COMMENTARII MATHEMATICI
UNIVERSITATIS SANCTI PAULI
Vol. 34, No. 2, 1985

ed. RIKKYO UNIV MATH
IKEBUKURO TOKYO
171 JAPAN

Generalized Direct Summands in an Abelian Category

by

T. H. FAY* and M. J. SCHOEMAN

(Received November 8, 1984)

Abstract

An abelian group A is said to be quasi-splitting if there exists an integer n such that $nA \leqslant \tau A \oplus C \leqslant A$ where τA denotes the torsion subgroup of A. In this paper we generalize this notion in two ways: first, we move the setting to an arbitrary abelian category by replacing nA by the subobject RA where R is a certain type of right exact radical which reduces to multiplication by n in the category of abelian groups, and second, we replace τA by a subobject B of A and call B a generalized direct summand of A provided $RA \leqslant B \oplus C \leqslant A$. A surprizing number of results in the classical quasi-splitting case are moved into this general setting. We show connections between $RA \leqslant B \oplus C \leqslant A$ and $0 \to B \to A \to A/B \to 0$ determining elements in kernels of certain naturally induced endomorphisms of $\mathrm{Ext}(A/B, B)$. Relationships between endomorphisms of A and of B and of A/B are explored and finally the question of when a generalized direct summand is a direct summand is considered.

Introduction

An abelian group A is said to be quasi-splitting if there exists an integer n such that $nA \leqslant \tau A \oplus C \leqslant A$ where τA is the torsion subgroup of A. This notion was introduced by C. P. Walker [8] and a nice exposition of results associated with this notion can be found in Chapter XIV of Volume II of L. Fuchs' *Infinite Abelian Groups* [2]. One is interested in when or under what conditions a group A is quasi-splitting, and when does quasi-splitting imply splitting. In this paper we generalize this notion of quasi-splitting in two ways: first, we move the setting to an arbitrary abelian category and replace the subgroup nA with a subobject RA where R is a right exact radical having a pointwise epic natural transformation from the identity functor to R. In the category of abelian groups such radicals are precisely the multiplications by integers n. Secondly, we replace the torsion subgroup τA by a subobject B and we say that B is a generalized direct summand of the object A provided $RA \leqslant B \oplus C \leqslant A$.

* The author wishes to express his appreciation to the University of Pretoria, and in particular to the Department of Mathematics, for providing a visiting appointment during which this work was initiated.
AMS(MOS): 20K40 18E40
Key Words: Generalized direct summand, direct summand, abelian group.

In [1] the authors studied a similar generalization of quasi-splitting. In that work, τA was replaced by SA where S is a preradical that is compatible with R in the sense that for each subobject B of A, RB contained in SA implies B was already contained in SA. Clearly the torsion subgroup functor is compatible with any n. The results in [1] showed that there is a very general and rich theory for which quasi-splitting of the torsion subgroup is just a special case.

In this work, the notion of generalized direct summand is more general than the notion developed in [1] and the results herein illuminate results in [1] as well as for the "classical" case of quasi-splitting. For example, we show the relationship between $RA \leqslant B \oplus C \leqslant A$ and certain kernels of naturally induced endomorphisms of $\mathrm{Ext}(A/B, B)$ which, when specialized to the category of abelian groups, show the relationship between $nA \leqslant B \oplus C \leqslant A$ and $0 \to B \to A \to A/B \to 0$ determining an element of finite order in $\mathrm{Ext}(A/B, B)$, which subsumes the classical result for the quasi-splitting of a group A. We show there is a connection between generalized direct summands and endomorphisms which are almost idempotent. We study the connections between endomorphisms of A which determine a generalized direct summand B, and endomorphisms of B and of A/B. Finally, we obtain some results on when a generalized direct summand is in fact a direct summand, these results, again, generalize the classical situation in the category of abelian groups.

Preliminaries

Throughout this paper, $\mathbf{A}$ shall be an abelian category and R shall denote a preradical (=subfunctor of the identity) having a pointwise epic natural transformation $\eta: \mathbf{1_A} \to R$. The natural inclusion transformation shall be denoted by $\mu: R \to \mathbf{1_A}$ and the composite transformation $\mu\eta: \mathbf{1_A} \to \mathbf{1_A}$ shall be denoted by ρ. The kernel of η induces a natural trnasformation and preradical $\kappa: K \to \mathbf{1_A}$, and so for each object A of $\mathbf{A}$ we have a short exact sequence:

$$0 \longrightarrow KA \overset{\kappa_A}{\rightarrowtail} A \overset{\eta_A}{\twoheadrightarrow} RA \longrightarrow 0$$

Here we have used $\rightarrowtail$ and $\twoheadrightarrow$ to denote κ_A is a monic and η_A is an epic for emphasis.

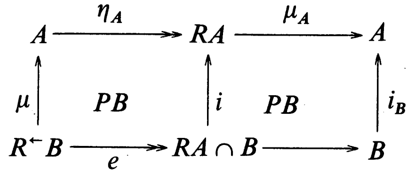
If $\mathbf{A}$ is the category of modules over a commutative ring with identity, then any such preradical R above is multiplication by some ring element; hence in the category of abelian groups, $R=n$ for some integer n, and, of course, $K=[n]$, the n-socle. The following result from [1] gives some further information.

PROPOSITION 1. *Given R and K as above, R is a right exact (preserves epimorphisms) radical and K is a left exact socle.*

Recall that a preradical S is a socle provided for all objects A, $S^2A = SA$ and is a radical provided $S(A/SA) = 0$ for all objects A.

