COMMENTARII MATHEMATICI
UNIVERSITATIS SANCTI PAULI
Vol. 35, No. 1, 1986

ed. RIKKYO UNIV/MATH
IKEBUKURO TOKYO
171 JAPAN

Hyper-Principle and the Functional Structure of Ordinal Diagrams
(*Continuation.*)

by

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(Received October 1, 1985)

Part II. Hyper-Principle

§1. Type-forms and hyper-types

DEFINITION 1.1. 1) We first set the language $\mathscr{L}_{tp}$ for the type-forms to be defined subsequently. $\mathscr{L}_{tp}$ consists of the language of $\mathscr{L}$-terms and $\mathscr{L}$-recursive formulas in Definitions 1.1 and 1.2 of Part I augmented by the new symbols below.

$$N_0,\ \text{ept},\ \Lambda,\ \{\ \ \},\ \rightarrow,\ \mathscr{C},\ [\ \ ],\ \Rightarrow,\ \Pi,\ T,\ \langle\ \ \rangle,\ \mathscr{R}.$$

Recall that the language of $\mathscr{L}$-terms contain the variables of at-type and those of fn-type.

2) We define type-forms, the variables in them, free and bound, and their reduction rules (where necessary) simultaneously. For a type-form $\mathfrak{s}$, the reduction rule is expressed in the form $\mathfrak{s} \Rightarrow \mathfrak{t}$, where $\mathfrak{t}$ is another type-form and will be called the immediate reduct of $\mathfrak{s}$. A special type-form, the T-type, will be singled out.

(0) ept (emptiness) is a type-form, which is atomic and has no variables.

(1) N_0 is the atomic type-form. It does not involve variables.

The subsequent type-forms are all non-atomic, viz., compound.

(2) If x is a variable (of at-type or of fn-type), and if $\mathfrak{t}$ is a type-form in which x is not bound, then $\Lambda x(\mathfrak{t})$ is a type-form. The free variables in this are the ones in $\mathfrak{t}$ except x, and the bound variables are those in t as well as x. (x may or may not occur in $\mathfrak{t}$.)

$\Lambda x(\mathfrak{t})$ may be abbreviated to $\Lambda x\mathfrak{t}$ when no confusion is anticipated.

(3) If $\mathfrak{s}$ and $\mathfrak{t}$ are type-forms, then $\mathfrak{s}\rightarrow\mathfrak{t}$ is a type-form; the variables are the corresponding ones in $\mathfrak{s}$ and $\mathfrak{t}$.

Note. $\Lambda x_1(\Lambda x_2(\cdots\Lambda x_n(\mathfrak{t})\cdots))$ may be abbreviated to $\Lambda x_1\cdots x_n\mathfrak{t}$; $\mathfrak{s}_1\rightarrow(\mathfrak{s}_2\rightarrow(\cdots\rightarrow(\mathfrak{s}_m\rightarrow\mathfrak{t})\cdots))$ may be abbreviated to

$$\mathfrak{s}_1, \cdots, \mathfrak{s}_m\rightarrow\mathfrak{t}.$$

$N_0\rightarrow N_0$ will be denoted by Sq.

(4) Let $A_1, \cdots, A_m$ be $\mathscr{L}$-recursive formulas (see Definition 1.1 of Part I), mutually exclusive, and let $\mathfrak{t}_1, \cdots, \mathfrak{t}_m, \mathfrak{t}_{m+1}$ be type-forms. Then

$$\mathscr{C}[A_1, \cdots, A_m; \mathrm{t}_1, \cdots, \mathrm{t}_m, \mathrm{t}_{m+1}],$$

abbreviated to $\mathscr{C}[(A); (\mathrm{t})]$, is a type-form with the reduction rule:

$$\mathscr{C}[(A); (\mathrm{t})] \Rightarrow \begin{cases} \mathrm{t}_l & \text{if } A_l, \quad \text{where } 1 \leq l \leq m, \\ \mathrm{t}_{m+1} & \text{if } \neg A_1 \wedge \cdots \wedge \neg A_m. \end{cases}$$

The free and the bound variables in this are the corresponding ones in (A) and (t).

(5) $T\langle i\rangle$ is a type-form if i is an $\mathscr{L}$-term of at-type. Its reduction rule is:

$$T\langle i\rangle \Rightarrow \Lambda l \mathscr{C}[l=0, l=1; T_1\langle i\rangle, T_2\langle i\rangle, \mathrm{ept}],$$

where $T_1\langle i\rangle$ abbreviates

$$\Lambda j \mathscr{C}[j<i; N_0 \rightarrow T\langle j\rangle, \mathrm{ept}],$$

and $T_2\langle i\rangle$ abbreviates

$$\mathrm{Sq} \rightarrow ((N_0 \rightarrow T_1\langle i\rangle) \rightarrow N_0).$$

$N_0 \rightarrow T_1\langle i\rangle$ will be abbreviated to $T_2(i)$, and l and j above are at-type variables. The variables in $T\langle i\rangle$ are those in i. The immediate reduct of $T\langle i\rangle$ may be expressed by $T'(i)$.

(6) Let $\mathfrak{s}$ be of the form $\Lambda x\mathrm{t}(x)$, where $\mathrm{t}(x)$ indicates all the occurrences of x in t, and let ϕ be an $\mathscr{L}$-term of the same type as x whose variables are not bound in t. Then $\Pi(\mathfrak{s}; \phi)$ is a type-form with the reduction rule:

$$\Pi(\mathfrak{s}; \phi) \Rightarrow \mathfrak{s}(\phi).$$

$\Pi(\mathfrak{s}; \phi)$ will be called the projection of $\mathfrak{s}$ to ϕ; its variables are the corresponding ones in $\mathfrak{s}$ and ϕ.

(7) Let $\mathfrak{s}$ be $T\langle i\rangle$ and let ϕ be an at-type $\mathscr{L}$-term. Then $\Pi(\mathfrak{s}; \phi)$ is a type-form. The variables are those in i and ϕ.

Note. 1) $\Pi(\cdots(\Pi(\mathfrak{s}; \phi_1), \phi_2); \cdots; \phi_n)$ will be abbreviated to $\Pi(\mathfrak{s}; \phi_1, \cdots, \phi_n)$.

2) We use l, i and j with the convention of usage specified in Part I, anticipating that the subsequent arguments make sense when l denotes natural numbers, while j and i denote the elements of $\mathbf{I}_\infty$. So, in fact $l=0$ is an abbreviated notation, and its precise form should be something like $l=\ulcorner 0\urcorner$, $\ulcorner 0\urcorner$ being the arithmetized notation for zero. We shall employ crude notations such as $l=0$ for such cases. We should also note that $j <_{\mathbf{I}} i$ is meaningful only if $\mathbf{I}_\infty(j)$ and $\mathbf{I}_\infty(i)$ hold.

