(X;\ S)$ *is read "S is a Cauchy sequence from X."*

PROPOSITION 6.3. *Define* $g^*(f,\ \xi,\ x,\ y)$ *to be* $\mathrm{cnv}\,(f(\xi(x),\ y))$. *Then*

$$\mathrm{cmpl}\,(Y)\,,\ \mathrm{unfcnt}\,(f,\ A,\ Y)\,,$$
$$\forall x \in \mathrm{cl}\,(A)(\mathrm{sq}(A,\ \xi(x)) \wedge \mathrm{cnv}\,(\xi(x),\ x)) \rightarrow$$
$$\mathrm{unfcnt}\,(g^*(f,\ \xi),\ \mathrm{cl}\,(A),\ Y)$$
$$\wedge \text{“}g^*(f,\ \xi) \equiv f \text{ on } A\text{”} \wedge \text{“such a map is unique on } \mathrm{cl}\,(A)\text{”},$$

where ξ serves as a relation between $x \in \mathrm{cl}\,(A)$ and a sequence associated with x.

§7. Subspaces

PROPOSITION 7.1. $\mathrm{ss}(X, C) \rightarrow$ “$(C,\ \rho \upharpoonright C)$ *is a metric space*”, *where $\rho \upharpoonright C$ represents the metric restricted to C.*

PROPOSITION 7.2. *There is a definable E^* such that*

$$\mathrm{ss}\,(X,\ C),\ \mathrm{MS}\,(X,\ E),\ \mathrm{sq}\,(C,\ \psi),\ \ \forall i(C \cap B(E(n),\ 1/m) \neq \phi$$
$$\vdash \psi(i) \in C \cap B(E(n),\ 1/m)) \rightarrow \mathrm{MS}\,(C,\ E^*)\,,$$

where $i = \kappa(m,\ n)$ and the parameters in E^* are omitted.

Proof. Let $P(i)$ represent $\psi(i) \in C \cap B(E(n),\ 1/m)$. Define ν as an application of DDI by:

$$\nu(0) = \mathbf{min}\,(i,\ P(i))\,,$$
$$\nu(j+1) = \mathbf{min}\,(i,\ i > \nu(j) \wedge P(i))\,.$$

Define $E^*(j,\ y)$ to be $y = \psi(\nu(j))$.

PROPOSITION 7.3. $\mathrm{cmpl}\,(C,\ \rho) \wedge\ \forall x \in \mathrm{cl}\,(C)\ \forall r > 0$

$$\exists! y(\psi(x,\ r,\ y) \wedge y \in C \wedge \rho(x,\ y) < r) \rightarrow \mathrm{clsd}\,(C)\,,$$

where ψ is a parameter.

Other properties concerning subspaces are stated and proved as usual.

§8. Baire category theorem

DEFINITION 8.1. $\mathrm{dns}\,(A)$ (A is dense in X.):

$$\mathrm{ss}\,(X,\ A) \wedge\ \forall \lambda \exists x \in A \cap B(\lambda)$$

$\mathrm{nwd}\,(A)$ (A is nowhere dense in X.):

$$\mathrm{ss}\,(X,\ A) \wedge \mathrm{cl}\,(X - \mathrm{cl}\,(A)) = X\,.$$

PROPOSITION 8.1. $\mathrm{nwd}\,(A) \leftrightarrow\ \forall \lambda \neg (B(\lambda) \subset \mathrm{cl}\,(A))\,.$

PROPOSITION 8.2. $\mathrm{cmpl}\,(X),\ \mathrm{MS}\,(X,\ E),\ \mathrm{opnsq}\,(\Psi),$

$$\forall i \, \mathrm{dns}\,(\Psi(i)) \rightarrow \mathrm{dns}\,(\bigcap \Psi)\,.$$

Proof. Suppose $u \in X$ and $s > 0$. We shall construct, as an application of DDI, a sequence from X, say ψ, and a sequence of positive rationals, say h, such that

$$\psi(0)=u,\ h(0)=s\,, \quad \forall n \geq 1(h(n)<1/n)\,,$$

$$(*) \qquad \forall n(\psi(n) \in \mathrm{cl}\,(B(\psi(n),\ h(n))) \subset B(\psi(n),\ 1/n) \subset$$

$$B(\psi(n-1),\ h(n-1)) \cap \psi(n))\,.$$

Then $\forall n\ \forall m \geq n \rho(\psi(m),\ \psi(n)) \leq 1/n$, and hence ψ is a Cauchy sequence. Thus