If A is an object of $\mathbf{A}$ and B is a subobject of A, then we denote the "inclusion" morphism by $i_B: B \to A$ and sometimes, when i_B is understood, we write $B \leqslant A$. If $B \leqslant A$ and $C \leqslant A$, then we write $C \leqslant B$ to mean i_C factors through i_B; that is, there exists a morphism $j\colon C \to B$ such that $i_B j = i_C$. If $B \leqslant C$ and $C \leqslant B$, then C and B are isomorphic as subobjects of A and we write $C \equiv B$ to denote this. Of course if we are in a module category where we identify subobjects with their image, "$\equiv$" becomes "$=$". The terms morphism and map shall be used interchangeably.

For an object A and subobject B, we define the subobject $R^{\leftarrow}B$ of A by the following pullback diagram:



In the category of abelian groups, where $R = n$, we have $n^{\leftarrow}B = \{a \in A : na \in B\}$.

It is clear that, for any subobject B of A, $\eta_A i_b = Ri_B \eta_B$ since η is a natural transformation. This implies that $B \leqslant R^{\leftarrow}B$ as $R^{\leftarrow}B$ is the pullback of i and η_A and Ri_B factors through i. On the other hand, it is also clear that $RR^{\leftarrow}B \leqslant B$.

In [1], the authors studied preradicals S in A having the property that for all objects A and subobjects B of A, the inclusion $RB \leqslant SA$ implies the inclusion $B \leqslant SA$. We call such preradicals *compatible* with R and one may use them to generate examples in this work.

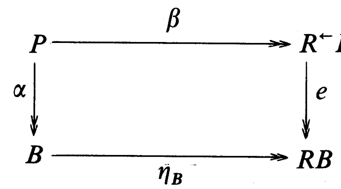
PROPOSITION 2. *If S is a preradical compatible with R, then $R^{\leftarrow}SA \equiv SA$ for every object A.*

Proof. We already have seen that $SA \leqslant R^{\leftarrow}SA$ and that $RR^{\leftarrow}SA \leqslant SA$. This latter inclusion, by compatibility, implies $R^{\leftarrow}SA \leqslant SA$ and so $R^{\leftarrow}SA \equiv SA$.

In the category of abelian group, with $R = n$ (n a non-zero integer), it is clear that the torsion subgroup functor τ is compatible with n and so $n^{\leftarrow}\tau A = \tau A$ for every group A. If A is a mixed group and B is a pure subgroup of A containing τA, then $n^{\leftarrow}B = B$. This example is generalized in the following proposition.

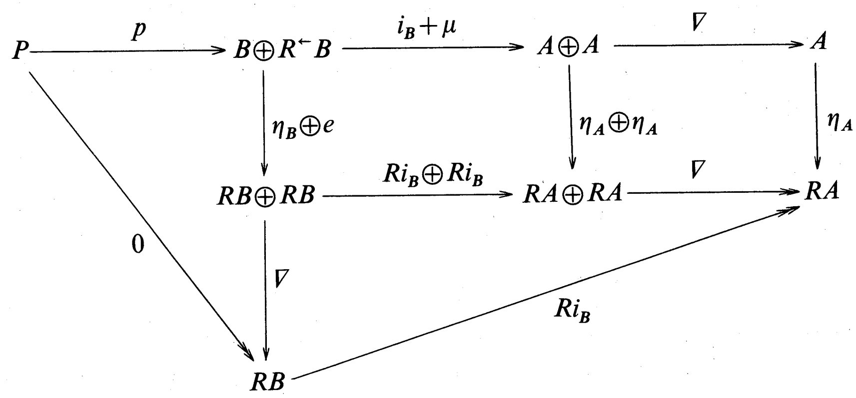
PROPOSITION 3. *Let B be a subobject of an object A having the property that $KA \leqslant B$ and that B is R-pure in A, that is, $RB \equiv B \cap RA$. Then $R^{\leftarrow}B \equiv B$.*

Proof. We have already seen that, for any subobject B of A, $B \leqslant R^{\leftarrow}B$. To see the reverse inclusion, we form the pullback diagram:

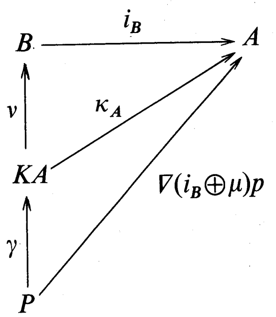


Here, because of the R-purity of B, $RB \equiv RA \cap B$ and so, without loss of generality, we may take $e\colon R^{\leftarrow}B \twoheadrightarrow RB$. Define $p = \{-\alpha, \beta\}\colon P \to B \oplus R^{\leftarrow}B$ by $\pi_1 p = -\alpha$ and

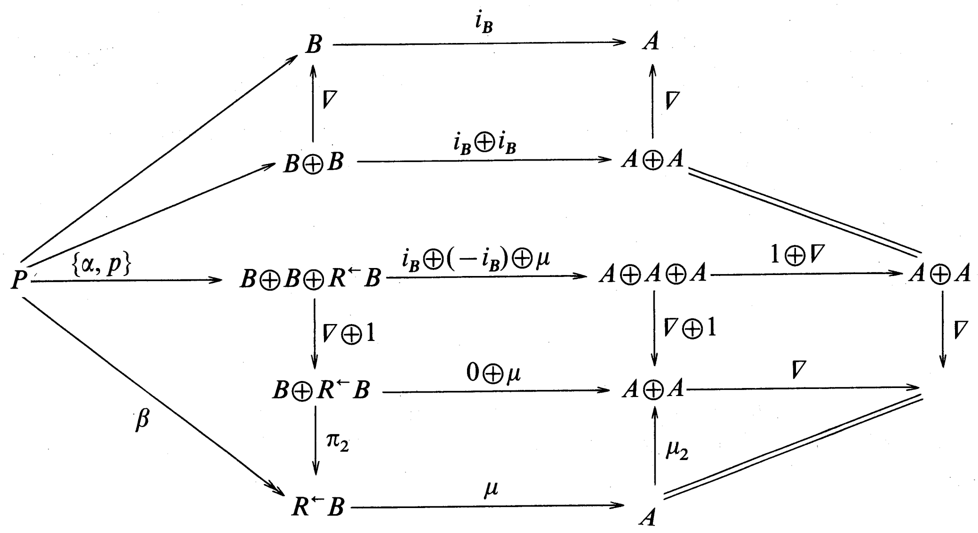
$\pi_2 p = \beta$ where π_1 and π_2 are the indicated projection maps. We have the commutative diagram:



It follows that $\eta_A \nabla(i_B \oplus \mu)p = 0$, so there exists a unique map $\gamma\colon P \to KA$ such that $\gamma_A \gamma = \nabla(i_B \oplus \mu)p$, and so we have the commutative diagram:



Putting all of this together, we have the commutative diagram:



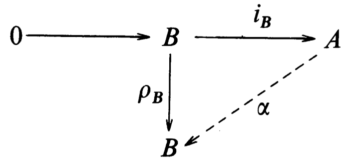
Here μ_2 denotes the unique map satisfying $\pi_1\mu_2=0$ and $\pi_2\mu_2=1_A$. The commutivity of this diagram shows that $R^{\leftarrow}B\leqslant B$ since μ is monic and β is epic and $\mu\beta$ factors through i_B.

Generalized Direct Summands. In this section we study a generalization of the notion of quasi-splitting in abelian groups (which was introduced by C. P. Walker [8], see also Section 102 of Volume II of Fuchs [2]) and which in turn generalizes and illuminates some of the results of [3].

For an object A and subobject B of A, we call B a *generalized direct summand* of A (with respect to R) provided $RA\leqslant B\oplus C\leqslant A$ for some subobject C of A.

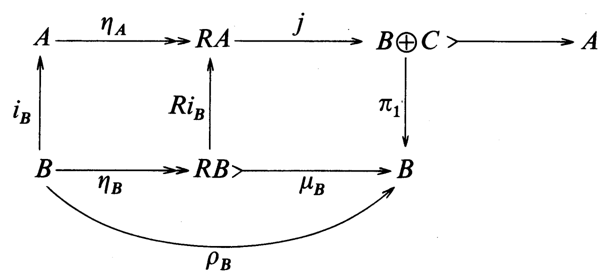
In the category of abelian groups, with $R=n$, the torsion subgroup τA of a group A is a generalized direct summand if and only if A quasi-splits; that is, $nA\leqslant\tau A\oplus C\leqslant A$. This happens if and only if $0\to\tau A\to A\to A/\tau A\to 0$ represents an element of $\mathrm{Ext}(A/\tau A, \tau A)[n]$ (see Theorem 102.2 of Fuchs [2]). The next two results generalize this theorem.

THEOREM 4. *Let A be an object and B a subobject of A. If B is a generalized direct summand of A, then there exists a map α: $A\to B$ such that $\alpha i_B=\rho_B$.*



Conversely, if such a morphism exists and $KB=0$, then $RA\leqslant B\oplus C\leqslant A$ where $C=$ ker α.

Proof. If $RA\leqslant B\oplus C\leqslant A$, let $\pi_1: B\oplus C\to B$ be the projection map and let $\alpha=\pi_1 j\eta_A$ where j: $RA\rightarrowtail B\oplus C$. Then $\alpha i_B=\rho_B$, as can be seen from the following commutative diagram



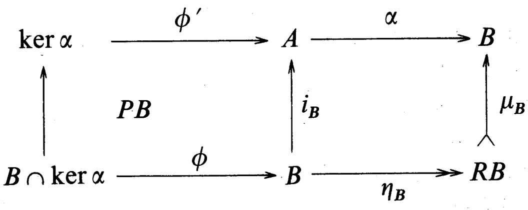
Conversely, if such an α exists, then define $f=\rho_A-i_B\alpha$. Observe that

$$f i_B=\rho_A i_B-i_B\alpha i_B=\rho_A i_B-i_B\rho_B=0\,.$$

Since $\rho_A=f+i_B\alpha$, we have

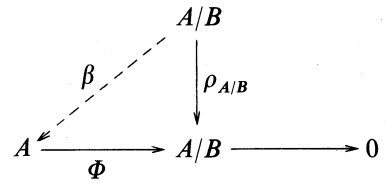
$$RA=\mathrm{Im}(\rho_A)=\mathrm{Im}(f+i_B\alpha)\leqslant \mathrm{Im}\, f+\mathrm{Im}\,\alpha\,.$$

But $\alpha f=\alpha\rho_A-\alpha i_B\alpha=\alpha\rho_A-\rho_B\alpha=0$, so $\mathrm{Im} f\leqslant\ker\alpha$. Thus we have $RA\leqslant\ker\alpha+\mathrm{Im}\,\alpha\leqslant A$. We have $\mathrm{Im}\,\alpha\leqslant B$, so it remains only to see that $\ker\alpha\cap B=0$. To that end, consider the following commutative diagram:



