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On h-Divisible Torsion Modules over Domains

by

B. DE LA ROSA and L. FUCHS

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1. Introduction

The concept of an h-divisible module over a domain R was introduced by Matlis in [4]; these modules are by definition the epimorphic images of the injective R-modules. The purpose of this paper is to continue the study of h-divisibles. We shall denote the field of quotients of R by Q and the R-module Q/R by K, and assume that $R\neq Q$.

It is well known that the torsion part of an h-divisible R-module is a direct summand and that the torsion-free part is isomorphic to a direct sum of copies of Q. This reduces our investigation to the torsion case.

Our basic tool is the concept of an *hd-exact sequence* which we introduce and study in Section 2. In this we rely heavily on the Matlis duality [3] between the category of h-divisible torsion R-modules T on the one hand, and the category of R-complete torsion-free R-modules M on the other hand, under the inverse correspondences

$$T\longmapsto \mathrm{Hom}_R(K, T) \quad \text{and} \quad M\longmapsto K\otimes_R M .$$

Based on the concept of *hd*-exactness our concept of *hd-projective* then amounts to the obvious adaptation of the definition of projective, and the corresponding notion of *hd-dimension* rests on the readily verifiable version of Schanuel's lemma [2] for *hd*-projective resolutions, which we tacitly assume. In a similar fashion we apply the appropriately modified version of Kaplansky's lemma [2] for *hd*-exact sequences in Section 3 where we prove as one of our main results that under the hypothesis $p.d.Q=1$, the *hd*-dimension of every h-divisible torsion R-module is 1 less than its projective dimension. In the final section we prove (under the same condition on Q) that this relationship is maintained between the global *hd*-dimension of R (defined in the obvious way) and the global dimension of R.

2. *hd*-exact sequences

An exact sequence $0\to A\to B\to C\to 0$ of R-modules will be called *hd-exact* if $K=$

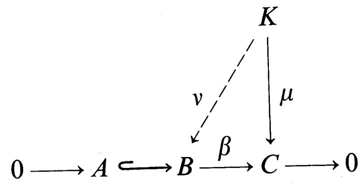
The authors are from the University of the Orange Free State (Bloemfontein) and Tulane University (New Orleans) respectively.

Q/R has the projective property with respect to it.

In this section we establish several characteristic features of *hd*-exact sequences of torsion modules over an arbitrary domain R.

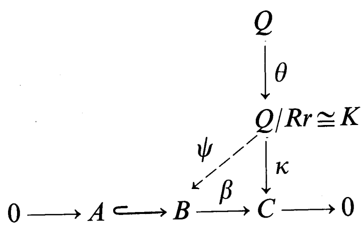
LEMMA 1. *Let* $0 \to A \hookrightarrow B \xrightarrow{\beta} C \to 0$ *be an hd-exact sequence of torsion R-modules. Then B is h-divisible if and only if A and C are h-divisible.*

Proof. Suppose that B is h-divisible. Then clearly C is h-divisible, and we may consider A. Let $a \in A$. Since $a \in B$, the h-divisibility of B ensures the existence of a homomorphism $\rho: Q \to B$ such that $\rho 1 = a$. For any $r \in R$ we have that $\beta\rho(r) = \beta(\rho r) = \beta(ra) = r\beta a = 0$. Thus we have that $\mu: K \to C$, $\mu(q+R) = \beta\rho(q)$ is a well defined homomorphism, and obviously $\mu\gamma = \beta\rho$, where $\gamma: Q \to K$ is the canonical map. For this μ there exists a $\nu: K \to B$ as in



such that the triangle commutes. We claim that $\rho - \nu\gamma$ maps Q into A and $(\rho - \nu\gamma)1 = a$. In fact, for any $q \in Q$, $(\rho - \nu\gamma)q \in B$; and $\beta(\rho - \nu\gamma)q = \beta\rho(q) - \beta\nu\gamma(q) = \beta\rho(q) - \mu\gamma(q) = 0$ shows that $(\rho - \nu\gamma)q \in \operatorname{Ker}\beta = A$. Furthermore, $(\rho - \nu\gamma)1 = \rho 1 - \nu(\gamma 1) = a - \nu(1+R) = a - 0 = a$. Thus we have shown that A is h-divisible.