$$\exists x(\mathrm{cnv}\,(\psi,\ x) \wedge x \in \bigcap \{\mathrm{cl}\,(B(\psi(n),\ h(n)));\ n=1,\ 2,\ \cdots\}$$

$$\subset B(u,\ s) \cap (\bigcap \Psi))\,.$$

Suppose we have constructed ψ and h to $n-1$. Since $\Psi(n)$ is dense and open, $A(n)$: $B(\psi(n-1),\ h(n-1)) \cap \Psi(n)$ is non-empty and open, and hence by Proposition 4.1 $\exists k \exists r > 0(B(E(k),\ r) \subset A(n))$. Put

$$\nu(n)=\mathbf{min}\,(k,\ \exists r>0(B(E(k),\ r) \subset A(n))\,,$$

and determine a rational $\theta(n)$ which satisfies $0<\theta(n) \leq 1/n$ and $B(E(\nu(n)),\ \theta(n)) \subset A(n)$. By Proposition 3.3, $\mathrm{cl}\,(B(E(\nu(n)),\ \theta(n)/2)) \subset B(E(\nu(n)),\ \theta(n))$. Thus, if we define $\psi(n)= E(\nu(n))$ and $h(n)=\theta(n)/2$, then $(*)$ will be satisfied.

PROPOSITION 8.3 (Baire category theorem).

$$\mathrm{cmpl}\,(X),\ \mathrm{MS}(X,\ E),\ \ \forall n\, \mathrm{nwd}\,(\Phi(n)) \rightarrow X \neq \bigcup \Phi\,.$$

Proof. Put $\Psi^*(n)=X-\mathrm{cl}\,(\Phi(n))$. By Proposition 8.2 applied to this Ψ^*,

$$\exists x \in \bigcap \Psi^* = X - \bigcup \{\mathrm{cl}\,(\Phi(n));\ n=1,\ 2,\ \cdots\}\,,$$

which implies $x \notin \bigcup \Phi$.

As an application of Propositions 8.1 and 8.3, we obtain

PROPOSITION 8.4 (The uniform boundedness principle).

$$\mathrm{cmpl}\,(X),\ \mathrm{MS}\,(X,\ E),\ \ \forall n\, \mathrm{cnt}\,(F(n),\ X,\ R),\ \mathrm{mp}\,(M,\ X,\ R)\,,$$

$$\forall x\ \forall n(|F(n,\ x)| \leq M(x)) \rightarrow \exists m \exists e \exists r>0\ \forall n\ \forall x \in B(e,\ r)(|F(n,\ x)| \leq m)\,.$$

§ 9. Compactness

For this section, we refer the reader to Section 6*.

PROPOSITION 9.1. *The elementary theorems of topology concerning the notion of sequential compactness* (scmp (X, Φ): *X is sequentially compact by Φ*) *are provable.*

DEFINITION 9.1. 1) MC (X, E, Φ) (X is metric compact by E and Φ.):

$$\mathrm{MS}\,(X, E) \wedge \mathrm{scmp}\,(X, \Phi)$$

2) TB (X, ψ, ν) (X is totally bounded by ψ and ν.):

$$\forall \varepsilon > 0(\forall k \leq \nu(\varepsilon) \exists ! y \psi(\varepsilon, k, y) \wedge \forall x \exists k$$
$$\leq \nu(\varepsilon)(\rho(x, \psi(\varepsilon, k)) < \varepsilon)) .$$

PROPOSITION 9.2. *There are definable ν^*, σ^* and ψ^* such that*

1) $\mathrm{MC}\,(X, E, \Phi) \rightarrow \forall \varepsilon > 0\ \forall x \exists k \leq \nu^*(\varepsilon)(\rho(x, E(\sigma^*(\varepsilon, k))) < \varepsilon)$,

and

2) $\mathrm{MC}\,(X, E, \Phi) \rightarrow \mathrm{TB}\,(X, \psi^*, \nu^*) \wedge \mathrm{cmpl}\,(X)$.

Proof. Define $\sigma^*(\varepsilon, 1) = 1$ and

$$\sigma^*(\varepsilon, n+1) = \mathbf{min}\,(m,\ \forall i \leq n(\rho(E(\sigma^*(\varepsilon, i)), E(m)) \geq \varepsilon/2)) ,$$

where we assume $\mathbf{min} = 1$ if the condition does not hold for any m. Put

$$\nu^*(\varepsilon) = \mathbf{min}\,(m,\ 1 < m \wedge \sigma^*(m) = 1) .$$

Such a ν^* is meaningfully defined under the assumption of scmp (X, Φ). $\psi^*(\varepsilon, n, y)$ is defined to be $E(\sigma^*(\varepsilon, n), y)$.