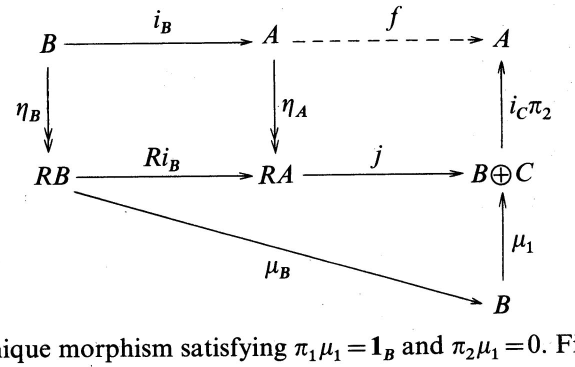
Since $\alpha\phi'=0$, it follows that $\rho_B\phi=\mu_B\eta_B\phi=0$. Thus there exists a unique morphism $\lambda\colon B\cap\ker\alpha\to KB$ such that $\kappa_B\lambda=\phi$. From the hypothesis of $KB=0$ and λ being necessarily monic, it follows that $B\cap\ker\alpha=0$.

THEOREM 5. *Let A be an object and B a subobject of A. If B is a generalized direct summand of A, then there exists a map $\beta\colon A/B\to B$ such that $\Phi\beta=\rho_{A/B}$.*



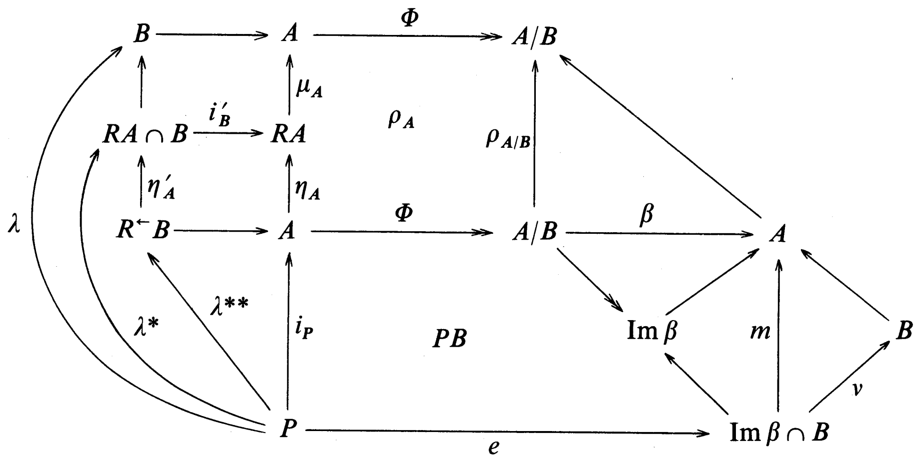
Conversely, if such a β exists and if $R^{\leftarrow}B\equiv B$, then $RA\leqslant B\oplus C\leqslant A$ where $C=\mathrm{Im}\,\beta$.

Proof. If $RA\leqslant B\oplus C\leqslant A$ and $\pi_2\colon B\oplus C\to C$ denotes the indicated projection map, then define $f\colon A\to A$ by $f=i_C\pi_2 j\eta_A$ as indicated in the following commutative diagram:



Here μ_1 is that unique morphism satisfying $\pi_1\mu_1=\mathbf{1}_B$ and $\pi_2\mu_1=0$. First observe that $\pi_2 jRi_B=\pi_2\mu_1\mu_B=0$; thus computing, $fi_B=i_C\pi_2 jRi_B\eta_B=0$. Consequently, there exists a unique map $\beta\colon A/B\to A$ such that $\beta\Phi=f$ (here Φ is the cokernel of i_B). Secondly, it is easy to see that $f+i_B\pi_1 j\eta_A=\rho_A$ and $\Phi f=\Phi(f+i_B\pi_1 j\eta_A)=\Phi\rho_A$. Thus computing, we have: $\Phi\beta\Phi=\Phi f=\Phi\rho_A=\rho_{A/B}\Phi$, and Φ being epic implies $\Phi\beta=\rho_{A/B}$ as was to be shown.

Conversely, if such a β exists, then $\Phi\beta=\rho_{A/B}$ implies $\mathrm{Im}(\Phi\rho)=\mathrm{Im}(\rho_{A/B})$, hence $R(A/B)=(\mathrm{Im}\,\beta+B)/B$, which in turn implies $RA\leqslant\mathrm{Im}\,\beta+B$. We have the following commutative diagram:



Computing we have: $\Phi\rho_A i_P=\rho_{A/B}\Phi i_P=\Phi\beta\Phi i_P=\Phi me=\Phi i_B\nu=0$. Thus there exists a unique map $\lambda: P\to B$ such that $\rho_A i_P=i_B\lambda$. But this implies that there exists a unique map $\lambda^*: P\to RA\cap B$ such that $i'_B\lambda^*=\eta_A i_P$ and $\mu'_A\lambda^*=\lambda$. This in turn implies that there exists a unique map $\lambda^{**}: P\to R^{\leftarrow}B$ such that $\mu\lambda^{**}=i_P$ and $\eta'_A\lambda^{**}=\lambda^*$. Thus we have $P\leqslant R^{\leftarrow}B$.

The hypothesis $R^{\leftarrow}B\equiv B$ implies $P\leqslant B$ and so $\Phi i_P=0$. Hence $\beta\Phi i_P=0$ and we have $\mathrm{Im}(\beta\Phi i_P)=\mathrm{Im}(me)=\mathrm{Im}\,\beta\cap B=0$.