Conversely, take any preassigned $b \in B$ and set $\beta b = c$. Let $\phi: Q \to C$ be a homomorphism with $\phi 1 = c$. Since C is torsion, $\operatorname{Ker}\phi \neq 0$. Let $0 \neq r \in \operatorname{Ker}\phi$. Then for the canonical map $\theta: Q \to Q/Rr \cong K$ and $\kappa: Q/Rr \to C$, $\kappa(q + Rr) = \phi q$, we have that $\phi = \kappa\theta$ so that $\kappa(1 + Rr) = c$. Since the row in the diagram below is *hd*-exact, there is a homomorphism ψ as inserted which makes the triangle commute:



Now $c = \kappa(1+Rr) = \beta\psi(1+Rr)$ together with $\beta b = c$ shows that $\psi(1+Rr) - b \in \operatorname{Ker}\beta = A$. Hence there exists a homomorphism $\eta: Q \to A$ such that $\lambda 1 = \psi(1+Rr) - b$. The homomorphism $\psi\theta - \eta: Q \to B$ maps 1 onto b. □

LEMMA 2. *For every h-divisible torsion R-module C there exists an hd-exact sequence of the form* $0 \to A \to \oplus K \to C \to 0$ *in which the kernel A is also h-divisible.*

Proof. For every $c \in C$ we may fix a homomorphism $\phi_c: Q \to C$ with $\phi_c 1 = c$ and a $0 \neq r_c \in \operatorname{Ker} \phi_c$, and construct a homomorphism $\psi_c \colon K \to C$ with $c \in \operatorname{Im} \phi_c$ as in

$$\begin{array}{ccc} Q & \xrightarrow{\ \phi_c\ } & C \\ \downarrow{\scriptstyle \text{canon}} & \nearrow & \\ Q/Rr_c \cong K & & \psi_c \colon q + Rr_c \longmapsto \phi_c(q)\,. \end{array}$$

This property of C ensures that the induced map

$\bigoplus_{\phi \in \operatorname{Hom}_R(K,\,C)} K \xrightarrow{\ \Phi\ } C$ is epic, and we have the exact sequence

$$0 \longrightarrow \operatorname{Ker} \Phi \longrightarrow \oplus K \xrightarrow{\ \Phi\ } C \longrightarrow 0\,.$$

That K has the projective property with respect to this sequence is guaranteed by the construction. Finally, since $\oplus K$ is torsion and h-divisible, $\operatorname{Ker} \Phi$ must also be, by Lemma 1. □

The following two lemmas assert that exact sequences in the category of R-complete torsion-free R-modules and hd-exact sequences in the category of h-divisible torsion R-modules are correspondents under the Matlis duality.

LEMMA 3. *If $0 \to A \to B \to C \to 0$ is exact, with A, B and C R-complete and torsion-free, then in the induced sequence*

$$0 \longrightarrow K \otimes_R A \longrightarrow K \otimes_R B \longrightarrow K \otimes_R C \longrightarrow 0$$

the modules $K \otimes_R A$, $K \otimes_R B$ and $K \otimes_R C$ are torsion and h-divisible, and this sequence is hd-exact.

Proof. By the Matlis duality the sequence is exact and the three tensor products in this sequence are h-divisible and torsion. To prove hd-exactness we must show that the induced left exact sequence

$$0 \longrightarrow \operatorname{Hom}_R(K, K \otimes_R A) \xrightarrow{\ \alpha_*\ } \operatorname{Hom}_R(K, K \otimes_R B) \xrightarrow{\ \beta_*\ } \operatorname{Hom}_R(K, K \otimes_R C)$$

ends up with β_* epic. This follows from the existence of natural isomorphisms (cf. [3]) as indicated in the commutative diagram

$$\begin{array}{ccccccc} 0 \longrightarrow & \operatorname{Hom}_R(K, K \otimes_R A) & \xrightarrow{\ \alpha_*\ } & \operatorname{Hom}_R(K, K \otimes_R B) & \xrightarrow{\ \beta_*\ } & \operatorname{Hom}_R(K, K \otimes_R C) & \\ & \downarrow \cong & & \downarrow \cong & & \downarrow \cong & \\ 0 \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow 0 \end{array}$$

and the exactness of the bottom row. □

LEMMA 4. *Let $0 \to S \to T \to U \to 0$ be an hd-exact sequence with S, T and U h-divisible and torsion. Then*

$$0 \longrightarrow \mathrm{Hom}_R(K, S) \longrightarrow \mathrm{Hom}_R(K, T) \longrightarrow \mathrm{Hom}_R(K, U) \longrightarrow 0$$

is exact, and $\mathrm{Hom}_R(K, S)$, $\mathrm{Hom}_R(K, T)$ *and* $\mathrm{Hom}_R(K, U)$ *are R-complete and torsion-free.*

Proof. By the Matlis duality, $\mathrm{Hom}_R(K, S)$, $\mathrm{Hom}_R(K, T)$ and $\mathrm{Hom}_R(K, U)$ are indeed R-complete and torsion-free; and we must only prove exactness. Now the exactness of $0 \to S \xrightarrow{\eta} T \xrightarrow{\chi} U \to 0$ implies that of

$$0 \longrightarrow \mathrm{Hom}_R(K, S) \xrightarrow{\eta_*} \mathrm{Hom}_R(K, T) \xrightarrow{\chi_*} \mathrm{Hom}_R(K, U)$$

and we must show that χ_* is epic. This, however, is an immediate consequence of the fact that K has the projective property with respect to $0 \to S \to T \to U \to 0$. □