(8) Let $\mathrm{t}=t(\Xi_1, \cdots, \Xi_n)$ be an expression in the language $\mathscr{L}_{tp}$, free of $\mathscr{R}$, with the parameters $\Xi_1, \cdots, \Xi_n$; let $v=v_1, \cdots, v_n$ be $\mathscr{L}$-terms of at-type; let $\mathfrak{s}$ be a type-form without $\mathscr{R}$; let l stand for an $\mathscr{L}$-term of at-type (which is supposed to be a numeral); for each p, $1 \leq p \leq n$, let x_p stand for a finite sequence of free variables different from l. Define

$$v_p^*(l, x_p) = \begin{cases} v_p(x_p) & \text{if } v_p(x_p) < l, \\ 0 & \text{otherwise}, \end{cases}$$

where $1 \leq p \leq n$. Define $\mathbf{M}(\mathfrak{s}, \mathrm{t}, v, x; l)$ and $\mathbf{N}_p(\mathfrak{s}, \mathrm{t}, v, x; l)$ as follows, where x stands for the sequence $x_1, \cdots, x_n$.

$$\mathbf{M}(\mathfrak{s}, \mathrm{t}, v, x; 0) = \mathfrak{s} ,$$

$$\mathbf{N}_p(\mathfrak{s}, \mathrm{t}, v, x; l) = \Lambda \xi_1 \cdots \xi_t \mathbf{M}(\mathfrak{s}, \mathrm{t}, v, x; v_p * (l, x_p)) ,$$

where $\xi_1, \cdots, \xi_t$ are some free variables in $\mathbf{M}(\mathfrak{s}, \mathrm{t}, v, x; v_p * (l, x_p))$ (depending on p).

$$\mathbf{M}(\mathfrak{s}, \mathrm{t}, v, x; l) = \mathrm{t}(\mathbf{N}(1, l), \cdots, \mathbf{N}(n, l)) ,$$

where $\mathbf{N}(p, l)$ abbreviates $\mathbf{N}_p(\mathfrak{s}, \mathrm{t}, v, x; l)$. Now, if $\mathbf{M}(\mathfrak{s}, \mathrm{t}, v, x; l)$ is a type-form for each numeral l, then $\mathscr{R}[\mathfrak{s}, \mathrm{t}, v, x; l]$ is a type-form for every l with the reduction rule:

$$\mathscr{R}[\mathfrak{s}, \mathrm{t}, v, x; l] \Rightarrow \mathbf{M}(\mathfrak{s}, \mathrm{t}, v, x; l) .$$

The variables in this are the ones in $\mathfrak{s}$, t, v, x and l.

(9) Type-forms are defined only by (1)$\sim$(8) above.

3) The t's in (2) and (3) above are called the "value" types of the type-forms defined there.

4) We write T for $\Lambda i T\langle i \rangle$, which is a type-form according to (2) and (7), and call it the T-type. Obviously the T-type has no free variables.

5) A type-form without free variables will be called a hyper-type. Examples of such are atomic ones and T.

PROPOSITION 1.1. 1) *Type-forms are closed under substitution of $\mathscr{L}$-terms for the free variables* (*of the same types*), *presuming that there are no "clashes" of variables. In particular, the immediate reduct of $\Pi(\mathfrak{s}; \phi)$ in* (6) *is a type-form.*

2) *The reduction rules are meaningful in the sense that the immediate reducts in the rules are type-forms.*

3) *Finite types based on N_0 are special cases of hyper-types.*

4) *In* (7), t *is not even assumed to be a type-form, but $\mathscr{R}[\mathfrak{s}, \mathrm{t}, v, x; l]$ can be regarded as a hyper-type if $\mathfrak{s}$ is a hyper-type, l is closed,* t *is closed except for the parameters $\Xi_1, \cdots, \Xi_n$, and if other free variables are bound by $\Lambda \xi_1 \cdots \xi_t$.*

5) *In any type-form, any symbol occurs only finitely many times.*

6) *If $\mathfrak{s}$ is a hyper-type, then the immediate reduct of $\mathfrak{s}$ in any of* (4), (5), (6) *and* (8) *is a hyper-type, which is uniquely determined.*

7) *$\Lambda n \mathscr{C}[n < m;$ t(n), ept$]$ is a type-form if* t(n) *is. See* (4) *and* (2). *This will be called the m-tuple determined by* t, *and will be written as*

$$\mathrm{TPL}(m; n, \mathrm{t}(n)) ;$$

n is here regarded as a bound variable. It follows that

$$\Pi(\mathrm{TPL}(m; n, \mathrm{t}(n)); l) \Rightarrow \mathscr{C}[l < m; \mathrm{t}(l), \mathrm{ept}] \Rightarrow \mathrm{t}(l)$$

*if $l < m$; $\Rightarrow$*ept *otherwise. That is, $\Pi(\mathrm{TPL}(m; n, \mathrm{t}(n)); l)$ functions just as the projection of* t (*if $l < m$*). *We may use* TPL *as if it were a primitive symbol.*

The same goes with $\mathrm{TPL}(i; j, \mathrm{t}(j)) = \Lambda j \mathscr{C}[j < i; \mathrm{t}(j), \mathrm{ept}]$.

$\Lambda l\mathscr{C}[l=0, l=1; \mathfrak{t}_1, \mathfrak{t}_2$, ept] may be written as $(\mathfrak{t}_1, \mathfrak{t}_2)$, and will be called the pairing of $\mathfrak{t}_1$ and $\mathfrak{t}_2$. In particular,

$$T\langle i\rangle \Rightarrow (T_1\langle i\rangle, T_2\langle i\rangle) .$$

Proof. of 4). The meaning of this is that, under the premises, $\mathbf{M}(\mathfrak{s}, \mathfrak{t}, v, x; l)$ is a hyper-type. This can be proved by induction on l.

DEFINITION 1.2. 1) We define the immediate reduct (if any) of each hyper-type and the normality of a hyper-type (if it is normal).

A hyper-type is said to be normal if and only if it has no immediate reduct.

The immediate reduct (in the extended sense) of a hyper-type will be defined to be a hyper-type. Refer to 2) of Definition 1.1.

The hyper-types defined in (0)$\sim$(2) are normal.