PROPOSITION 9.3. cmpl (X), TB $(X, \psi, \nu) \rightarrow \mathrm{MC}\,(X, E^*, \Phi^*)$ *with definable E^* and Φ^*.*

Proof. Define E^* to be

$$\bigcup \{\psi(1/m, n);\ m = 1, 2, \cdots, n \leq \nu(1/m)\} .$$

Suppose sq (ξ). First define μ by:

$$\mu(1) = \mathbf{min}\,(m,\ m \leq \nu(1) \wedge\ \forall n \exists l \geq n(\xi(l) \in B(\psi(1, m), 1))) ,$$
$$\mu(k+1) = \mathbf{min}\,(m,\ m \leq \nu(1/(k+1)) \wedge$$
$$\forall n \exists l \geq n(\xi(l) \in \bigcap \{B(\psi(1/i, \mu(i)), 1/i);\ i \leq k\} \cap$$
$$B(\psi(1/(k+1), m), 1/(k-1)))) .$$

Then define

$$\tau(1) = \mathbf{min}\,(i,\ \xi(i) \in B'(\psi(1, \mu(1)), 1)) ,$$
$$\tau(k+1) = \mathbf{min}\,(i,\ \tau(k) < i \wedge$$
$$\xi(i) \in \bigcap \{B(\psi(1/j, u(j)), 1/j);\ j \leq k+1\}) .$$

Now $\Phi^*(\xi)$ is defined to be $\{k, x\}\xi(\tau(k), x)$.

PROPOSITION 9.4. *There is a definable ψ^* such that*

$$\mathrm{MC}\,(X, E, \Phi),\ \mathrm{clsd}\,(F),\ F \neq \phi \rightarrow \psi^*(E, \Phi, F) \subset F \wedge$$
$$\exists x \in F\, \mathrm{cnv}\,(\psi^*(E, \Phi, F), x)\,,$$

where clsd (F) *is read "F is a closed set."*

Proof. Recall that $\rho(x, F)$ is a definable real. Put

$$\nu(m) = \mathbf{min}\,(n, \rho(E(n), F) < 1/m)\,,$$

and define $\psi^*(E, \Phi, F)$ to be $\Phi(\{m, x\}E(\nu(m), x))$.

PROPOSITION 9.5. *There is a definable Φ^* such that*

$$\mathrm{MC}\,(X, E, \Phi),\ \mathrm{clsq}\,(\beta),\ \forall i(\phi \neq \beta(i+1) \subset \beta(i)) \rightarrow$$
$$\mathrm{sq}\,(\Phi^*(E, \Phi, \beta)) \wedge \exists y \in \bigcap \beta\ \mathrm{cnv}\,(\Phi^*(E, \Phi, \beta), y)\,,$$

where clsq (β) *is read "β is a sequence of closed sets"; see Definition* 6.2*.

Proof. Put

$$\mu(i) = \max\,(i, \mathbf{min}\,(j, \exists x(\ \mathrm{cnv}\,(\psi^*(E, \Phi, \beta(i)), x) \wedge$$
$$\rho(x, \psi^*(E, \Phi, \beta(i))(j)) < 1/i)))$$

with the ψ^* defined in Proposition 9.4. Then define $\Phi^*(E, \Phi, \beta)$ to be $\{i\}\psi^*(E, \Phi, \beta(i))(\mu(i))$.

PROPOSITION 9.6. MC $(X, E, \Phi) \rightarrow$ ccmp (X), *where* ccmp (X) *is read "X is countably compact"; see Definition* 6.2.*.

Proof. By virtue of Proposition 6.2*, it suffices to show MC $(X, E, \Phi) \rightarrow$ FIP, where FIP stands for the finite intersection property. Assume clsq (γ) and $\forall n \exists x \in \bigcap\{\gamma(i);\ i \leq n\}$, and write $G(\gamma, n)$ for $\bigcap\{\gamma(i);\ i \leq n\}$. $G(\gamma)$: $\{n\}G(\gamma, n)$ is a decreasing sequence of non-empty closed sets. So, Propositon 9.5 yields a Φ^* such that sq $(\Phi^*(E, \Phi, G(\gamma)))$ and $\exists y \in \bigcap G(\gamma)$ cnv $(\Phi^*(E, \Phi, G), y)$. But $\bigcap G(\gamma) = \bigcap \gamma$.

