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Generalizations of Boolean Rings, Boolean-like Rings and von Neumann Regular Rings

by

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1. Introduction

Throughout this paper all rings will be commutative with identity. A ring R is called von Neumann regular if for each $x\in R$ there exists an $a\in R$ with $xax=x$. Such rings (not necessarily commutative or with identity) were introduced by von Neumann in his coordinization theorems for complemented modular lattices (see [9] for details). von Neumann regular rings are important in multiplicative ideal theory since they are exactly the rings that are locally fields, i.e., R is von Neumann regular if and only if R_M is a field for each maximal ideal M of R. Equivalently, R is von Neumann regular if and only if R is zero dimensional and reduced or if and only if every element of R is the product of an idempotent and unit. From a homological point of view R is von Neumann regular if and only if every R-module is flat (hence the Bourbaki terminology absolutely flat). We define a ring R to be n-von Neumann regular if given $x_1, \cdots, x_n\in R$, there exist $a_1, \cdots, a_n\in R$ with $(x_1a_1x_1-x_1)\cdots(x_na_nx_n-x_n)=0$. Thus 1-von Neumann regular is just von Neumann regular. We show that R is n-von Neumann regular if and only if $\dim R=0$ and $\mathrm{nil}(R)^n=0$ (where $\dim R$ is the Krull dimension of R and $\mathrm{nil}(R)$ is the nilradical of R). We also show that $\dim R=0$ if and only if for each $x\in R$, there exists and $a\in R$ and a natural number n such that $(xax-x)^n=0$.

R is called a Boolean ring if every element is idempotent. It is well known that R is a Boolean ring if and only if R_M is isomorphic to Z_2 (the ring of integers modulo 2) for each maximal ideal M of R. It is also well known that Boolean rings, Boolean algebras and complemented distributive lattices are essentially the "same things." Thus the duality for Boolean algebras can also be stated for Boolean rings. Less well known ([1], [3], [4]) is that a general such duality theory can be given for arbitrary commutative rings with the Boolean ring duality as a special case. From this duality, one is naturally led to the definition of a Boolean-like ring ([1, page 149]). In [2] it is shown that Boolean-like rings are characterized as the commutative rings R with 1 satisfying $2x=0$ and $xy(1+x)(1+y)=0$ for all $x, y\in R$. We take this for our definition of a Boolean-like ring. More generally we define a ring R to be n-Boolean if $\mathrm{char}\, R=2$ and $x_1\cdots x_n(1+x_1)\cdots(1+x_n)=0$ for all $x_1, \cdots, x_n\in R$. Thus Boolean rings are just the 1-Boolean rings and Boolean-like rings are the 2-Boolean rings. We show that R is n-Boolean if and only if $\operatorname{char} R=2$, $R/\operatorname{nil}(R)$ is Boolean and $\operatorname{nil}(R)^n=0$. Thus R is a Boolean-like ring if and only if $\operatorname{char} R=2$, $R/\operatorname{nil}(R)$ is Boolean and $\operatorname{nil}(R)^2=0$. We also show that every Boolean-like ring has the form $R \oplus M$ where R is a Boolean ring and M is an R-module with the multiplication given by $(r, m)(r_1, m_1)=(rr_1, rm_1+r_1m)$. ($R \oplus M$ is called the idealization of R and M.)

Our general reference for commutative ring theory will be [5]. R will always denote a commutative ring with identity. We will use $\operatorname{nil}(R)$ to denote the nilradical of R. Recall that $\operatorname{nil}(R)$ is said to be T-nilpotent if given any sequence $\{x_i\} \subseteq \operatorname{nil} R$, there exists an n with $x_1 \cdots x_n=0$. Many of the results of this paper can be generalized to noncommutative rings or rings without an identity. For simplicity we have made the assumption that all rings are commutative with identity.

2. Generalized von Neumann regular rings

Let R be a commutative ring with 1 and n a natural number. R is said to be *n-von Neumann regular* if given $x_1, \cdots, x_n \in R$, there exist $a_1, \cdots, a_n \in R$ such that $(x_1a_1x_1-x_1)\cdots(x_na_nx_n-x_n)=0$. Thus 1-von Neumann regular is just von Neumann regular. R is said to the *T-von Neumann regular* if given any sequence $\{x_i\} \subseteq R$, there exists an n and $a_1, \cdots, a_n \in R$ with $(x_1a_1x_1-x_1)\cdots(x_na_nx_n-x_n)=0$. Thus if R is n-von Neumann regular and $m \geq n$, R is m-von Neumann regular and is T-von Neumann regular. Our first result gives an ideal-theoretic characterization of n-von Neumann regular rings which generalizes the well-known result that R is von Neumann regular if and only if R is zero dimensional and reduced.