COROLLARY 6. *Let A be an object and B be a subobject of A. If B is a generalized direct summand of A, then the short exact sequence $0\to B\to A\to A/B\to 0$ determines an element of the kernel of ρ_{B*}: $\mathrm{Ext}(A/B, B)\to\mathrm{Ext}(A/B, B)$ and of the kernel of $\rho^*_{A/B}$: $\mathrm{Ext}(A/B, B)\to\mathrm{Ext}(A/B, B)$.*

If $KB=0$ and $0\to B\to A\to A/B\to 0$ determines an element of the kernel of ρ_{B*} or if $R^{\leftarrow}B=B$ and $0\to B\to A\to A/B\to 0$ determines an element of the kernel of $\rho^*_{A/B}$, then B is a generalized direct summand of A.

To conclude this section we give an analogue of the connection between direct summands and indepotent endomorphisms.

PROPOSITION 7. *Let A be an object and B be a subobject of A, If B is a generalized direct summand of A, then there exist endomorphisms ϕ, ψ : $A\to A$ such that $\phi^2=\rho_A\phi$ and $\psi^2=\psi\rho_A$.*

Conversely, if $\alpha: A\twoheadrightarrow B$ is epic and $\phi=i_B\alpha: A\to A$ satisfies $\phi^2=\rho_A\phi$ and if either $KB=0$ or $R^{\leftarrow}B=B$, then B is a generalized direct summand of A. Dually, if $\beta: A/B\to A$ is monic, $\psi=\beta\Phi$ satisfies $\psi^2=\psi\rho_A$, and either $KB=0$ or $R^{\leftarrow}B=B$, then B is a generalized direct summand of A.

Proof. If B is a generalized direct summand, then by Theorem 4 there exists a morphism $\alpha: A \to B$ such that $\alpha i_B = \beta_B$. Define $\phi = i_B\alpha$ and compute: $\phi^2 = i_B\alpha i_B\alpha = i_B\rho_B\alpha = \rho_A i_B\alpha = \rho_A\phi$. From Theorem 5 there exists a morphism $\beta: A/B \to A$ such that $\Phi\beta = \rho_{A/B}$. Define $\psi = \beta\Phi: A \to A$ and compute:

$$\psi^2 = \beta\Phi\beta\Phi = \beta\rho_{A/B}\Phi = \beta\Phi\rho_A = \psi\rho_A .$$

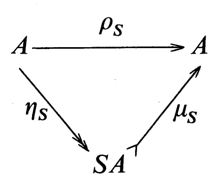
Conversely, computing: $\phi^2 = i_B\alpha i_B\alpha = \rho_A i_B\alpha = i_B\rho_B\alpha$; α epic and i_B monic imply $\alpha i_B = \rho_B$. If $KB = 0$, then by Theorem 4, B is a generalized direct summand. On the other hand, note that $(\rho_A - \phi)i_B = \rho_A i_B - i_B\rho_B = 0$, hence there exists a unique morphism $\delta: A/B \to A$ such that $\delta\Phi = \rho_A - \phi$. Computing again: $\Phi\delta\Phi = \Phi\rho_A - \Phi\phi = \rho_{A/B}\Phi - \Phi i_B\alpha = \rho_{A/B}\Phi$, and Φ being epic implies $\Phi\delta = \rho_{A/B}$. Thus if $R^{\leftarrow}B = B$, then by Theorem 5, B is a generalized direct summand of A.

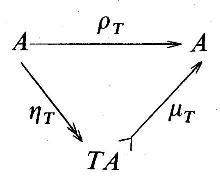
If $\beta: A/B \to B$ is monic and $\psi = \beta\Phi$ satisfies $\psi^2 = \psi\rho_A$, then Φ epic and β monic imply that $\Phi\beta = \rho_{A/B}$ for, computing, we have:

$$\psi^2 = \beta\Phi\beta\Phi = \beta\Phi\rho_A = \beta\rho_{A/B}\Phi .$$

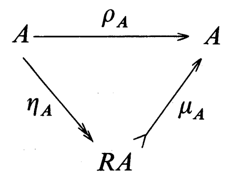
Thus if $R^{\leftarrow}B = B$, then by Theorem 5, B is a generalized direct summand of A. On the other hand, define $\phi = \rho_A - \beta\phi$. Then computing $\Phi\phi = \Phi\rho_A - \Phi\beta\Phi = \Phi\rho_A - \rho_{A/B}\Phi = 0$, so ϕ factors through i_B. That is, there exists a unique morphism $\gamma: A \to B$ such that $i_B\gamma = \phi$. Thus, if $KB = 0$, then by Theorem 4, B is a generalized direct summand.

Relations between Endomorphism Rings. By way of motivation we first discuss the situation for abelian groups. Let B be a subgroup of a group A and suppose that there exists an endomorphism α of A such that $\alpha b = nb$ for all $b \in B$, and for every $a \in A$, $\alpha a - ma \in B$ where m and n are distinct integers. Then we can describe this situation by saying that $\alpha \in E(A)$, the endomorphism ring of A, induces the pair $(\bar{n}, \bar{m})$ in $E(B) \times E(A/B)$ where $\bar{n}$ and $\bar{m}$ denote the endomorphisms of B and A/B defined by multiplication by n and by m respectively. That there is a connection with generalized direct summands and this situation is given by a result of [6, 7] which states that given such an α and n and m, if either $B[n-m] = 0$ or $B = (n-m)^{\leftarrow}B$, then B is a quasi-direct summand. Other results along this line for abelian groups and quasi-splitting of the torsion subgroups are to be found in [3]. Throughout this section we shall fix the following notation. We have preradicals S and T and natural transformations such that for each object A we have the commutative diagrams:





(We assume that $S \neq T$). We define R by taking the epi-mono factorization of $\rho = \rho_S - \rho_T$, so that for each object A we have the commutative diagram:



Thus S, T and R are right exact radicals each having a pointwise epic natural transformation from the identity to themselves. As before, K denotes the kernel functor defined by ρ.