PROPOSITION 9.7. ccmp $(X) \rightarrow$ scmp (X, Ψ^*) *with a definable Ψ^*.*

Proof. By virtue of Proposition 6.2*, it suffices to show that BW $\rightarrow$ scmp (X, Ψ^*), where BW stands for the Bolzano-Weierstrass property.

$$\mathrm{BW},\ \mathrm{sq}\,(S) \rightarrow \{x\}\mathrm{clst}\,(S, x) \neq \phi\,.$$

Define τ and H simultaneously as follows.

$$H(S, 0, x)\colon \mathrm{clst}\,(S, x)\,,$$
$$\tau(S, 1) = \mathbf{min}\,(n, \exists x(H(S, 0, x) \wedge \rho(x, S(n)) \leq 1))$$
$$\tau(S, m+1) = \mathbf{min}\,(n, n > \tau(S, m) \wedge$$
$$\exists x(H(S, m, x) \wedge \rho(x, S(n)) \leq 1/(m+1)))\,,$$

$$H(S, m+1, x)\colon H(S, m, x) \wedge \rho(x, S(\tau(S, m+1))) \leq 1/(m+1)\,.$$

Now let $\Psi^*(S, m, x)$ be $S(\tau(S, m), x)$.

Summing up, Proposition 6.2* and Propositions 9.2 to 9.7 claim that, in the presence of the metric separability, various notions of compactness are all mutually definably interpretable.

PROPOSITION 9.8. 1) $\mathrm{TB}(X, \psi, \nu) \wedge \mathrm{unfcnt}(f, X, Y) \to \mathrm{TB}(f(X), f(\psi), \nu)$, *where the conclusion asserts the total boundedness of* Y.

2) $\mathrm{TB}(X, \psi, \nu)$, $\mathrm{unfcnt}(f, X, R) \to$ "sup f and inf f exist".

PROPOSITION 9.9. $\mathrm{MC}(A, E, \Phi) \to \mathrm{clsd}(A)$, *where* $\mathrm{MC}(A, E, \Phi)$ *represents the metric compactness of* A *with respect to* $\rho \restriction A$.

Proof. There are definable Ψ^* and ν^* such that

$$\mathrm{MC}(A, E, \Phi) \to \mathrm{TB}(A, \psi^*, \nu^*) \wedge \mathrm{cmpl}(A)$$

by Proposition 9.2. Modifying ψ^* a little (using E), we obtain a ψ' so that

$$\forall x \in \mathrm{cl}(A)\, \forall r > 0 \exists ! y \in A(\psi'(x, r, y) \wedge \rho(x, y) < r)\,.$$

Proposition 7.3 then implies $\mathrm{clsd}(A)$.

PROPOSITION 9.10. *Let* $\Gamma(E, A, \psi)$ *be the condition on* A *as stated below.*

$$\forall j\, \forall i (\exists x \in A(\rho(E(i), x) < 1/2j \vdash$$
$$\exists ! y \in A(\psi(i, j, y) \wedge \rho(E(i), y) < 1/2j))\,.$$

Then

$$\mathrm{MC}(X, E, \Phi),\ \mathrm{clsd}(A),\ \Gamma(E, A, \psi) \to \mathrm{MC}(A, E^*, \Phi)$$

with a definable E^*. (*In particular,* $A = \mathrm{cl}(B)$ *satisfies* Γ *if* $\mathrm{opn}(B)$.)

Proof. $\mathrm{MC}(X, E, \Phi)$, $\mathrm{clsd}(A) \to \mathrm{scmp}(A, \Phi)$ by Proposition 6.1*. Let $\{l\}\nu(j, l)$ be an enumeration of the i's such that $\exists x \in A(\rho(E(i), x) < 1/2j)$. Then

$$\forall l \exists x \in A(\rho(E(\nu(j, l)), x) < 1/2j)\,.$$

Let κ_1 and κ_2 be respectively the first and the second inverse of the pairing function, and define $E^*(n, y)$ to be $\psi(\nu(\kappa_1(n), \kappa_2(n), y)$.

We defined the one-point compactification of the spaces of a certain type in Section 10*. The same argument goes through here, since the conditions there are satisfied by the separable metric space. For the metric compactification, we refer the reader to Bishop [1] and Bridges [2].