Finally, in this section we briefly introduce our notion of projectivity with respect to *hd*-exact sequences: an *h*-divisible torsion R-module is said to be *hd-projective* if it has the projective property with respect to *hd*-exact sequences. Standard arguments lead to the following characterization of the *hd*-projectives:

LEMMA 5. *The hd-projective R-modules are exactly the direct summands of direct sums of copies of K.* □

3. The *hd*-dimension of an *h*-divisible torsion module

An *hd*-projective resolution of an *h*-divisible torsion R-module H is an *hd*-exact sequence

$$\longrightarrow P_n \xrightarrow{\delta_n} P_{n-1} \longrightarrow \cdots \longrightarrow P_1 \xrightarrow{\delta_1} P_0 \xrightarrow{\delta_0} H \longrightarrow 0$$

of *hd*-projective modules P_i. The validity of Schanuel's lemma for these resolutions secures the following concept of dimension as an invariant for *h*-divisible torsion modules: the *hd-dimension* of H, in notation $hd.d.H$, is equal to n if there is a smallest index n with $\mathrm{Im}\,\delta_n$ *hd*-projective, or to ∞ if no such n exists. The theorem below compares *hd*-dimension with projective dimension under the hypothesis $p.d.Q=1$.

THEOREM 6. *Let R be a domain such that* $p.d.Q=1$, *and let T be an h-divisible torsion R-module. Then* $p.d.T=k$ $(\geqq 1)$ *if and only if* $hd.d.T=k-1$.

Proof. Corollary 10.10 of [3] provides our basis for induction: $p.d.T.=1$ if and only if $hd.d.T=0$. Assume that the theorem is true for $k \geqq 1$. Let T be an h-divisible torsion R-module with $p.d.T=k+1$. Tensoring the exact sequence $0 \to R \to Q \to K \to 0$ by the R-complete torsion-free R-module $A=\mathrm{Hom}_R(K, T)$ and keeping in mind that $A \otimes_R Q$ is torsion-free and divisible, we obtain an exact sequence

$$0 \longrightarrow A \longrightarrow \oplus Q \longrightarrow T \longrightarrow 0\,.$$

Since $p.d.\oplus Q=1$, Kaplansky's lemma applied to this sequence gives $p.d.A=k$. Now consider a free resolution of A:

$$0 \longrightarrow H \longrightarrow F \longrightarrow A \longrightarrow 0\,.$$

By [3], the induced sequence $0 \to \tilde{H} \to \tilde{F} \to \tilde{A}$ of completions is exact, and as $\tilde{A} = A$ in the present case, we have a short exact sequence $0 \to \tilde{H} \to \tilde{F} \to \tilde{A} \to 0$ of R-complete torsion-free R-modules. In view of Lemma 3, the induced sequence

$$0 \longrightarrow K \otimes \tilde{H} \longrightarrow K \otimes \tilde{F} \cong K \otimes F \longrightarrow K \otimes A \cong T \longrightarrow 0$$

is hd-exact. Here $K \otimes F$ is a direct sum of copies of K, whence $hd.T = hd.d.(K \otimes \tilde{H}) + 1$ follows provided that $hd.d.T \neq 0$, (which is true because of $p.d.T > 1$). By induction, $hd.d.(K \otimes H) = p.d.H = p.d.A - 1 = k - 1$, and therefore $hd.d.T = k$ follows. □

4. The global *hd*-dimension of *R*

Following the definition of *global dimension* of R we define the *global hd-dimension of R* by

$$gl.hd.d.R = \sup\{hd.d.T \mid T \text{ an } h\text{-divisible torsion } R\text{-module}\}\,.$$

As our final result we prove that under the hypothesis $p.d.Q = 1$, the relationship between hd-dimension and projective dimension established in Section 3 is maintained by the corresponding global dimensions.

THEOREM 7. *Let R be a domain such that $p.d.Q = 1$. Then $gl.hd.d.R = gl.d.R - 1$.*

Proof. This theorem is an immediate consequence of our Theorem 6 and Propositon 3.5 in [3]. A direct proof within our framework runs as follows: Let $J \neq 0$ be an ideal of R. From the exact sequence $0 \to J \to Q \to Q/J \to 0$ we obtain that either $p.d.J = 1 = p.d.Q/J$ or $p.d.Q/J = p.d.J + 1$. The first alternative would imply $p.d.R/J = 2$ and a contradiction after examining the projective dimensions in the exact sequence $0 \to R/J \to Q/J \to K \to 0$. From Theorem 6 we infer that $hd.d.Q/J = p.d.Q/J - 1 = p.d.J$. Since R is not semisimple (as $R \neq Q$), we have by Auslander's result (cf. e.g. [1]) that $gl.d.R = 1 + \sup\{p.d.J \mid J \text{ an ideal of } R\}$. The claim is now immediate. □

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Department of Mathematics
University of the Orange Free State
Bloemfontein, 9301
Republic of South Africa

Department of Mathematics
Tulane University
New Orleans
Louisiana 70118