THEOREM 1. *For a commutative ring R the following statements are equivalent.*
(1) *R is n-von Neumann regular (T-von Neumann regular).*
(2) $\dim R=0$ *and* $\operatorname{nil}(R)^n=0$ ($\operatorname{nil}(R)$ *is T-nilpotent*).
(3) $R/\operatorname{nil}(R)$ *is von Neumann regular and* $\operatorname{nil}(R)^n=0$ ($\operatorname{nil}(R)$ *is T-nilpotent*).

Proof. Clearly (2) and (3) are equivalent since $\dim R=\dim R/\operatorname{nil}(R)$ and $\dim R/\operatorname{nil}(R)=0$ if and only if $R/\operatorname{nil}(R)$ is von Neumann regular. (1) $\Rightarrow$ (2). Suppose that $\dim R>0$, so that there are primes $P \subsetneq M$. Let $x \in M-P$. Then there exist $a_1, \cdots, a_n \in R$ with $(xa_1x-x)\cdots(xa_nx-x)=0 \in P$. Since $x \notin P$, we have some $a_ix-1 \in P \subseteq M$. But $x \in M$, so $1 \in M$, a contradiction. Hence $\dim R=0$. Let $x_1, \cdots, x_n \in \operatorname{nil}(R)$. Then $0=(x_1a_1x_1-x_1)\cdots(x_na_nx_n-x_n)=x_1\cdots x_n(a_1x_1-1)\cdots(a_nx_n-1)$ for some $a_1, \cdots, a_n \in R$. But since $x_i \in \operatorname{nil}(R)$, a_ix_i-1 is a unit. Hence $x_1\cdots x_n=0$, so $\operatorname{nil}(R)^n=0$. (3) $\Rightarrow$ (1). Let $x_1, \cdots, x_n \in R$ be given. Since $\bar{R}=R/\operatorname{nil}(R)$ is von Neumann regular, there exists an $a_i \in R$ with $\bar{x}_i\bar{a}_i\bar{x}_i=\bar{x}_i$, that is, $x_ia_ix_i-x_i \in \operatorname{nil}(R)$. Since $\operatorname{nil}(R)^n=0$, we have $(x_1a_1x_1-x_1)\cdots(x_na_nx_n-x_n)=0$.

The proof of the equivalence of (1), (2) and (3) in the T-nilpotent case is similar.

In a similar manner, we can give a characterization of zero dimensional rings. The proof being similar to the proof of Theorem 1 will be omitted.

THEOREM 2. *For a commutative ring R, the following statements are equivalent.*

(1) *Given* $x \in R$, *there exists a positive integer* n *and an* $a \in R$ *such that* $(xax-x)^n=0$.

(2) *Given* $x \in R$, *there exists a positive integer* n *and* $a_1, \cdots, a_n \in R$ *such that* $(xa_1x-x)\cdots(xa_nx-x)=0$.

(3) $\dim R=0$.

(4) $R/\mathrm{nil}(R)$ *is von Neumann regular.*

In the definition of a von Neumann regular ring, the element a with $xax=x$ depends on x. Suppose that there exists an $a \in R$ with $xax=x$ for all $x \in R$. Then for $x=1$, we have $a=1a1=1$, so $x^2=x1x=x$ for all $x \in R$ so R is a Boolean ring. Conversely a Boolean ring is a von Neumann regular ring with $x1x=x$. It is easily seen that R is n-von Neumann regular if and only if given $x_1, \cdots, x_n \in R$, there exist $a_1, \cdots, a_n \in R$ with $x_1\cdots x_n(1+a_1x_1)\cdots(1+a_nx_n)=0$ and that R is Boolean if and only if there exists an $a \in R$ such that $x(1+ax)=0$ for all $x \in R$. Thus we are led to consider rings satisfying the following condition: there exists a $b \in R$ with $x_1\cdots x_n(1+bx_1)\cdots(1+bx_n)=0$ for all $x_1, \cdots, x_n \in R$. Our next theorem shows that there is no loss in generality in taking $b=1$.