§ 10. Product space

DEFINITION 10.1. 1) We consider a sequence of metric spaces $\{(X_n, d_n, \rho_n);$

$n=1, 2, \cdots\}$ with $0 \le \rho_n \le 1$ for every n, where d_n denotes a designated element of X_n and the elements of the spaces are supposed to be in a universe Z. The axioms on the spaces are assumed to be presented uniformly in n; thus we write $X(n)$ for X_n, $\rho(n)$ for ρ_n and $d(n)$ for d_n, where X, ρ and d are constant predicates.

2) $\mathscr{D}$ will stand for the axiom system on a sequence of metric spaces.

$$\forall x(Z(x) \vdash\dashv \exists m X(m, x))$$

$$\forall m\, \forall n\, \forall x(X(m, x) \wedge X(n, x) \vdash m=n)$$

The equivalence relations on $=$ of the elements of Z

$$\forall m\, \forall x\, \forall y(x=y \wedge X(m, x) \vdash X(m, y))$$

$$\forall m \exists ! x(X(m, x) \wedge d(m, x))$$

$$\forall m\, \forall x\, \forall y(X(m, x) \wedge X(m, y) \vdash R(\rho(m, x, y)))$$

The metric properties of $\rho(n)$ for all n

3) The product space is defined as follows. (See also Definitions 8.2* and 8.3*.)

$$\xi \in \prod X(n):\ \forall m\, \forall x\, \forall y(x=y \wedge \xi(m, x) \vdash \xi(m, y)) \wedge$$

$$\forall m \exists ! x(X(m, x) \wedge \xi(m, x))$$

$$\xi = \eta:\ \forall m\, \forall x\, \forall y(\xi(m, x) \wedge \eta(m, y) \vdash x=y)$$

We may write (x_n) for $\xi \equiv \{n, x\}\xi(n, x)$.

4) $\sigma((x_n), (y_n)) = \Sigma\{\exp(2, -n)\rho(n, x_n, y_n);\ n=1, 2, \cdots\}$, where $\exp(2, m) = 2^m$.

PROPOSITION 10.1. 1) *σ is a metric on $\prod X(n)$.*

2) *The product topology and the metric topology induced by σ are definably equivalent.* (*See Proposition* 8.4*.)

Proof. of 2). Given $\xi \in \prod X(n)$ and $r>0$. Let V be $\{\eta\}(\sigma(\xi, \eta)<r)$. Define U to be $\{\xi\}\ \forall n \le p+2(\rho(n, x_n, z_n) < \exp(2, -p-n-2))$, where $\xi \equiv (x_n)$, $\zeta \equiv (z_n)$ and p is the least natural number satisfying $1/p<r$. Then $U \subset V$.

Consider conversely W an open set in $X(n)$ and let U be $\{\xi\}(x_n \in W)$.

$$\xi \in U \rightarrow \exists r>0\ \forall y(\rho(n, x_n, y)<r \vdash y \in W),$$

and hence $\{\eta\}(\sigma(\xi, \eta) < r \exp(2, -n)) \subset U$.

PROPOSITION 10.2. *The product space is metric compact if each space is.*

Proof. The sequential compactness of the product space was established in the general setting; see Proposition 9.2*. As for the separability, suppose $\forall k$ MS $(X(k), E(k))$. Define F^* as follows. Suppose natural numbers $l, m, n, m_1, \cdots, m_n$ satisfy the relations $l=(m, n)$ and $m=(m_1, \cdots, m_n)$.

$$F^*(l, i) = \begin{cases} \mathscr{E}(i, m_i) & \text{if } i \leq n\,, \\ d(i) & \text{if } i > n\,. \end{cases}$$

Then MS ($\prod X(n)$, F^*).

§ 11. Function space

In this section we assume

[Assumption 1] MS (X, E)

throughout. Then, by virtue of Proposition 4.2, $\{n, m\}B(E(n), 1/m)$ forms a countable base for the metric topology. This we write Z.

[Assumption 2] X is countably compact with respect to Z.

DEFINITION 11.1. 1) $C(X)$: $\{f\}\,\mathrm{cnt}\,(f, X, R)$, where cnt is defined with respect to Z. (See Definition 4.2*.)

