COMMENTARII MATHEMATICI
UNIVERSITATIS SANCTI PAULI
Vol. 35, No. 1, 1986

ed. RIKKYO UNIV/MATH
IKEBUKURO TOKYO
171 JAPAN

On $p^{\omega+n}$-Projective p-Groups

by

John IRWIN, Tom SNABB and Doyle CUTLER

(Received September 18, 1984)

Introduction

All groups considered in this paper will be separable abelian p-groups. The notation and terminology will be, for the most part, the same as that in [4]. The symbol $\oplus_c$ will denote a direct sum of cyclic p-groups, and $\overset{v}{\oplus}$ will denote a direct sum as valuated groups.

Recall that a p-group G is $p^{\omega+n}$-projective if there exists a p^n-bounded subgroup P of G such that $G/P=\oplus_c$. G is said to be proper $p^{\omega+n}$-projective if it is not $p^{\omega+t}$-projective for any $t<n$.

In [2], Benabdallah, Irwin, and Rafiq were able to show that for a separable p-group G, if G is not a direct sum of cyclics (G not p^{ω}-projective) then there exists a subgroup H of G such that H is proper $p^{\omega+1}$-projective. In [6], Nunke has shown that subgroups of $p^{\omega+n}$-projective groups are again $p^{\omega+n}$-projective. One is led to consider the following question.

Question 1. Does there exist a separable p-group G, not $p^{\omega+1}$-projective, (thus G contains a proper $p^{\omega+1}$-projective subgroup) which contains no proper $p^{\omega+n}$-projective subgroup for $n>1$?

To restate the question in slightly more general terms we define, for each $n\geq 1$, the class of n-groups, $\mathscr{C}_n$, as follows.

DEFINITION. $G\in\mathscr{C}_n$ if G has no proper $p^{\omega+t}$-projective subgroups for $t>n$.

From Nunke's result we note that if G is $p^{\omega+t}$-projective, $t\leq n$, then $G\in\mathscr{C}_n$. We can now pose the more general question.

Question 2. Does there exist, for any n, a separable p-group G belonging to $\mathscr{C}_n$ such that G is not $p^{\omega+n}$-projective?

If there does exist such a group G we shall call it a proper n-group.

We shall now consider some results related to the question of the existence of proper n-groups. In [3] we were able to show that if G is not fully starred then for each n there exists a proper $p^{\omega+n}$-projective H_n such that $H_n\leq G$. This result implies that if G is an n-group, it must be fully starred. In the same paper we also showed for G fully starred and not $p^{\omega+n}$-projective, if G is C-decomposable (has a summand that is a direct sum of cyclic groups and has the same final rank as G) then it has a proper $p^{\omega+n+1}$-projective subgroup. Thus to be a proper n-group G cannot be $p^{\omega+n}$-projective but must be fully starred and non C-decomposable.

This leads one to ask, if G is fully starred but not proper $p^{\omega+n}$-projective does there exist $A \leq G$ such that $A = H \oplus C$, H not $p^{\omega+n}$-projective and the final rank of C equal to the final rank of H? If this holds then A and thus G would not be a proper n-group. We would then have that the answer to question 2 is no.

A straightforward generalization of Lemma 2.8 in [2] shows that if the following chain condition holds, then such an above A exists.

Chain condition

If G is a p-group and $\{S_k\}$ is a countable sequence of pure dense subgroups of G such that $S_k \leq S_{k+1}$, S_k is $p^{\omega+n}$-projective for every K, and $G = \cup S_k$ then is G $p^{\omega+n}$-projective?

That the chain condition fails to hold for an uncountable chain is shown by the following example: Let $G = \bar{B} \oplus P$, $S_a = B_a \oplus P$, B_a basic in $\bar{B}$ such that $\cup B_a = \bar{B}$, P the prufer group. G is the union of an ascending chain of $p^{\omega+1}$-projectives but is itself not p^{α}-projective for any α.

In pursuing this chain condition we have obtained some partial results. We first need two lemmas.

LEMMA 1. *Let G be a p-group such that $p^{\omega}G = 0$. Let S and P be subsocles of G such that $S + P = S \overset{v}{\oplus} P$ where the sum is direct as a valuated vector space. Let $\bar{P}$ be the closure of P in G with respect to the p-adic toplogy. Then $S + \bar{P} = S \overset{v}{\oplus} \bar{P}$ as a valued vector space.*

Proof. Suppose that $t \in \bar{P}$ and $s \in S$ with $h(t+s) > h(t) = h(s)$. Let $x_i \in P$, $i \in \omega$, such that $h(x_i) = h(t)$ for all $i \in \omega$ and $\{x_i\}_{i \in \omega} \to t$. Then $\{x_i + s\}_{i \in \omega} \to t + s$. Now $h(x_i + s) = h(x_i) = h(s)$ for all $i \in \omega$ since $x_i + s \in S \overset{v}{\oplus} P$. Therefore $h((x_i + s) - (t + s)) = h(x_i + s)$ for all $i \in \omega$. This contradicts $\{x_i + s\}_{i \in \omega} \to t + s$ and thus $h(t+s) = \min\{h(t), h(s)\}$ which implies $S \overset{v}{\oplus} \bar{P}$ is direct as a valuated vector space.