(3) If $\mathfrak{s}$ and $\mathfrak{t}$ are normal, then $\mathfrak{s}\to\mathfrak{t}$ is normal. Otherwise, let $\mathfrak{s}'$ and $\mathfrak{t}'$ be the immediate reducts of $\mathfrak{s}$ and $\mathfrak{t}$ respectively. (One of them may be the given object itself.) Then $\mathfrak{s}'\to\mathfrak{t}'$ is the immediate reduct of $\mathfrak{s}\to\mathfrak{t}$.

For (4)$\sim$(6) and (8), the immediate reducts have been defined.

(7) Let ψ be the normal form of ϕ. Then the immediate reduct of $\Pi(T\langle i\rangle; \phi)$ is defined to be $\Pi(T'(i); \psi)$.

2) If $\mathfrak{t}$ is the immediate reduct of $\mathfrak{s}$, then we write $\mathfrak{s}\Rightarrow\mathfrak{t}$, and call this the reduction rule (in the extended sense).

3) A hyper-type $\mathfrak{t}$ is said to be a reduct of another one $\mathfrak{s}$ if there is a sequence of hyper-types $\mathfrak{s}_0, \cdots, \mathfrak{s}_m$ such that $\mathfrak{s}_0=\mathfrak{s}$, $\mathfrak{s}_m=\mathfrak{t}$ and $\mathfrak{s}_{n+1}$ is the immediate reduct of $\mathfrak{s}_n$, $0\le n\le m-1$.

4) The constructional complexity of a type-form $\mathfrak{s}_0$ (relative to $\mathscr{L}$-terms) will be denoted by $\#(\mathfrak{s}_0)=\omega c_1+c_2$, where $c_1, c_2<\omega$ and $c_2\ge 1$.

$$\#(\mathfrak{s}_0)=1 \quad \text{if} \quad \mathfrak{s}_0 \quad \text{is atomic} .$$

$$\#(\Lambda x\mathfrak{t})=\begin{cases}\#(\mathfrak{t})+1 & \text{if } x \text{ is of at-type} ,\\ \#(\mathfrak{t})+2 & \text{if } x \text{ is of fn-type} .\end{cases}$$

$$\#(\mathfrak{s}\to\mathfrak{t})=\#(\mathfrak{s})+\#(\mathfrak{t})$$

$$\#(\mathscr{C}[(A); (\mathfrak{t})]=\#(\mathfrak{t}_1)+\cdots+\#(\mathfrak{t}_{m+1})$$

$$\#(\Pi(\mathfrak{s}; \phi))=\#(\mathfrak{s})+1$$

$$\#(T\langle i\rangle)=2$$

$$\#(\mathscr{R}[\mathfrak{s}, \mathfrak{t}, v, x; l])=\omega$$

PROPOSITION 1.2. 1) *For any type-form $\mathfrak{s}_0(x)$ and for any $\mathscr{L}$-term ϕ of the same type as x,*

$$\#(\mathfrak{s}_0(x))=\#(\mathfrak{s}_0(\phi)) .$$

2) *If $\mathfrak{s}_0$ does not contain $\mathscr{R}$, then $\#(\mathfrak{s}_0)<\omega$.*

3) *For any hyper-type* $\mathfrak{s}_0$, *there is a unique, finite sequence of hyper-types* $\mathfrak{s}_0 = \mathfrak{r}_0, \mathfrak{r}_1, \cdots, \mathfrak{r}_m = \mathfrak{r}$, *such that* $\mathfrak{r}_{n+1}$ *is the immediate reduct of* $\mathfrak{r}_n$, $0 \leq n \leq m-1$ *and* $\mathfrak{r}$ *is normal.* $\mathfrak{r}$ *will be called the normal-form of* $\mathfrak{s}_0$ *and the sequence as above will be called the reduction sequence of* $\mathfrak{s}_0$.

Proof. 1) By induction on $\#(\mathfrak{s}_0(x))$.

3) By induction on $\#(\mathfrak{s}_0)$. It suffices to consider the cases (3)$\sim$(8) for the transition from $\mathfrak{r}_0$ to $\mathfrak{r}_1$.

(3) $\mathfrak{r}_0$ is of the form $\mathfrak{s} \to \mathfrak{t}$, and $\mathfrak{r}_1$ is of the form $\mathfrak{s}' \to \mathfrak{t}'$. (3) repeats for this case, and since $\#(\mathfrak{s}), \#(\mathfrak{t}) < \#(\mathfrak{r}_0)$, there are unique reduction sequences of $\mathfrak{s}$ and $\mathfrak{t}$, which determines the one of $\mathfrak{r}_0$. The length of the latter is the maximum of the lengths of the former two.

(4) $\mathfrak{r}_1$ is one of $\mathfrak{t}_l$, $1 \leq l \leq m+1$, and l is uniquely determined. $\#(\mathfrak{t}_l) < \#(\mathfrak{r}_0)$, and hence the proposition holds for $\mathfrak{t}_l$, and so for $\mathfrak{r}_0$ also.

(5) The immediate reduct of $T\langle i \rangle$ is normal since it is of the form in (2). So $m=1$.

(6) If $\mathfrak{s}$ is of the form $\Lambda x \mathfrak{t}(x)$, then $\#(\Pi(\mathfrak{s}; \phi)) = \#(\mathfrak{t}(x)) + 1 = \#(\mathfrak{t}(\phi)) + 1$. So $\#(\mathfrak{r}_1) < \#(\mathfrak{r}_0)$, and the induction hypothesis applies.

(7) $\Pi(T\langle i \rangle; \phi) \Rightarrow \Pi(T'(i); \psi)$,
where $T'(i)$ abbreviates (as before)

$$\Lambda l \mathscr{C}[l=0, l=1; T_1\langle i \rangle, T_2\langle i \rangle, \text{ept}].$$

So (6) applies, and

$$\Pi(T'(i); \psi) \Rightarrow \mathscr{C}[\psi=0, \psi=1; T_1\langle i \rangle, T_2\langle i \rangle, \text{ept}]$$
$$\Rightarrow \begin{cases} T_1\langle i \rangle & \text{if } \psi = 0 \\ T_2\langle i \rangle & \text{if } \psi = 1 \\ \text{ept} & \text{otherwise} \end{cases}$$

by virtue of (4). Any of the reducts is normal by virtue of (0), (2) and (5) in Definition 1.2. So $m=3$.

(8) Let $\mathfrak{s}_0$ be $\mathscr{R}[\mathfrak{s}, \mathfrak{t}, v, x; l]$. Recall that $\mathfrak{s}, \mathfrak{t}, v, x$ and l are free of $\mathscr{R}$, and hence $\mathfrak{r}_1 = \mathbf{M}(\mathfrak{s}, \mathfrak{t}, v, x; l)$ also (see the definition of $\mathbf{M}$). So

$$\#(\mathfrak{r}_1) < \omega = \#(\mathfrak{s}_0),$$

and hence $\mathfrak{r}_1$ has the unique reduction sequence, from which the reduction sequence for $\mathfrak{s}_0$ can be induced.