2) $f = g$: $\forall x(f(x) = g(x))$

3) $\|f\| = \sup\{f(x);\ x \in X\}$

4) $\sigma(f, g) = \|f - g\|$

5) $\Sigma[F(i);\ i \leq k]$: $\{x\}\Sigma\{F(i, x);\ i \leq k\}$
$\prod[F(i);\ i \leq k]$: $\{x\}\prod\{F(i, x);\ i \leq k\}$

6) $\max[F(i);\ i \leq k]$: $\{x\}\sup\{F(i, x);\ i \leq k\}$
$\min[F(i);\ i \leq k]$: $\{x\}\inf\{F(i, x);\ i \leq k\}$

PROPOSITION 11.1. 1) *The* $=$ *in* 2) *of Definition* 11.1 *defines an equivalence relation on* $C(X)$. *So, we shall call* $(C(X), =)$ *the space of real-valued continuous functions on* X.

2) *There is a definable* f^* *such that*

$$x, y \in X,\ x \neq y \to f^*(x, y) \in C(X)$$
$$\wedge f^*(x, y, x) = 1 \wedge f^*(x, y, y) = 0\,.$$

3) $a \in R, f \in C(X) \to af \in C(X) \wedge |f| \in C(X)$.

4) $\forall i \leq k(F(i) \in C(X)) \to \Sigma[F(i);\ i \leq k] \in C(X) \wedge \prod[F(i);\ i \leq k] \in C(X)$.

5) $\forall i \leq k(F(i) \in C(X)) \to \max[F(i);\ i \leq k] \in C(X) \wedge \min[F(i);\ i \leq k] \in C(X)$.

6) $f \in C(X) \to R(\|f\|)$.

7) $(C(X), =, \sigma)$ *forms a metric space.*

Proof. 2) follows immediately from Urysohn's lemma applied to the singletons $\{x\}$ and $\{y\}$; see Propositions 5.2* and 4.3. As for 4), assume $\forall i \leq k(F(i) \in C(X))$ and consider $P(j)$: $j \leq k \vdash \Sigma[F(i);\ i \leq j] \in C(X)$. Then $\Sigma[F(i);\ i \leq k] \in C(X)$ is proved by induction applied to $P(j)$ (which is a definable formula). 5) can be derived in a similar manner using 4). 6) follows from 6.5*.

PROPOSITION 11.2. *$(C(X), =, \sigma)$ is a (definably) complete space.*

Proof. Suppose F represents a Cauchy sequence from $C(X)$. Then $\{F(n, x)\}_n$ is a Cauchy sequence of real numbers for every $x \in X$, and by [6] there is a definable $\alpha^* \equiv \alpha^*(F)$ such that $\lim \{F(n, x);\ n=1, 2, \cdots\} = \alpha^*(x)$. It can easily be seen that $\alpha^* \in C(X)$.

DEFINITION 11.2. 1) $\mathscr{F} \subset C(X)$: $\forall f(f \in \mathscr{F} \vdash f \in C(X))$

2) unfcl $(\mathscr{F}, G, f)$ (f is in the uniform closure of $\mathscr{F}$ by G.): $\forall n(G(n) \in \mathscr{F}) \wedge f \in C(X) \wedge \forall n(\sigma(f, G(n)) < 1/n)$

3) ltc $(\mathscr{F})$($\mathscr{F}$ is a lattice.):

$$\forall F\, \forall k(\, \forall i \leq k(F(i) \in \mathscr{F}) \vdash \min([F(i);\ i \leq k] \in \mathscr{F} \wedge \max[F(i);\ i \leq k] \in \mathscr{F})$$

PROPOSITION 11.3. $\mathscr{F} \subset C(X)$, ltc $(\mathscr{F})$, $f \in C(X)$,

$$\forall k\, \forall l\, \forall n(k \neq l \vdash F(f, k, l, n) \in \mathscr{F} \wedge |F(f, k, l, n, E(k)) - f(E(k))| < 1/n \wedge |F(f, k, l, n, E(l)) - f(E(l))| < 1/n) \rightarrow \text{unfcl}(\mathscr{F}, G^*, f)$$

with a definable $G^* \equiv G^*(\mathscr{F}, f, F)$.