THEOREM 8. *Let A be an object and B a subobject of A. Let $\alpha: A\to A$ be an endomorphism of A satisfying $\alpha i_B=\rho_B i_B$ and $\alpha-\rho_T$ factors through i_B. If B has $KB=0$ or $R^{\leftarrow}B\equiv B$, then B is a generalized direct summand of A.*

Proof. Define $\beta: A\to A$ by $\beta=\alpha-\rho_T$. Then computing,

$$\beta i_B=\alpha i_B-\rho_T i_B=(\rho_S-\rho_T)i_B=\rho_A i_B=i_B\rho_B\,.$$

The hypothesis that $\alpha-\rho_T$ factors through i_B implies there exists a morphism $\bar{\beta}: A\to B$ such that $i_B\bar{\beta}=\beta$. Thus we have $i_B\rho_B=\beta i_B=i_B\bar{\beta}i_B$ and i_B monic implies $\bar{\beta}i_B=\rho_B$. Thus if $KB=0$, then by Theorem 4, B is a generalized direct summand.

On the other hand, define $\gamma: A/B\to A$ to be that unique morphism induced from $\alpha i_B=\rho_S i_B$, and hence we have $\gamma\Phi=\rho_S-\alpha$. Comupting again: $\Phi\gamma\Phi=\Phi(\rho_S-\alpha)$. But since $\alpha-\rho_T$ factors through i_B, it follows that $\Phi(\alpha-\rho_T)=0$ or, in other words, $\Phi\alpha=\Phi\rho_T$. Thus $\Phi\gamma\Phi=\Phi(\rho_S-\rho_T)=(\rho_S-\rho_T)\Phi$ since $\rho_S-\rho_T$ is a natural transformation. It follows from Φ being epic and $\rho_A=\rho_S-\rho_T$, that $\gamma\Phi=\rho_A$. Thus if $R^{\leftarrow}B=B$, then by Theorem 5, B is a generalized direct summand of A.

Thus the existence of an endomorphism $\alpha: A\to A$ satisfying $\alpha i_B=\rho_S i_B$ and $\alpha-\rho_T=i_B\bar{\beta}$ for some $\bar{\beta}: A\to B$, and B having either $KB=0$ or $R^{\leftarrow}B=B$ implies $RB\leqslant B\oplus C\leqslant A$. In the abelian group case, of couse, S is multiplication by n, T is multiplication by m and R is multiplication by $n-m$. We have a converse to Theorem 8 as well.

THEOREM 9. *If A is an object, B a subobject of A, and $RA\leqslant B\oplus C\leqslant A$, then there is an endomorphism $\alpha: A\to A$ of A such that $\alpha i_B=\rho_S i_B$ and $\alpha-\rho_T$ factors through i_B.*

Proof. From Theorems 4 and 5, we have morphisms $\phi: A\to B$ and $\psi: A/B\to A$ such that $\phi i_B=\rho_B$ and $\Phi\psi=\rho_{A/B}$. Define $\alpha=i_B\phi+i_B\phi\psi\Phi+\rho_T$. Then clearly $\alpha-\rho_T$ factors through i_B computing:

$$\alpha i_B=i_B\rho_B+0+\rho_T i_B=\rho_A i_B+\rho_T i_B=(\rho_S-\rho_T)i_B+\rho_T i_B=\rho_S i_B\,.$$

When a Generalized Direct Summand is a Summand. The Problem of when the torsion subgroup of an abelian group is a direct summand is an important question and has been considered by many authors (see Chapter XIV of Fuchs [2]). One useful results is that if A is a mixed group and nA is splitting (for some integer $n \neq 0$), then A is splitting (Proposition 100.2, Fuchs [2]). In this section we see how this result is a consequence of the general theory being developed herein and of a special property enjoyed by the category of abelian groups. We will give results which will tell when a generalized direct summand is a direct summand.

If B is a subgroup of the abelian group A, then a subgroup C of A is called *B-high* if C is maximal with respect to $C \cap B = 0$. The existence of B-high subgroups is guaranteed by Zorn's lemma. An important result for our work is the following (Lemma 9.8, Fuchs [2]): if B is a subgroup of A, C is B-high in A, then for $a \in A$, $pa \in C$ implies $a \in B \oplus C \leqslant A$ (p is a prime).

We can extend some of the known results for abelian groups if we assume that our category satisfies the following axiom.

Axiom $*$: We say that the category **A** enjoys Axiom $*$ provided for every subobject B of an object A, if C is a B-high subobject of A, then $R^{\leftarrow}C \leqslant B \oplus C \leqslant A$.

Although, in the work that follows we shall only use the containment $R^{\leftarrow}C \leqslant B \oplus C \leqslant A$, it is of some interest to note that there are two other equivalent statements. We list these in the following lemma; the proof is straightforward "categorics" and is omitted.

LEMMA 10. *Let A be an object in an abelian category* **A**, *B a subobject of A, and C a B-high subobject. Then the following are equivalent*:

(1) $R^{\leftarrow}C \leqslant B \oplus C \leqslant A$

(2) $K(A/B) = R^{\leftarrow}C/C \leqslant B \oplus C/C \leqslant A/C$

(3) *there exists a map ϕ making the following triangle commute.*

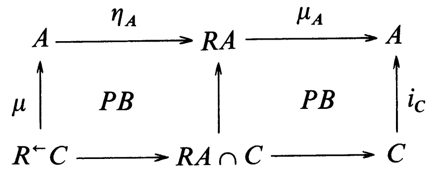
$$\begin{array}{ccc} A/C & \xrightarrow{\rho_{A/C}} & R(A/C) \\ & \searrow & \downarrow \phi \\ & & A/B+C \end{array}$$

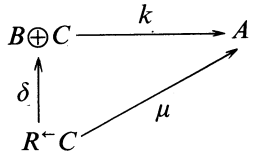
THEOREM 11. *If* **A** *is the category of modules over a Principal Ideal Domain, then* **A** *enjoys Axiom* $*$.