COROLLARY. *If we define* $\simeq$ *for hyper-types by:*
$\mathfrak{s} \simeq \mathfrak{t}$ *if and only if* $\mathfrak{s}$ *and* $\mathfrak{t}$ *have the same normal form when the bound variables are appropriately altered and the* $\mathscr{L}$*-terms are "normalized," then* $\simeq$ *becomes an equivalence relation.*

Note. The normalization of an $\mathscr{L}$-term can be explained as follows. Any $\mathscr{L}$-

term occurring in a hyper-type is of at-type, since it is either the i in (5) or $v_p(x_p)$ or l in (7), or one occurring in A_l in (4), and hence in the form $s=t$ or $s<t$. In any case, such a term can be transformed into another successively according to the defining equations in Definition 1.3 of Part I, until no such can be applied any further. We shall not go into details on this matter.

DEFINITION 1.3. 1) We define the "objects" of respective hyper-types, according to the definitions in Definition 1.1.

(0) ept represents the emptiness, which will also be denoted by ept.

(1) An object of type N_0 is a natural number.

(2) An object of hyper-type $\Lambda x\mathfrak{t}$ is a method to associate with each x_0, which is either a natural number or an $\mathscr{L}$-recursive function as the case may be, an object of hyper-type $\mathfrak{t}(x_0)$.

(3) An object Q of hyper-type $\mathfrak{s}\rightarrow\mathfrak{t}$ is a method to associate with each object R of hyper-type $\mathfrak{s}$ an object of hyper-type $\mathfrak{t}$.

The result of applying Q to R will be denoted by ap(Q; R), or $Q(R)$. If $\mathfrak{s}$ is ept, then Q is regarded as a method to specify an object of hyper-type $\mathfrak{t}$. If $\mathfrak{t}$ is ept, then Q is ept.

(4) A hyper-type $\mathscr{C}[(A); (\mathfrak{t})]$ can be identified with $\mathfrak{t}_l$ if A_l, $1\leq l\leq m+1$, where A_{m+1} is defined to be $\neg A_1\wedge\cdots\wedge\neg A_m$. (Recall that a hyper-type does not contain free variables, and $A_1, \cdots, A_m$ are $\mathscr{L}$-recursive.) So an object of hyper-type $\mathscr{C}[(A); (\mathfrak{t})]$ is in fact an object of hyper-type $\mathfrak{t}_l$.

(5) The objects of hyper-types $T\langle i\rangle$ for all $i\in\mathbf{I}_\infty$ can be defined by transfinite induction on i, according to (0)$\sim$(4) above and the reduction rule in (5). The details will be explained later in the proof.

(6) An object of hyper-type $\Pi(\mathfrak{s}; \phi)$ is defined to be an object of hyper-type $\mathfrak{t}(\phi)$.

(7) An object of hyper-type $\Pi(\mathfrak{s}; \phi)$ is defined to be an object of hyper-type $\Pi(T'(i); \phi)$.

(8) An object of hyper-type $\mathscr{R}[\mathfrak{s}, \mathfrak{t}, v, x; l]$ is an object of hyper-type $\mathbf{M}(\mathfrak{s}, \mathfrak{t}, v, x; l)$.

2) An object of a hyper-type $\mathfrak{s}$, say M, will be called a hyper-method of $\mathfrak{s}$; we may express this fact by writing $M^{\mathfrak{s}}$ for M, or $[M]=\mathfrak{s}$.

The set of all hyper-methods of $\mathfrak{s}$ will be denoted by $MT(\mathfrak{s})$, and will be called the $\mathfrak{s}$-universe. The set of all hyper-methods will be called the hyper-universe and will be denoted by MT.

PROPOSITION 1.4. 1) *The definition of hyper-methods is consistent and complete.*

2) *If* $\mathfrak{s}\simeq\mathfrak{t}$ {*see Corollary of Proposition* 1.2), *then a hyper-method of* $\mathfrak{s}$ *can be regarded as a hyper-method of* $\mathfrak{t}$, *and vice versa, and hence we can identify* $\mathfrak{s}$ *and* $\mathfrak{t}$.

Proof. We work on 1) and 2) simultaneously by induction on $\#(\mathfrak{s}_0)$ for any $\mathfrak{s}_0$. (5) can be dealt with locally by transfinite induction on i. For 2), it suffices to deal with the case where t is the immediate reduct of $\mathfrak{s}$.

(3) For $\mathfrak{s}$ and t, the objects are well-defined and can be identified with the objects of $\mathfrak{s}'$ and t' respectively, where $\mathfrak{s}'$ and t' are the immediate reducts of $\mathfrak{s}$ and t respectively. So the same holds for $\mathfrak{s}\to\mathrm{t}$.

(4) The objects of $\mathscr{C}[(A); (\mathrm{t})]$ are defined in terms of those of t_l, the immediate reduct, and hence the propositions hold.

(5) We define an object of $T\langle i\rangle$ to be one of $T'(i)$. Let $\mathbf{M}(i)$, $\mathbf{L}(i)$, $\mathbf{N}_1(i)$ and $\mathbf{N}_2(i)$ be the hyper-universes of hyper-types $T'(i)$, $T_2(i)$, $T_1\langle i\rangle$ and $T_2\langle i\rangle$ respectively. Then $\mathbf{N}_1(i)$ consists of the single method $\mathbf{N}_1(0)$ which associates with each j the emptiness; $\mathbf{L}(0)$ consists of the method $L(0)$ to associate $N_1(0)$ with each natural number; $\mathbf{N}_2(0)$ consists of the methods N_n to associate n with each sequence and $L(0)$ for all n. Now $\mathbf{M}(0)$ consists of the methods of pairing $N_1(0)$ and N_n's, $(N_1(0), N_n)$.

Suppose the $T'(j)$-universe $\mathbf{M}(j)$ have been defined for all $j<i$. Then an object of $\mathbf{N}_1(i)$ is a method to associate with each $j<i$ a sequence of objects from $\mathbf{M}(j)$ and with each $j\geq i$ the emptiness. An object of $\mathbf{L}(i)$ is a method to associate with each natural number an object of $\mathbf{N}_1(i)$. An object of $\mathbf{N}_2(i)$ is a method to associate a natural number with each pair of a sequence and an object of $\mathbf{L}(i)$. Now an object of $\mathbf{M}(i)$ is the method of pairing the objects from $\mathbf{N}_1(i)$ and $\mathbf{N}_2(i)$.