Proof. We omit the parameter f in the expressions that follow. Define

$$V(k, l, n):\ \{z\}(F(k, l, n, z) < f(z) + 1/n)$$

and

$$W(k, l, n):\ \{z\}(F(k, l, n, z) > f(z) - 1/n)\,.$$

Then

$$\bigcup\{V(k, l, n);\ k=1, 2, \cdots\} = X$$

by the property of E and the continuity of f and F. Also, $\forall k$ opn $(V(k, l, n))$. The countable compactness then implies

$$\exists i(\bigcup\{V(k, l, n):\ k \leq i\} = X)\,.$$

Let i_0 be the least such i, and define $W(l, n)$: $\bigcap\{W(k, l, n);\ k \leq i_0\}$ and $H(l, n)$: $\min[F(k, l, n);\ \ k \leq i_0]$. Then $H(l, n) \in \mathscr{F}$, $H(l, n)(z) < f(z) + 1/n$ on X and $H(l, n)(z) > f(z) - 1/n$ on $W(l, n)$.

$$\bigcup\{W(l, n);\ l=1, 2, \cdots\} = X$$

and, as above,

$$\exists i(\bigcup\{W(l, n);\ l \leq j\} = X)\,.$$

Let j_0 be the least such j, and define $G^*(n)$ to be $\max[H(l, n);\ l \leq j_0]$. Then $G^*(n) \in \mathscr{F}$ and

$$f(z)-1/n<G^*(n)(z)<f(z)+1/n$$

on X, which means unfcl $(\mathscr{F}, G^*, f)$.

DEFINITION 11.3. 1) alg $(\mathbf{A})$: $\{x\}1\in\mathbf{A}\wedge$
$\forall a\in R\ \forall f\in\mathbf{A}(af\in\mathbf{A})\wedge\ \forall F\,\forall k(\ \forall i\leq k(F(i)\in\mathbf{A})\vdash$
$\Sigma[F(i);\ i\leq k]\in\mathbf{A}\wedge\prod[F(i);\ i\leq k]\in\mathbf{A})$
2) $B(X)$: $\{f\}(\mathrm{mp}\,(f,\ X,\ R)\wedge R(\|f\|))$
3) $\phi(n,\ b)=\exp\,(b+(1/\exp\,(n,\ 2)),\ 1/2)$
$\psi(i,\ n,\ a)=\exp\,(\exp\,(a,\ 2)-(1/2),\ i)\ \phi(n,\ 1/n)/i!$
$m_0=\mathbf{min}\,(m,\ \ \forall r\in[-1,\ 1](|\,\phi(n,\ \exp\,(r,\ 2))-$
$\Sigma\{\psi(i,\ n,\ r);\ i\leq m\}\,|<1/\exp\,(2,\ n)))$
$\chi(n,\ a)=\Sigma\{\Sigma(i,\ n,\ a);\ i\leq m_0\}$
$\pi(n,\ a)=\chi(n,\ a)-\chi(n,\ 0)$
4) clsd $(\mathbf{A},\ \pi)$: $\forall n\ \forall f\in\mathbf{A}(\{x\}\pi(n,\ f(x))\in\mathbf{A})$
5) unfclsd $(\mathbf{A},\ \pi)$: $\forall f\in\mathbf{A}(\text{unfcl}\,(\mathbf{A},\ \{n,\ x\}\pi(n,\ f(x)))\subset\mathbf{A})$

PROPOSITION 11.4.
1) $\forall n\ \mathrm{cnt}\,(\pi(n),\ [-1,\ 1],\ R)\wedge\ \forall n\ \forall a\in R(|\,\pi(n,\ a)-|\,a\,|\,|<4/n)$.
2) alg $(\mathbf{A})\rightarrow\ \forall n(\pi(n)\in\mathbf{A})$.
3) alg $(\mathbf{A})$, $\mathbf{A}\subset\{f\}$ mp $(f,\ X,\ R)\rightarrow$clsd $(\mathbf{A},\ \pi)$.
4) alg $(\mathbf{A})$, unfclsd $(\mathbf{A},\ \pi)$, $\mathbf{A}\subset B(X)\rightarrow$ltc $\{\mathbf{A}\}$.

Proof. 1) and 2). The Taylor's series for ϕ about $a=1/2$ converges uniformly in $[0,\ 1]$ (see [6]), and

$$\phi(n,\ \exp\,(a,\ 2))=\Sigma\{\psi(i,\ n,\ a);\ i=0,\ 1,\ 2,\ \cdots\}\,.$$

So the definition of m_0 is meaningful, and the desired properties of π follow from the construction of π.

3) From the construction of π.

