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A Note on p-adic Integration

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In this note we construct a continuous function $\phi: \mathbf{Z}_p \to \mathbf{C}_p$ such that

$$\lim_{n\to\infty} p^{-n}(\phi(0)+\phi(1)+\cdots+\phi(p^n-1))$$

exists but $\sum d_n \frac{(-1)^n}{n+1}$, where d_n is the nth Mahler coefficient of ϕ, does not. This example provides a negative solution of the problem posed in [1], Proposition 3.

Construction. Let $f: \mathbf{Z}_p \to \mathbf{C}_p$ be continuous, such that f is not differentiable at 0 yet $\lim_{n\to\infty} p^{-n}(f(p^n)-f(0))$ exists. (For example, let $p>2$ let $\sigma: \{0, 1, 2, \cdots, p-1\} \to \{0, 1, 2, \cdots, p-1\}$ be defined by $\sigma(0):=0$, $\sigma(k):=1$ if $k\neq 0$ and set $f\left(\sum_{n=0}^{\infty} a_n p^n\right) := \sum_{n=0}^{\infty} \sigma(a_n)p^n$.) Let $f(x)=\sum_{n=0}^{\infty} b_n \binom{x}{n}$ $(x\in \mathbf{Z}_p)$ be the Mahler expansion of f. From

$$\frac{f(x)-f(0)}{x} = \sum_{n=1}^{\infty} b_n x^{-1}\binom{x}{n} = \sum_{n=1}^{\infty} \frac{b_n}{n}\binom{x-1}{n-1} \qquad (x\in \mathbf{Z}_p \setminus \{0\})$$

we infer that $\langle b_n/n\rangle$ is not a null sequence (otherwise the right hand side would represent a continuous function of $x\in \mathbf{Z}_p$ and f would be differentiable at 0). Now set

$$\phi(x) := f(x+1)-f(x) \qquad (x\in \mathbf{Z}_p)\,.$$

Then

$$\lim_{n\to\infty} p^{-n}\left(\sum_{k=0}^{p^n-1} \phi(k)\right) = \lim_{n\to\infty} p^{-n}(f(p^n)-f(0))$$

exists. Let d_n $(n\in\{0, 1, 2, \cdots\})$ be the Mahler coefficients of ϕ. Then $d_n = b_{n+1}$ for all n so that $\langle d_n/(n+1)\rangle = \langle b_{n+1}/(n+1)\rangle$ is not a null sequence and $\sum d_n \frac{(-1)^n}{n+1}$ diverges.

Reference

[1] ENDO, M.; An extension of p-adic integration, *Comment. Math. Univ. St. Pauli*, **32** (1983), 109–130.

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