2) is obvious for this case from the discussion above.

(6) Obvious from the definition.

(7) $\Pi(T\langle i\rangle; \phi) \Rightarrow \Pi(T'(i); \psi)$

$$\Rightarrow \mathscr{C}[\psi=0, \psi=1;\ T_1\langle i\rangle, T_2\langle i\rangle, \mathrm{ept}]$$

$$\Rightarrow \begin{cases} T_1\langle i\rangle & \text{if } \psi=0 \\ T_2\langle i\rangle & \text{if } \psi=1 \\ \mathrm{ept} & \text{otherwise.} \end{cases}$$

By (6) and (4), an object of $\Pi(T'(i); \psi)$ is defined to be an object of

$$\mathscr{C}[\psi=0, \psi=1;\ T_1\langle i\rangle, T_2\langle i\rangle, \mathrm{ept}],$$

which is defined to be an object of $T_1\langle i\rangle$, $T_2\langle i\rangle$ or ept as the case may be. In any case, such an object has been defined in (5). 2) is obvious from the definition.

(8) An object of hyper-type $\mathscr{R}[\mathfrak{s}, \mathrm{t}, v, x; l]$ is defined to be an object of the hyper-type $\mathbf{M}(\mathfrak{s}, \mathrm{t}, v, x; l)$. # decreases.

§ 2. Term-forms and hyper-functionals

DEFINITION 2.1. 1) We first set the language $\mathscr{L}_{tm}$ for the term-forms to be defined subsequently. $\mathscr{L}_{tm}$ consists of the language $\mathscr{L}_{tp}$ (Definition 1.1) as well as the new symbols below.

Let n be any natural number. Then the "variable-form of the associated type-form $\mathfrak{s}$", written as $X_n^{\mathfrak{s}}$, is prepared for every $\mathfrak{s}$. (The variables in the language of $\mathscr{L}$-

terms are distinguished from the variable-forms.) Other symbols are:

$$\mathscr{J}_0,\ \eta_0,\ \zeta_0,\ \mu_0,\ \mathscr{B},\ \lambda,\ \Pi,\ \mathscr{C},\ [\ \ ]\,.$$

2) Φ, Ψ, $\cdots$ will be mostly used for the term-forms to be defined.

3) We now define the term-form of a certain type-form, free and bound variables and variable-forms in it and the associated variables (in type-forms).

In most cases, we include the variables in the variable-forms for the economy of expressions.

We write $[\Phi]=\mathfrak{s}$ if Φ is of type-form $\mathfrak{s}$.

(1) $\mathscr{J}_0$ is an atomic term-form of hyper-type N_0, N_0, $N_0 \to N_0$ without any variables or variable-forms. We replace $\mathscr{J}$ in an $\mathscr{L}$-term by $\mathscr{J}_0$, and call the resulting expression again an $\mathscr{L}$-term. Each $\mathscr{L}$-term is a term-form (of at-type or fn-type), whose variables are those in it. There are no associated variables or variable-forms in an $\mathscr{L}$-term.

In particular, a constant or a variable is said to be atomic.

(2) Each variable-form is an atomic term-form of its type-form. It is free in itself. The associated variables are those in its type-form.

(3) η_0 is an atomic term-form of hyper-type $\mathfrak{t}_\eta$, which is defined as follows via several auxiliary objects.

$$k_0(i, \gamma)=k_0 \qquad \text{if} \quad \varepsilon(i, \gamma)=(k_0, b_0, \beta_0)\,.$$

$$\text{If} \quad \gamma=\gamma_0\#\cdots\#\gamma_{m-1}\,, \quad \text{where} \quad m \geq 2,$$

and the components are arranged in the i-non-increasing order, then $m_0(i, \gamma)=m$.

$$T_3(i, \gamma, l)=\mathscr{C}[l<m_0(i, \gamma);\, k_0(i, \gamma_l),\, \mathrm{ept}]$$

$$T_4(i, \gamma)=\Lambda l T_3(i, \gamma, l)$$

$$\mathfrak{t}_\eta=\Lambda i \Lambda \gamma(T<k_0(i, \gamma)> \to T_4(i, \gamma))$$

η_0 has no variable-forms, and the associated variables are the bound ones in $\mathfrak{t}_\eta$.

(4) ζ_0 is an atomic term-form of hyper-type $\mathfrak{t}_\zeta$, which is defined by

$$\mathfrak{t}_\zeta=\Lambda i \Lambda \gamma(T<k_0(i, \gamma)> \to \Lambda \delta(T_4(i, \delta) \to N_0))\,,$$

where T_4 was defined above. ζ_0 has no variable-forms. The associated variables are the bound variables in $\mathfrak{t}_\zeta$.

(5) μ_0 is an atomic term-form of hyper-type $\mathfrak{t}_\mu$, where $\mathfrak{t}_\mu=\mathrm{Sq} \to N_0$. μ_0 has no variable-forms and no associated variables.

(6) If Φ is a term-form of type-form $\mathfrak{s} \to \mathfrak{t}$, and if Ψ is a term-form of type-form $\mathfrak{s}$, then $\Pi(\Phi; \Psi)$ is a term-form of type-form $\mathfrak{t}$. The variable-forms in it are those in Φ and Ψ, and the associated variables are those of Φ and Ψ.

(7) If Φ is a term-form of type-form $\Lambda x\, \mathfrak{t}(x)$ and if ϕ is an $\mathscr{L}$-term of the same type as x, then $\Pi(\Phi, \phi)$ is a term-form of type-form $\Pi(\Lambda x \mathfrak{t}(x); \phi)$. The variable-forms are those of Φ and ϕ; the associated variables are the ones for Φ as well as the variables in ϕ.

(8) If Φ is a term-form of type-form $T\langle i\rangle$ and if ϕ is an $\mathscr{L}$-term of at-type, then $\Pi(\Phi; \phi)$ is a term-form of type-form $\Pi(T\langle i\rangle; \phi)$. The variable-forms are those of Φ and of ϕ; the associated variables are those for Φ as well as the variables in ϕ.

(9) If X is a variable or a variable-form of type-form $\mathfrak{s}$ and if Φ is a term-form of type-form $\mathfrak{t}$, where X is not bound in Φ or $\mathfrak{t}$, then $\lambda X\Phi$ is a term-form. If X does not occur in $\mathfrak{t}$, then the associated type-form of $\lambda X\Phi$ is $\mathfrak{s}\rightarrow\mathfrak{t}$; otherwise it is $\lambda X\mathfrak{t}$. The variable-forms in $\lambda X\Phi$ are the corresponding ones in Φ except that X becomes bound. The associated variables are the ones for Φ and the variables in $\mathfrak{s}$ if the former is the case; they are the ones for Φ, where X becomes bound, if the latter is the case.