THEOREM 12 (F. Minnaar). *If R is a ring with identity and enjoys the property that every prime ideal is maximal, then the category of left R-modules enjoys Axiom* $*$.

This result, proved in a similar manner as for Lemma 9.8 of [2], is particularly nice as it shows that the category of left modules over any Boolean ring or over any Dedekind domain enjoys the axiom.

We shall fix the following notation:

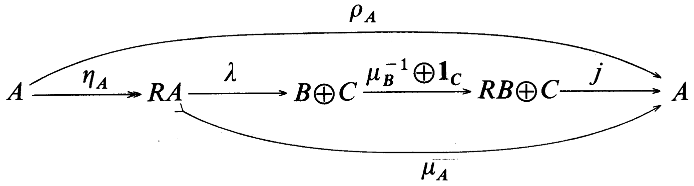




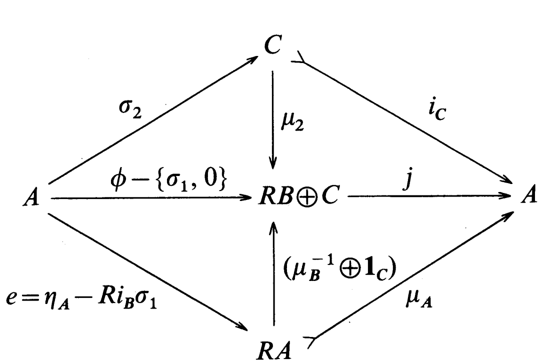
The following is a generalization of Proposition 100.2 of Fuchs [2].

THEOREM 13. *If* **A** *enjoy Axiom* $*$ *and* A *is an object with subobject* B *having* $\mu_B\colon RB \to B$ *an isomorphism, and if* $RA = B \oplus G$, *then* $A = B \oplus C$ *where* C *is* B*-high and* $G \leqslant C$.

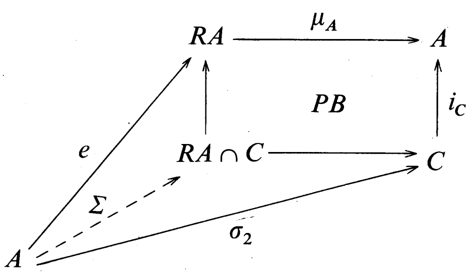
Proof. Choose C to be a B-high subobject of A with $G \leqslant C$. We have a commutative diagram:



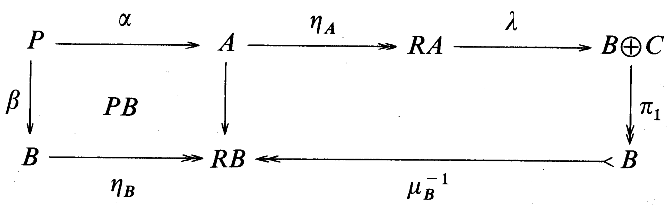
Let $\phi = (\mu_B^{-1} \oplus \mathbf{1}_C)\lambda\eta_A$ and observe that $\phi = \{\rho_1, \sigma_2\}$ where $\sigma_1 = \pi_1\phi$ and $\sigma_2 = \pi_2\phi$, π_1 and π_2 being the projections of $RB \oplus C$. Let $\mu_2\colon C \to RB \oplus C$ be the canonical injection and note that $\phi - \{\sigma_1, 0\} = \{0, \sigma_2\} = \mu_2\sigma_2$. We have the commutative diagram:



Consequently we have a unique map $\Sigma\colon A \to RA \cap C$ such that the following diagram commutes.



Next, let $\{P, \alpha, \beta\}$ be the pullback of η_B and $\mu_B^{-1}\pi_1\lambda\eta_A$ as shown in the next commutative diagram.



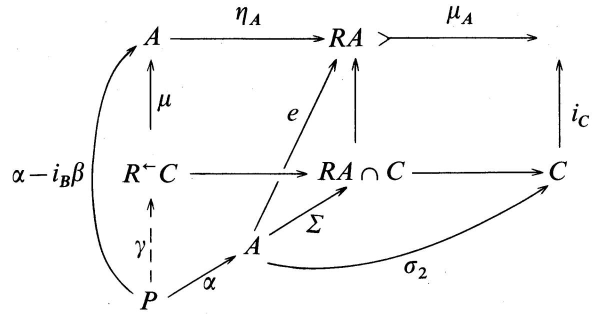
Note that α is epic.

Computing we have:

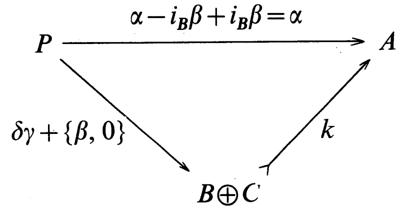
$$\rho_A(\alpha - i_B\beta) = j\phi\alpha - \mu_A R i_B \eta_B \beta = j\phi\alpha - \mu_A R i_B \pi_1(\mu_B^{-1} \oplus \mathbf{1}_C)\lambda\eta_A\alpha$$

$$= j\phi\alpha - \mu_A R i_B \sigma_1 \alpha = j\phi\alpha - i_B \mu_B \sigma_1 \alpha = j\phi\alpha - j\mu_1\sigma_1\alpha = i_C\sigma_2\alpha\ .$$

Thus it follows that we have a unique map $\gamma: P \to R^{\leftarrow} C$ so that the following diagram commutes:



A is a consequence of Axiom $*$, we have $k\delta\gamma = \mu\gamma = \alpha - i_B\beta$. We also have that $k\{\beta, 0\} = i_B\beta$, so that the following diagram commutes.