THEOREM 3. *For a commutative ring R and natural number n the following statements are equivalent.*

(1) *There exists a* $b \in R$ *with* $x_1\cdots x_n(1+bx_1)\cdots(1+bx_n)=0$ *for all* $x_1, \cdots, x_n \in R$.

(2) $x_1\cdots x_n(1+x_1)\cdots(1+x_n)=0$ *for all* $x_1, \cdots, x_n \in R$.

(3) *For any unit* $u \in R$, $x_1\cdots x_n(1+ux_1)\cdots(1+ux_n)=0$. *Moreover, any element* $b \in R$ *satisfying condition* (1) *must be a unit.*

Proof. (3) $\Rightarrow$ (2). Take $u=1$. (2) $\Rightarrow$ (1). Take $b=1$. (1) $\Rightarrow$ (3). We first show that b must be a unit. Let $x_1=\cdots=x_n=1$, so $(1+b)^n=0$. Then $1+b$ is nilpotent and hence is contained in every maximal ideal, so b is a unit. Let $u \in R$ be a unit and let $x_i'=b^{-1}ux_i$. Then $1+bx_i'=1+ux_i$ and $0=x_1'\cdots x_n'(1+bx_1')\cdots(1+bx_n')=(b^{-1}u)^n x_1\cdots x_n(1+ux_1)\cdots(1+ux_n)$. Since $(b^{-1}u)^n$ is a unit, we have $0=x_1\cdots x_n(1+ux_1)\cdots(1+ux_n)$.

Analogous to Theorem 1 we have the following result.

THEOREM 4. *For a commutative ring R and natural number n the following conditions are equivalent.*

(1) $x_1\cdots x_n(1+x_1)\cdots(1+x_n)=0$ *for all* $x_1, \cdots, x_n \in R$. (*Given any sequence of elements* $\{x_i\} \subseteq R$, *there exists an* m *such that* $x_1\cdots x_m(1+x_1)\cdots(1+x_m)=0$.)

(2) $R/\mathrm{nil}(R)$ *is a Boolean ring and* $\mathrm{nil}(R)^n=0$ ($\mathrm{nil}(R)$ *is T-nilpotent*).

(3) $\dim R=0$, *for each maximal ideal* M *of* R, $R/M \cong Z_2$ *and* $\mathrm{nil}(R)$ *is T-nilpotent*).

Proof. (1) $\Rightarrow$ (2). Let $x=x_1=\cdots=x_n$, so $x^n(1+x)^n=0$. Hence in $\bar{R}=R/\mathrm{nil}(R)$, $(\bar{x}(\bar{1}+\bar{x}))^n=\bar{0}$. Since $\bar{R}$ is reduced, $\bar{x}(\bar{1}+\bar{x})=\bar{0}$. Replacing $\bar{x}$ by $-\bar{x}$ shows that $\bar{x}=\bar{x}^2$

and hence $\bar{R}$ is Boolean. Let $x_1, \cdots, x_n \in \operatorname{nil}(R)$. Then $1+x_1, \cdots, 1+x_n$ are units, so $0=x_1\cdots x_n(1+x_1)\cdots(1+x_n)$ implies that $x_1\cdots x_n=0$, so $\operatorname{nil}(R)^n=0$. (2) $\Rightarrow$ (3). Since $\bar{R}$ is a Boolean ring, so is $R/M \cong \bar{R}/\bar{M}$. Since R/M is a field, we must have $R/M \cong Z_2$. (3) $\Rightarrow$ (2). Since $\bar{R}$ is zero dimensional and reduced, $\bar{R}$ is von Neumann regular. Moreover, for each maximal ideal M of R, $\bar{R}_{\bar{M}} \cong \bar{R}/\bar{M} \cong R/M \cong Z_2$. Hence $\bar{R}$ is a Boolean ring. (3) $\Rightarrow$ (1). Let $x_1, \cdots, x_n \in R$. Since $\bar{R}$ is a Boolean ring, $\bar{x}_i(\bar{1}+\bar{x}_i)=\bar{0}$, so $x_i(1+x_i) \in \operatorname{nil}(R)$. Since $\operatorname{nil}(R)^n=0$, we have $x_1\cdots x_n(1+x_1)\cdots(1+x_n)=0$.

The proof of the T-nilpotent case is similar.