(10) Let $A_1, \cdots, A_m$ be $\mathscr{L}$-recursive formulas, mutually exclusive, and let $\Phi_1, \cdots, \Phi_m, \Phi_{m+1}$ be term-forms of type-forms $\mathfrak{t}_1, \cdots, \mathfrak{t}_m, \mathfrak{t}_{m+1}$ respectively. Then $\mathscr{C}[A_1, \cdots, A_m; \Phi_1, \cdots, \Phi_m, \Phi_{m+1}]$, or abbreviated to $\mathscr{C}[(A); (\Phi)]$, is a term-form of type-form $\mathscr{C}[(A); (\mathfrak{t})]$. The variable-forms are those in (A) and (Φ), while the associated variables are the variables in (A) and those for Φ.

(11) Let $\mathfrak{t}_0$ be the $\mathfrak{t}_\mu$ as above, let $\mathfrak{t}_1(z), \cdots, \mathfrak{t}_b(z)$ be type-forms with a free at-type variable z, let S be an fn-type variable and let m and l be at-type variables. Define the following.

$$\mathfrak{r}_d = \Lambda z \mathfrak{t}_d(z)\,, \qquad 1\leq d\leq b$$

$$\mathfrak{q}_d = \Lambda z(\Lambda s \mathfrak{t}_d(z * s)\rightarrow \mathfrak{t}_d(z))\,, \qquad 1\leq d\leq b$$

$$\mathfrak{p}_d = \mathfrak{t}_0, \mathfrak{r}_1, \cdots, \mathfrak{r}_b, \mathfrak{q}_1, \cdots, \mathfrak{q}_b$$
$$\rightarrow \Lambda m \Lambda S \mathfrak{t}_d(S\restriction m)\,, \qquad 1\leq d\leq b\,.$$

For any such type-form $\mathfrak{p}=\mathfrak{p}_d$, $\mathscr{B}^{\mathfrak{p}}$ is an atomic term-form with $\mathfrak{p}_d$ as its associated type-form, and with the associated variables which are free variables in $\mathfrak{t}_1, \cdots, \mathfrak{t}_b$. (There are no variable-forms involved.)

$\mathscr{B}^{\mathfrak{p}}$ will be called the bar constant of type-form $\mathfrak{p}=\mathfrak{p}_d$, $1\leq d\leq b$.

The case where z does not occur in $\mathfrak{t}$ can be dealt with similarly.

(12) The term-forms are defined only by (1)∼(11) above.

2) A term-form which does not have associated free variables will be called a hyper-term.

3) A hyper-term which does not have free variables or variable-forms will be called a hyper-functional.

4) If two term-forms Φ_1 and Φ_2 are defined in the same manner except that at each construction stage the associated type-forms are "equivalent" in the sense of $\simeq$ (see Corollary of Proposition 1.2), they will be "identified", and such a fact will be expressed as $\Phi_1 \simeq \Phi_2$. For example, suppose $[X]=\Pi(\mathfrak{s}, \phi)$, $\mathfrak{s}=\Lambda x \mathfrak{t}(x)$, $[Y]=\mathfrak{t}(\phi)$, ϕ is an $\mathscr{L}$-term, and Φ' is obtained from Φ by replacing X by Y everywhere. Then $\lambda X\phi \simeq \lambda Y\Phi'$.

5) An $\mathscr{L}$-term (which is a hyper-term) of type Sq will be called a sequence term; a hyper-term of type $\mathfrak{t}_\mu$ (or $\mathfrak{t}_0$: $Sq\rightarrow N_0$) will be called a super-sequential term.

6) $\lambda X_1 \cdots \lambda X_n \Phi$ may be abbreviated to $\lambda X_1 \cdots X_n \Phi$; $\Pi(\Pi(\cdots \Pi(\Pi(\Phi; \Psi_1);$

$\Psi_2)\cdots)$; $\Psi_n)$ may be abbreviated to $\Pi(\Phi; \Psi_1, \cdots, \Psi_n)$. $\Pi(\Phi; \Psi)$ will be called the projection of Φ to Ψ.

7) Let S denote a sequence term. Define $\upharpoonright$ and $*$ as follows.

$$S\upharpoonright x = \lambda y \mathscr{C}[y<x; \Pi(S; x), \text{ept}]$$

$$(S\upharpoonright x) * s = \lambda y \mathscr{C}[y<x, y=x; \Pi(S; x), s, \text{ept}]$$

8) $\lambda n \mathscr{C}[n<m; \Phi(n), \text{ept}]$ will be called the m-tuple determined by Φ, and will be written as

$$\text{TPL}(m; n, \Phi(n));$$

n is here regarded as a bound variable. The same goes with $\text{TPL}(i; j, \Phi(j)) = \lambda j \mathscr{C}[j<i, \Phi(j), \text{ept}]$.

$\lambda l \mathscr{C}[l=0, l=1; \Phi_1, \Phi_2, \text{ept}]$ may be written as (Φ_1, Φ_2) and will be called the pairing of Φ_1 and Φ_2.

9) The constructional complexity of a term-form Φ will be denoted by $\#(\Phi)$.

$$\#(\Phi) = 1 \quad \text{if} \quad \Phi \quad \text{is atomic}.$$

$$\#(\Pi(\Phi; \Psi)) = \#(\Phi) + \#(\Psi)$$

$$\#(\lambda X \Phi) = \#(\Phi) + 1$$

$$\#(\mathscr{C}[(A); (\Phi)]) = \#(\Phi_1) + \cdots + \#(\Phi_{m+1})$$

PROPOSITION 2.1. 1) *Term-forms are closed under substitutions of* $\mathscr{L}$*-terms (of the same types) for the free variables, presuming that there are no "clashes" of variables.*

2) *Term-forms are closed under substitutions of term-forms for the free variable-forms of the same type-forms (up to the equivalence* $\simeq$*).*

3) *In any term-form, any symbol occurs only finitely many times.*

§ 3. Semantics of term-forms

We shall henceforth work in the hyper-universe (Definition 1.3), and assume the "continuity principle" CNPR for the objects of type $t_0 = t_\mu$ (which will be called the super-sequential methods).

CNPR(L, S):

$$\forall S'(S'\upharpoonright \text{ab}(L; S) = S\upharpoonright \text{ap}(L; S) \vdash \text{ap}(L; S') = \text{ap}(L; S))$$

This is an informal principle which has a formal appearance for the brevity of expression.