But recall that α is an epic and so k is epic and thus is an isomorphism.

An analysis of the previous proof shows that we obtained the following result as well.

THEOREM 14. *If the category* **A** *enjoys Axiom* $*$ *and* A *is an object with subobject* B *satisfying* $RB \equiv B$ *and* B *is a generalized direct summand of* A $(RA \leqslant B \oplus C \leqslant A)$, *then* B *is a direct summand of* A.

The next proposition does not require Axiom $*$.

PROPOSITION 15. *If* A *is an object,* B *a subobject of* A *and* $R(A/B) \equiv A/B$, *then* B *a generalized direct summand implies* B *is a direct summand.*

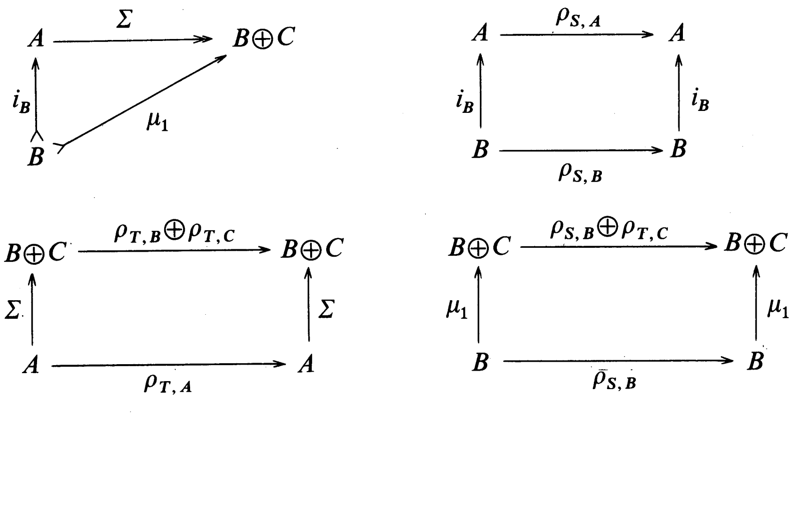
Proof. Recall that R being right exact means that R preserves epimorphisms and thus $R(A/B) \equiv B + RA/B \equiv A/B$. Consequently, we have

$$A \equiv B + RA \leqslant B + B \oplus C \equiv B \oplus C\,.$$

Now putting all of this together we obtain the result:

THEOREM 16. *If* A *enjoys Axiom* $*$, A *is an object with subobject* B, $\rho = \rho_S - \rho_T$ *as in the previous section and either* $KB = 0$ *or* $R^{\leftarrow}B \equiv B$, *and either* $RB \equiv B$ *or* $R(A/B) \equiv A/B$, *then* B *is a direct summand if and only if there exists an endomorphism* $\alpha\colon A \to A$ *such that* $\alpha i_B = \rho_B i_B$ *and* $\alpha - \rho_T$ *factors through* i_B.

Proof. Suppose B is a direct summand. Then we have an isomorphism $\Sigma\colon A \to B \oplus C$. We have the following commutative diagrams:



Define $\alpha = \Sigma^{-1}(\rho_{S,B} \oplus \rho_{T,C})\Sigma : A \to A$. Then computing:

$$\alpha i_B = \Sigma^{-1}(\rho_{S,B} \oplus \rho_{T,C})\mu_1 = \Sigma^{-1}\mu_1 \rho_{S,B} = i_B \rho_{S,B} = \rho_{S,A} i_B .$$

Furthermore, computing again:

$$\begin{aligned} \alpha - \rho_{T,A} &= \Sigma^{-1}(\rho_{S,B} \oplus \rho_{T,C})\Sigma - \Sigma^{-1}(\rho_{T,B} \oplus \rho_{T,C})\Sigma \\ &= \Sigma^{-1}((\rho_{S,B} - \rho_{T,B}) \oplus 0)\Sigma = \Sigma^{-1}(\rho_B \oplus 0)\Sigma \\ &= \Sigma^{-1}\mu_1 \rho_B \pi_1 \Sigma = i_B \rho_B \pi_1 \Sigma . \end{aligned}$$

Thus $\alpha - \rho_{T,A}$ factors through i_B.

Conversely, if such an α exists and if $RB \equiv B$, then Theorem 14 implies $A \equiv B \oplus C$. If $R(A/B) \equiv A/B$, then Proposition 15 implies $A \equiv B \oplus C$.

To put this last result in perspective we cite the following corollary valid in the category of abelian groups:

COROLLARY 17 (Mader [5]). *If A is a torsion group, B a subgroup of A with either $A[p]=0$ or $A/B[p]=0$, p a prime, then B is a direct summand of A if and only if there exists an endomorphism $\alpha: A \to A$ such that $\alpha b = nb$ for all $b \in B$, $\alpha a - ma \in B$ for all $a \in A$ where n and m are integers with $n - m = p$.*

Proof. Recall that a torsion group having no elements of prime order p is necessarily divisible by p.

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Department of Mathematics
University of Southern Mississippi
Hattiesburg, MS 39406
U.S.A.

Department Wiskunde
Universiteit van Pretoria
0002 Pretoria
R.S.A.