If R satisfies any of the equivalent conditions of Theorem 4, then setting $x_1=\cdots=x_n=1$, shows that $\operatorname{char} R=2^n$. For any natural number n, the ring $Z/2^nZ$ satisfies the conditions of Theorem 4 for n, but not for $n-1$. In particular the ring $Z/4Z$ satisfies $xy(1+x)(1+y)=0$ for all x, y but has characteristic 4.

Analogous to Theorem 2, we have the following result whose proof will be omitted.

THEOREM 5. *For a commutative ring R the following statements are equivalent.*

(1) *Given $x \in R$, there exists a natural number n with $x^n(1+x)^n=0$.*

(2) *$R/\operatorname{nil}(R)$ is a Boolean ring.*

(3) *$\dim R=0$ and for each maximal ideal M of R, $R/M \cong Z_2$.*

3. Boolean-like rings

Let R be a commutative ring with identity and n a natural number. Then R is called an *n-Boolean ring* if $\operatorname{char} R=2$ and $x_1\cdots x_n(1+x_1)\cdots(1+x_n)=0$ for all $x_1, \cdots, x_n \in R$. Thus R is a 1-Boolean ring if and only if R is a Boolean ring and R is a 2-Boolean ring if and only if R is a Boolean-like ring. We further define R to be a *T-Boolean ring* if given any sequence $\{x_i\} \subseteq R$, there exists an m with $x_1\cdots x_m(1+x_1)\cdots(1+x_m)=0$. Before giving our characterization of n-Boolean rings (Theorem 7), we need a preliminary result which generalizes [2, Theorems 17, 18].

THEOREM 6. *Let R be a ring with $\operatorname{char} R=2$. Then $B=\{b \in R \mid b=b^2\}$ is a Boolean subring of R. $R=B+\operatorname{nil}(R)$ (and hence $R=B\oplus\operatorname{nil}(R)$) if and only if $\bar{R}=\bar{R}/\operatorname{nil}(R)$ is a Boolean ring. In this case, $\bar{R}\cong B$.*

Proof. The fact that $\operatorname{char} R=2$ easily yields that B is a Boolean subring of R. If $R=B+\operatorname{nil}(R)$, then the map $B\to R\to R/\operatorname{nil}(R)$ is an isomorphism since $B\cap\operatorname{nil}(R)=0$. Conversely, suppose that $\bar{R}$ is a Boolean ring. Let $r\in R$. Then in $\bar{R}$, $\bar{r}=\bar{r}^2$. Let $x=r-r^2$, so $r=r^2+x$ and x is nilpotent. Since x is nilpotent, $x^{2^l}=0$ for some natural number l. Since $\operatorname{char} R=2$, $r^{2^l}=(r^2+x)^{2^l}=(r^2)^{2^l}+x^{2^l}=(r^2)^{2^l}=(r^{2^l})^2$, so r^{2^l} is idempotent. But $\bar{r}=\bar{r}^{2^l}$, so $y=r-r^{2^l}$ is nilpotent and $r=r^{2^l}+y\in B+\operatorname{nil}(R)$, so $R=B+\operatorname{nil}(R)$.

THEOREM 7. *For a commutative ring R the following conditions are equivalent.*

(1) *R is n-Boolean (T-Boolean).*

(2) $R/\mathrm{nil}(R)$ *is Boolean*, char $R=2$ *and* $\mathrm{nil}(R)^n=0$ ($\mathrm{nil}(R)$ *is T-nilpotent*).

(3) $\dim R=0$, char $R=2$, $R/M\cong Z_2$ *for each maximal ideal M of R and* $\mathrm{nil}(R)^n=0$ ($\mathrm{nil}(R)$ *is T-nilpotent*).

(4) char $R=2$, *every element of R is the sum of an idempotent element and nilpotent element and* $\mathrm{nil}(R)^n=0$ ($\mathrm{nil}(R)$ *is T-nilpotent*).

Proof. The equivalence of (1)–(3) follows from Theorem 4. The equivalence of (2) and (4) follows from Theorem 6.

As an immediate corollary to Theorem 7, we get the following characterizations of Boolean-like rings. The implication (1) implies (2) and the equivalence of (1) and (4) is given by Foster [2].

THEOREM 8. *For a commutative ring R the following statements are equivalent.*

(1) *R is a Boolean-like ring.*

(2) $R/\mathrm{nil}(R)$ *is a Boolean ring*, char $R=2$ *and* $\mathrm{nil}(R)^2=0$.