DEFINITION 3.1. 1) Assignment to the variables (of at-type and of fn-type). Let $x_1, \cdots, x_n$ denote a finite sequence of distinct variables, and let $\phi_1, \cdots, \phi_n$ be a finite sequence of (arbitrary) closed $\mathscr{L}$-terms (including the numerals) such that ϕ_l is of the same type as x_l. Then $(x_1/\phi_1, \cdots, x_n/\phi_n)$ will express the assignment of ϕ_l to x_l, $1 \leq l \leq n$, and will be called a first assignment. It will be denoted by **a** etc. for short.

For any type-form t, **a**t will express the result of substitution of ϕ_l for x_l in t, $1 \leq l \leq n$, presuming that x_l is not bound in t and there are no "clashes" of variables. (x_l needs not occur in t, and $x_1, \cdots, x_n$ need not exhaust all the free variables in t.) **a**t will be called the assignment **a** for t.

If $x_1, \cdots, x_n$ exhaust all the free variables in t, then we say that **a**t (or simply **a**) is a complete assignment for t. Note that **a**t is again a type-form, and will be a hyper-type if **a** is complete for t.

Let Φ be a term-form. $\mathbf{a}\Phi$ will denote the result of substitution of ϕ_l for x_l a variable in Φ as well as for x_l an associated variable of Φ, presuming that there occur no clashes of variables and that x_l is not bound. $\mathbf{a}\Phi$ will be called a first assignment for Φ.

If $x_1, \cdots, x_n$ exhaust all the free associated variables of Φ, then $\mathbf{a}\Phi$ will become a hyper-term and will be called a complete first assignment for Φ.

2) Assignment to the variable-forms (including variables). Let $Y_1, \cdots, Y_m$ be a finite sequence of distinct variable-forms of hyper-types, and let $Q_1, \cdots, Q_m$ be a finite sequence of hyper-methods such that Q_l is of the same hyper-type as Y_l, $1 \leq l \leq m$. Then $(Y_1/Q_1, \cdots, Y_m/Q_m)$ will express the assignment of Q_l to Y_l, $1 \leq l \leq m$. We shall call any such tuple a second assignment, and will denote it by a letter such as **b**. For any hyper-term Ψ, $\mathbf{b}\Psi$ will express assignment of Q_l to Y_l in Ψ, presuming that Y_l is not bound in Ψ, $1 \leq l \leq m$, and will be called the assignment **b** for Ψ. If $Y_1, \cdots, Y_m$ exhaust all the free variable-forms in Ψ, then we say that **b** is a complete assignment for Ψ. (Note that here **b** is merely an assignment, and not a substitution.)

3) If Φ is a term-form and **a** is complete for Φ, then $\mathbf{ba}\Phi$ can be defined according to 2) above, and we denote the successive assignments of **a** and **b** by **ba**. If **b** is complete for $\mathbf{a}\Phi$, then we say that **ba** is a complete assignment for Φ.

COROLLARY.

$$\mathbf{a}(\mathfrak{s} \to \mathrm{t}) = \mathbf{a}\mathfrak{s} \to \mathbf{a}\mathrm{t}\,; \qquad \mathbf{a}\Lambda x\mathrm{t} = \Lambda x\mathbf{a}\mathrm{t}\,;$$

$$\mathbf{a}\mathscr{C}[(A); (\mathrm{t})] = \mathscr{C}[(\mathbf{a}A); (\mathbf{a}\mathrm{t})]\,; \qquad \mathbf{a}T\langle i\rangle = T\langle \mathbf{a}i\rangle\,;$$

$$\mathbf{a}\Pi(\mathfrak{s}; \phi) = \Pi(\mathbf{a}\mathfrak{s}; \mathbf{a}\phi)\,; \qquad \mathbf{a}X^{\mathfrak{s}} = X^{\mathbf{a}\mathfrak{s}}\,.$$

DEFINITION 3.2. Let **ba** be a complete assignment for Φ a term-form of type-form $\mathfrak{s}$. The "interpretation of Φ at **ba**", written as $I(\Phi, \mathbf{b}, \mathbf{a})$, or $I\Phi$ for short, will be defined to be a method of hyper-type $\mathbf{a}\mathfrak{s}$ determined by **b**. It is defined according to the construction of Φ in Definition 2.1, in a manner similar to the reduction in Definition 4.5 of [10].

(1) Closed $\mathscr{L}$-terms can be interpreted naturally as objects of type N_0 or $N_0, \cdots, N_0 \to N_0$ (see Definition 1.1 of Part I).

(2) Φ is a variable-form $X^{\mathfrak{s}}$. Since **ba** is complete, **b** determines an assignment Q

to $\mathbf{a}X^{\mathfrak{s}}$, or $X^{\mathbf{a}\mathfrak{s}}$. So $I(\Phi, \mathbf{b}, \mathbf{a})=Q$.

(3)~(5) We assume the hyper-methods of hyper-types t_η, t_ζ and t_μ, corresponding respectively to η_0, ζ_0 and μ_0, which we write respectively η^*, ζ^* and μ^*.

(6) Φ is of type-form $\mathfrak{s}\to\mathrm{t}$ and Ψ is of type-form $\mathfrak{s}$. Define

$$I(\Pi(\Phi; \Psi), \mathbf{b}, \mathbf{a})=\mathrm{ap}(I\Phi; I\Psi)$$

(see (3) of Definition 1.3), which is of hyper-type $\mathbf{a}\mathrm{t}$.

(7) Φ is of type-form $\Lambda x\mathrm{t}$ and ϕ is an $\mathscr{L}$-term.

$$I(\Pi(\Phi; \phi), \mathbf{b}, \mathbf{a})=\mathrm{ap}(I\Phi; I\phi)\,,$$

which is of hyper-type $(\mathbf{a}, x/I\phi)\mathrm{t}$.

(8) Φ is of type-form $T\langle i\rangle$ and ϕ is an at-type $\mathscr{L}$-term.

$$I(\Pi(\Phi; \phi), \mathbf{b}, \mathbf{a})=\mathrm{ap}(I\Phi; I\phi)\,,$$

which is of hyper-type

$$\mathscr{C}[I\phi=0,\ I\phi=1;\ T_1\langle \mathbf{a}i\rangle,\ T_2\langle \mathbf{a}i\rangle,\ \mathrm{ept}]\,;$$

that is $T_1\langle \mathbf{a}i\rangle$ if $I\phi=0$, $T_2\langle \mathbf{a}i\rangle$ if $I\phi=1$, and ept otherwise.