LEMMA 2. *Let H be a pure subgroup of G, P a subgroup of $H[p]$ such that $H/P = \oplus_c$, and $\bar{P}$ the closure of P in G with respect to the p-adic topology in G. Then $(H + \bar{P})/\bar{P}$ is a pure subgroup of $G/\bar{P}$.*

Proof. Note that by [5] $H[p] = S \overset{v}{\oplus} P$ as a valuated vector space. Let $g \in G$ and $h \in H$ such that $p^n g + \bar{P} = h + \bar{P}$. Thus $p^n g = h + t$ for some $t \in \bar{P}$. Hence $p^{n+1} g = ph$. Since H is pure in G, there exists $h_1 \in H$ such that $p^{n+1} h_1 = ph$. Therefore $p^n h_1 - h \in H[p]$ and we may write $p^n h_1 - h = s + u$ with $s \in S$ and $u \in P$. Thus $p^n h_1 - p^n g = s + (u - t)$ with $s \in S$ and $u - t \in \bar{P}$. By Lemma 1 $S \overset{v}{\oplus} \bar{P}$ is direct as a valued vector space hence there exists $h_2 \in H$ such that $p^n h_2 = s$. Thus $p^n(h_1 - h_2) - p^n g = u - t \in \bar{P}$ from which we obtain $p^n(h_1 - h_2) + \bar{P} = h + \bar{P}$. Therefore $H + \bar{P}/\bar{P}$ is pure in $G/\bar{P}$.

We will also need the following result from [1].

THEOREM. *Let B be a basic subgroup of a p-group G, $p^\omega G = 0$. Then all B-high subgroups of G are $\oplus_c$ if and only if $G = \oplus_c$.*

COROLLARY. *Let G be a p-group such that $p^\omega G = 0$. Suppose there exists a basic subgroup H of G such that G/H is countable. Then $G = \oplus_c$.*

With the two lemmas and the above corollary we are now able to prove

THEOREM. *Let G be a separable p-group and H a subgroup of G such that*

i) *H is pure and dense in G*
ii) $\| G/H \| \leq \aleph_0$
iii) *H is $p^{\omega+1}$-projective*

then G is $p^{\omega+1}$-projective.

Proof. Let $P \leq H[p]$ such that $H/P = \oplus_c$. Let $\bar{P}$ be the closure of P in $G[p]$ with respect to the relative p-adic topology. Note that $\bar{P} \cap H = P$ since $p^\omega(H/P) = 0$. Thus $(H + \bar{P})/\bar{P} \cong H/H \cap \bar{P} = H/P = \oplus_c$ and is pure in $G/\bar{P}$ by Lemma 2. Note also that $p^\omega(G/\bar{P}) = 0$ since $\bar{P}$ is closed in G. Since $(G/\bar{P})/[(H + \bar{P})/\bar{P}] \cong G/(H + \bar{P})$ is a homomorphic image of G/H we have $(G/\bar{P})/[H + \bar{P}]$ divisible and countable. By the above corollary $G/\bar{P}$ is a direct sum of cyclic groups. Thus G is $p^{\omega+1}$-projective.

The authors hope that the discussion in this paper will shed some light on the question as to whether there are proper n-groups. If there are proper n-groups then we have a new class of groups to look at. If there are not then we have a new characterization of $p^{\omega+n}$-projectives. This is one of those rare instances in which either alternative is not unfavorable.

References

[1] BENABDALLAH, K. and IRWIN, J. M.; An application of B-high subgroups of abelian p-groups, *Journal of Algebra*, **34** (1975), 213–216.

[2] BENABDALLAH, K., IRWIN, J. M. and RAFIQ, M.; A core class of abelian p-groups, *Symposia Mathematica*, **13** (1974), 195–206.

[3] CUTLER, D., IRWIN, J. M. and SNABB, T.; Abelian p-groups containing proper $p^{\omega+n}$-projective subgroups, submitted.

[4] FUCHS, L.; *Infinite Abelian Groups*, Vol. 1 and 2, Academic Press, New York, 1970 and 1973.

[5] FUCHS, L. and IRWIN, J. M.; On $p^{\omega+1}$-projective p-groups, *Proc. London Math. Soc.*, **30** (1975), 459–470.

[6] NUNKE, R. J.; Purity and subfunctors of the identity, *Topics in Abelian Groups*, Chicago, 1963, pp. 121–171.

Department of Mathematics
Wayne State University
Detroit, Michigan 48202
U.S.A.

Department of Mathematics
University of California
Davis, California 95616
U.S.A.

Department of Mathematics and Statistics
The University of Michigan
Dearborn, Michigan 48128
U.S.A.