(3) $\dim R=0$, char $R=2$, $R/M\cong Z_2$ *for each maximal ideal M of R and* $\mathrm{nil}(R)^2=0$.

(4) char $R=2$, *every element of R is (uniquely) the sum of an idempotent element and nilpotent element and* $\mathrm{nil}(R)^2=0$.

Suppose that R is a commutative ring with identity and M is an R-module. Then $R^*=R\oplus M$ is a commutative ring with identity under the sum $(r, m)+(r_1, m_1)=(r+r_1, m+m_1)$ and the product $(r, m)(r_1, m_1)=(rr_1, rm_1+r_1m)$. This is the so-called method of idealization. Note that $R\cong R\oplus 0$ is a subring of R^* and $M\cong 0\oplus M$ is now an ideal of R^*. It is easily seen that $\mathrm{nil}(R\oplus M)=\mathrm{nil}(R)\oplus M$.

THEOREM 9. *Suppose that R is an n-Boolean ring and N is an R-module. Then* $R^*=R\oplus N$ *is an* $n+1$*-Boolean ring.* R^* *is an n-Boolean ring if and only if* $\mathrm{nil}(R)^{n-1}N=0$.

Proof. Since char $R=2$, $2x=0$ for all $x\in N$, so char $R^*=2$. Since $R^*/\mathrm{nil}(R^*)=R^*/\mathrm{nil}(R)\oplus N\cong R/\mathrm{nil}(R)$, $R^*/\mathrm{nil}(R^*)$ is a Boolean ring. It is easily proved that $\mathrm{nil}(R^*)^m=\mathrm{nil}(R)^m\oplus\mathrm{nil}(R)^{m-1}N$ for each natural number m. Hence if R is n-Boolean, then $\mathrm{nil}(R)^n=0$ so $\mathrm{nil}(R^*)^{n+1}=0$. If R^* is n-Boolean, then $\mathrm{nil}(R)^{n-1}N=0$ and conversely.

The proof of the previous theorem shows that the analogous theorem for n-von Neumann regular rings is also true. If R is an n-von Neumann regular ring, then $R^*=R\oplus N$ is $n+1$-von Neumann regular and R^* is n-von Neumann regular if and only if $\mathrm{nil}(R)^{n-1}N=0$.

Thus if R is a Boolean ring and N is an R-module, then ring $R^*=R\oplus N$ is a Boolean-like ring. In Theorem 10 we show that every Boolean-like ring is of this form. Suppose that R is a Boolean ring and M is a fixed maximal ideal of R. Then $R/M\cong Z_2$ and any 2-elementary abelian group N (every element has order 2) has a natural R-module structure induced by the homomorphism $R\to R/M\cong Z_2$. Thus

$R^*=R\oplus N$ is a Boolean-like ring. This was already observed (in different terminology) by Foster [2] and slightly generalized by Harary [6]. (In our terminology, Harary allowed N to be a direct sum of simple R-modules, i.e., semisimple.) The Boolean-like rings constructed by Harary turn out to be the so-called atomic-based Boolean-like rings. (See [8] for details.) However, not all Boolean-like rings are atomic-based (since an R-module need not be semisimple). A complete structure theory of Boolean-like rings is given by the next theorem.

THEOREM 10 (Structure Theory of Boolean-like Rings). *If B is a Boolean ring and N is an B-module, then the idealization $R=B\oplus N$ is a Boolean-like ring. Conversely, suppose that R is a Boolean-like ring. Then $\bar{R}=R/\mathrm{nil}(R)$ is a Boolean ring and $R\cong\bar{R}\oplus\mathrm{nil}(R)$ where $\mathrm{nil}(R)$ has the natural $\bar{R}$-module structure induced by the homomorphism $R\to\bar{R}$. Equivalently, if we let $B=\{b\in R\,|\,b=b^2\}$, then B is a Boolean ring (with $B\cong\bar{R}$) and $R\cong B\oplus\mathrm{nil}(R)$ where $\mathrm{nil}(R)$ is given the natural B-module structure induced by the inclusion $B\to R$.*