4) Assume $\forall i\leq k(F(i)\in\mathbf{A})$, and consider $P(j)$: $i\leq k\vdash\max\,[F(i);\ i\leq j]\in\mathbf{A}$. We shall show $\forall jP(j)$, hence $P(k)$, by iduction appleid to $P(j)$. If one notices that $\max\,(f,\ g)=(f+g+|\,f-g\,|)/2$ in general, one sees that, by letting f be $\max\,[F(i);\ i\leq j]$ and g be $F(j+1)$ (presuming that $j+1\leq k$), it suffices to claim $\max\,(f,\ g)\in\mathbf{A}$ when $f,\ g\in\mathbf{A}$. For that purpose it suffices to prove $|\,f\,|\in\mathbf{A}$ assuming $f\in\mathbf{A}$. Without loss of generality, we may assume $\|f\|\leq 1$ since $\mathbf{A}\subset B(X)$.

By virtue of the property of π and the premises, we have that $\pi(n,\ f)\in\mathbf{A}$ and $\|\pi(n,\ f)-|\,f\,|\|\leq 4/n$ (by 1) and 3) above). This implies $|\,f\,|\in\text{unfcl}\,(\mathbf{A},\ \{n,\ x\}\pi(n,\ f(x)))\subset\mathbf{A}$.

DEFINITION 11.4. spr $(\mathbf{A},\ \phi)$: $\forall k\ \forall l(k\neq l\vdash$

$$(\phi(k,\ l)\in\mathbf{A}\wedge 0\neq\phi(k,\ l,\ E(k))\neq\phi(k,\ l,\ E(l))\neq 0))$$

PROPOSITION 11.5 (The Stone-Weierstrass theorem). *There is a definable G^* such that*

$$\mathbf{A} \subset C(X),\ \text{alg}(\mathbf{A}),\ \text{unfclsd}(\mathbf{A}, \pi),$$

$$\text{spr}(\mathbf{A}, \phi) \rightarrow \text{unfcl}(\mathbf{A}, G^*) = C(X).$$

Proof. 1°. Under the premises, $\mathbf{A} \subset B(X)$, and hence ltc $(\mathbf{A})$ by Proposition 11.4.

2°. Define $F(f, k, l, z)$ to be $c\phi(k, l, z) + d \exp(\phi(k, l, z), 2)$, where c and d are the solutions of the equations below.

$$f(E(k)) = c\phi(k, l, E(k)) + d \exp(\phi(k, l, E(k)), 2),$$

$$f(E(l)) = c\phi(k, l, E(l)) + d \exp(\phi(k, l, E(l)), 2).$$

(We assume $E(k) \neq E(l)$.) Then $F(f, k, l) \in \mathbf{A}$, $f(E(k)) = F(f, k, l, E(k))$ and $f(E(l)) = F(f, k, l, E(l))$. Thus, the condition on F and f in Proposition 11.3 is trivially satisfied (without involving n), and hence $f \in C(X)$ implies unfcl $(\mathbf{A}, G^*, f)$ with the G^* there (for $\mathbf{A}$). The converse follows from Proposition 11.2.

References

[1] BISHOP, E.; *Foundations of Constructive Analysis*, McGraw-Hill Book Co., New York, 1967

[2] BRIDGES, D. S.; *Constructive Functional Analysis*, Pitman, London, 1979.

[3] KELLEY, J. L.; *General Topology*, Van Nostrand Co., New York, 1969.

[4] LOOMIS, L. H.; *An Introduction to Abstract Harmonic Analysis*, Van Nostrand Co., New York, 1953.

[5] ROYDEN, H. L.; *Real Analysis*, Collier-Macmillan Limited, London, 1968.

[6] TAKEUTI, G.; *Two Applications of Logic to Mathematics*, Iwanami Shoten and Princeton University Press, Tokyo, 1978.

[7] YASUGI, M.; Arithmetically definable analysis, *Proc. Res. Inst. Math. Sci.*, **180** (1973), 39–51.

[8] YASUGI, M.; The Hahn-Banach theorem and a restricted inductive definition, Lecture Notes in Math., No. 891 Springer-Verlag, 1981, pp. 359–394.

[9] YASUGI, M.; Definability problems in metric spaces; a summary, *Proc. Res. Inst. Math. Sci.*, **441** (1981), 66–82.

[10] YASUGI, M.; Definability problems in elementary topology, *J. Australian Math. Soc. Series A*, **34** (1982), 399–420.

[11] YASUGI, M.; Definability in the abstract theory of integration, *Comment. Math. Univ. Sancti Pauli*, **32** (1983), 137–161.

[12] YASUGI, M.; Definability in L^p-spaces, *Comment. Math. Univ. Sancti Pauli*, **32** (1983), 1–18.

Institute of Information Science
University of Tsukuba
Sakuramura, Ibaraki 305
Japan