Proof. The first statement follows from Theorem 9. Suppose that R is a Boolean-like ring. By Theorems 6 and 8 $\bar{R}\cong B$ is a Boolean ring. Since $\mathrm{nil}(R)^2=0$, $\mathrm{nil}(R)$ has a natural $\bar{R}=R/\mathrm{nil}(R)$-module structure. Now by Theorem 8 each element r of R can be written uniquely in the form $r=b+n$ where b is idempotent and n is nilpotent. Define the maps $\varphi_1: R\to\bar{R}\oplus\mathrm{nil}(R)$ and $\varphi_2: R\to B\oplus\mathrm{nil}(R)$ by $\varphi_1(r)=(\bar{b}, n)=(\bar{r}, n)$ and $\varphi_2(r)=(b, n)$. Clearly the maps φ_1 and φ_2 preserve addition and using the fact that $\mathrm{nil}(R)^2=0$ it easily follows that φ_1 and φ_2 preserve multiplication. Hence φ_1 and φ_2 are ring homomorphisms. The existence and uniqueness of the representation $r=b+n$ gives that φ_2 is an isomorphism. If $b+n=r\in\ker\varphi_1$, then $\bar{b}=\bar{0}$ in $\bar{R}$ and $n=0$. But then $r=b$ is idempotent and $b\in\mathrm{nil}(R)$, so $r=b=0$. Hence φ_1 is injective. Let $(x, n)\in\bar{R}\oplus N$. By Theorem 6, there exists an idempotent element $b\in R$ with $\bar{b}=x$ in $\bar{R}$. Let $r=b+n$. Then $\varphi_1(r)=(\bar{b}, n)=(x, n)$. Hence φ_2 is surjective.

A natural question is whether Theorem 10 can be extended to n-Boolean rings for $n>2$ and to n-von Neumann regular rings. If R is von Neumann regular and N is an R-module, then $R^*=R\oplus N$ is 2-von Neumann regular. However, if $R^*=Z/4Z$, then R^* is 2-von Neumann regular but R^* does not have the form $R^*=R\oplus N$ where R is von Neumann regular and N is an R-module. For since R^* is not von Neumann regular, we must have $R=Z_2$ and hence R^* would be a Boolean-like ring which isn't the case since $Z/4Z$ has characteristic 4. This same example shows that a ring satisfying the identity $x_1x_2(1+x_1)(1+x_2)=0$ need not be the idealization of a ring satisfying $x_1(1+x_1)$ with a module.

Let $R^*=Z_2[X]/(X^3)\times Z_2$, so R^* is 3-Boolean, but not 2-Boolean. Now $|R^*|=16$ and $|\mathrm{nil}(R^*)|=4$. Suppose that $R^*\cong R\oplus N$ where R is 2-Boolean and N is an R-module. Now R can't be a Boolean ring for then R^* would be 2-Boolean, so $|R|>2$. Also $|R|\neq 16$, so $|R|=4$ or 8. Now $|R|=4$ implies $|N|=4$. In this case, since $0\oplus N\subseteq\mathrm{nil}(R\oplus N)$ and $|\mathrm{nil}(R^*)|=4$, R is reduced. But then R is a

Boolean ring and hence R^* would be a Boolean-like ring, a contradiction. Hence $|R|=8$ and $|N|=2$, so N is isomorphic as an abelian group to Z_2. Since R^* is the direct product of two local rings, the same is true of R. It is easily seen that R must be isomorphic to $Z_2[X]/(X^2)\times Z_2$. What is the action of R on $N\cong Z_2$? Now $(\bar{1},\bar{1})$ and $(\bar{1}+\bar{x},\bar{1})$ are units of R, so $(\bar{1},\bar{1})\bar{1}=\bar{1}$ and $(\bar{1}+\bar{x},\bar{1})\bar{1}=\bar{1}$. Hence $(\bar{x},\bar{0})\bar{1}=\bar{0}$, so $\operatorname{nil}(R)\ N=0$. It follows from Theorem 9 that R^* is 2-Boolean, a contradiction.

If $\{X_\alpha\}_{\alpha\in\Lambda}$ is any set of indeterminates over Z_2, then $R=Z_2[\{X_\alpha\}]/(\{X_\alpha\})^2$ is a quasi-local Boolean-like ring. Let $M=(\{X_\alpha\})/(\{X_\alpha\})^2$, the maximal ideal of R. Then $M=M/M^2$ is a vector space over $Z_2=R/M$ of dimension $|\Lambda|$. If we consider M as a Z_2-module, then R is isomorphic to the idealization $Z_2\oplus M$. The next theorem gives a complete characterization of Boolean-like rings with only finitely many maximal ideals. This of course includes Noetherian Boolean-like rings. More generally, any Boolean-like ring is a suitable homomorphic image of a polynomial ring over a Boolean ring.

THEOREM 11. *A semi-quasi-local Boolean-like ring is a finite direct product of quasi-local Boolean-like rings. Let* (R,M) *be a quasi-local Boolean-like ring. Let* $\{X_\alpha\}_{\alpha\in\Lambda}$ *be a set of indeterminates with* $|\Lambda|=\dim_{R/M} M/M^2$. *Then* $R\cong Z_2[\{X_\alpha\}]/(\{X_\alpha\})^2$.

Proof. Let R be a semi-quasi-local Boolean-like ring with maximal ideals $M_1,\cdots,M_n$. Since $M_1^2\cdots M_n^2=\operatorname{nil}(R)^2=0$, we have $R\cong R/M_1^2\times\cdots\times R/M_n^2$ by the Chinese Remainder Theorem and each R/M_i^2 is a quasi-local Boolean-like ring. Let (R,M) be a quasi-local Boolean-like ring. Let $B=\{b\,|\,R\,|\,b=b^2\}$. Since R is quasi-local, $B=Z_2$. Thus by Theorem 10, $R=Z_2\oplus M$. Let $\{x_\alpha\}_{\alpha\in\Lambda}$ be a basis for M considered as a Z_2-module and let $\{X_\alpha\}_{\alpha\in\Lambda}$ be a set of indeterminates over Z_2 in one-to-one correspondence with the basis elements. Define the map $\varphi\colon Z_2[\{X_\alpha\}]\to R$ by $\varphi(f(X_\alpha))=f(x_\alpha)$. Clearly φ is an epimorphism with $\ker\varphi=(\{X_\alpha\})^2$, so $Z_2[\{X_\alpha\}]/(\{X_\alpha\})^2\cong R$.

More generally, it can be shown that a quasi-local n-Boolean ring has the form $Z_2[\{X_\alpha\}]/I$ where I is an ideal with $(\{X_\alpha\})^n\subseteq I$.

Recall that an R-module N is said to be arithmetical if the lattice of R-submodules of N is distributive. It is well known that N is arithmetical if and only if for each maximal ideal M of R, the R_M-submodules of N_M are totally ordered. From Theorem 11 it is easily seen that a quasi-local arithmetical Boolean-like ring is isomorphic to either Z_2 or $Z_2[X]/(X)^2$. Thus a Boolean-like ring R is arithmetical if and only if for each maximal ideal M of R, R_M is isomorphic to either Z_2 or $Z_2[X]/(X^2)$ or equivalently $\dim_{R/M} M/M^2\le 1$ for each maximal ideal M of R. This is observed in [8] where a sheaf-theoretic characterization of arithmetical Boolean-like rings is given. We end this paper with an alternate structure theory for arithmetical Boolean-like rings.

THEOREM 12. *If B is a Boolean ring and N is an arithmetical B-module, then the idealization $R=B\oplus N$ is an arithmetical Boolean-like ring. Conversely if R is an arithmetical Boolean-like ring,* $\mathrm{nil}(R)$ *is an arthmetical $B=R/\mathrm{nil}(R)$-module and $R=B\oplus\mathrm{nil}(R)$:*

Proof. Suppose that B is a Boolean ring and N is an arithmetical B-module. By Theorem 10, $R=B\oplus N$ is a Boolean-like ring. Let M^* be a maximal ideal of R. Then $M^*=M\oplus N$ where M is a maximal ideal of B. Then $B_M\cong Z_2$ and N_M is a B_M-module whose submodules are totally ordered. Hence N_M is isomorphic to 0 or Z_2, so $R_{M^*}=B_M\oplus N_M$ is isomorphic to Z_2 or $Z_2[X]/(X^2)$ both of which are chained rings. Hence R is an arithmetical ring. Conversely suppose that R is an arithmetical Boolean-like ring. Then by Theorem 10, $R=B\oplus\mathrm{nil}(R)$. Since R is an arithmetical rıng, $\mathrm{nil}(R)$ is an arithmetical R-module. However, since the structure of $\mathrm{nil}(R)$ as an R-module is essentially the same as the structure of $\mathrm{nil}(R)$ has a B-module, $\mathrm{nil}(R)$ is also an arithmetical B-module.

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