(9) Consider $\lambda X\Phi$ where X is of type-form $\mathfrak{s}$ and Φ is of t, free of X. For each Q a hyper-method of $\mathbf{a}\mathfrak{s}$,

$$Q'=I(\Phi,(\mathbf{b}, \mathbf{a}X/Q), \mathbf{a})$$

has been defined as a hyper-method of hyper-type $\mathbf{a}\mathrm{t}$. ($\mathbf{a}X$ is the variable-form of hyper-type $\mathbf{a}\mathfrak{s}$ induced from X.) So define $I(\lambda X\Phi, \mathbf{b}, \mathbf{a})$ to be the hyper-method (of hyper-type $\mathbf{a}\mathfrak{s}\to\mathbf{a}\mathrm{t}$) which associates with any such Q the Q' as above, so that

$$\mathrm{ap}(I\lambda X\Phi; Q)=Q'\,.$$

Suppose next X occurs (free) in t. Then $\lambda X\Phi$ is of type-form $\Lambda X\mathrm{t}$. For each ϕ_0 an object of the same type as X, put $\mathbf{c}=(\mathbf{a}, X/\phi_0)$. Then

$$Q'(\phi_0)=I(\Phi, \mathbf{b}, \mathbf{c})$$

has been defined as a hyper-method of $\mathbf{c}\mathrm{t}$. Define $I(\lambda X\Phi, \mathbf{b}, \mathbf{a})$ to be the method Q_0 so that

$$\mathrm{ap}(Q_0; \phi_0)=Q'(\phi_0)$$

for each ϕ_0. Q_0 is of hyper-type $\mathbf{a}\Lambda X\mathrm{t}=\Lambda X\mathbf{a}\mathrm{t}$.

(10) First note that for an $\mathscr{L}$-recursive A, $\mathbf{ba}A$ can be naturally interpreted. So, given $\mathscr{C}[(A); (\Phi)]$, an l can be singled out so that $\mathbf{a}A_l$ holds, $1\le l\le m+1$, where A_{m+1} is understood to be $\neg A_1\wedge\cdots\wedge\neg A_m$. $I(\mathscr{C}[(A); (\Phi)], \mathbf{b}, \mathbf{a})$ is then defined to be $I\Phi_l$, which is of hyper-type $\mathbf{a}\mathrm{t}_l$ ($\simeq\mathscr{C}[(A); (\mathrm{t})]$). (See (4) of Definition 1.1 and the corollary of Proposition 1.2.)

(11) Let us first note that, if Q is a hyper-method of $(\mathrm{t}_1\to\cdots(\mathrm{t}_{k-1}\to(\mathrm{t}_k\to\mathrm{t}))\cdots)$, then it can be regarded as a method to associate with any k-tuple of hyper-methods of

$\mathfrak{t}_1, \cdots, \mathfrak{t}_k$ a hyper-method of $\mathfrak{t}$, and this characterization determines Q. So, for $I\mathscr{B}^{\mathfrak{p}} = I(\mathscr{B}^{\mathfrak{p}}, \mathbf{d}, \mathbf{a})$, it suffices to determine $I(\Pi(\mathscr{B}^{\mathfrak{p}}; \mathscr{X}), \mathbf{d}, \mathbf{a})$, where $\mathscr{X}$ consists of the variable forms, say,

$$L,\ \Phi_1,\ \Phi_2,\ x,\ S$$

of type-forms respectively

$$\mathfrak{t}_0,\ \mathfrak{r},\ \mathfrak{q},\ N_0,\ \mathrm{S}\mathfrak{q}\ ,$$

$\mathfrak{r}$ being an abbreviation of $\mathfrak{r}_1, \cdots, \mathfrak{r}_b$ and $\mathfrak{q}$ being an abbreviation of $\mathfrak{q}_1, \cdots, \mathfrak{q}_b$. Φ_1 and Φ_2 stand for sequences of variable-forms. The $\mathfrak{p}$ above stands for $\mathfrak{p}_d$. $\mathbf{da}$ is supposed to be an arbitrary complete assignment for $\Pi(\mathscr{B}^{\mathfrak{p}}; \mathscr{X})$. (We shall omit $\mathfrak{p}$ in $\mathscr{B}^{\mathfrak{p}}$.) Now $I(\Pi(\mathscr{B}; \mathscr{X}), \mathbf{d}, \mathbf{a})$ is defined as follows.

(1°) $$I\Pi(\Phi_1; S\restriction x) \quad \text{if} \quad IL(S) \leq \mathbf{d}x\ ,$$

(2°) $$I\Pi(\Phi_2; S\restriction x, \lambda s\Pi(\mathscr{B}; L, \Phi_1, \Phi_2, x+1, (S\restriction x)*s)) \quad \text{if} \quad \mathbf{d}x < IL(S)\ .$$

Here, for example, $I\Pi(\Phi_1; S\restriction x)$ in fact stands for $T\Pi(\Phi_{1,d}; S\restriction x)$ when $\mathscr{B}$ is of $\mathfrak{p}_d$, $1 \leq d \leq b$.

This completes the definition of I.

PROPOSITION 3.1. *The I above is well-defined.*

Proof. By induction on # according to 3) of Definition 2.1.

(1) The computations of $\mathscr{L}$-terms are assumed.

(2) Let X be a variable-form, where $[X]=\mathfrak{s}$. Then $\mathbf{ba}X$ determines a hyper-method of $\mathbf{a}\mathfrak{s}$.

(3)~(5) The objects are assumed.

(6) $\#(\Phi)$, $\#(\Psi) < \#(\Pi(\Phi; \Psi))$, and hence $I\Phi$ and $I\Psi$ are well-defined.

(7) $I\Phi$ is well-defined to be a hyper-method of $\Lambda x\mathbf{a}\mathfrak{t}$. So $\mathrm{ap}(I\Phi; I\phi)$ is defined for every $I\phi$.

(8) Similarly.

(9) $I\Phi$ is well-defined for any appropriate assignment, and hence the correspondence of Q' to Q is well-defined.

(10) $\mathbf{ba}A_l$ is well-defined, and hence an l can be chosen and $I\Phi_l$ is well-defined.

(11) By the induction hypothesis, $IL(S)$ is well-defined, and so the two case conditions in (1°) and (2°) are well-determined, exclusive and exhaustive. To show that

$$I(\Pi(\mathscr{B}^{\mathfrak{p}}; \mathscr{X}), \mathbf{d}, \mathbf{a})$$

is well-defined, we use an informal reasoning of the bar theorem. Fix $\mathbf{b}$ an assignment to the variables of $\mathscr{X}$, and $\mathbf{a}$. Let $R(z)$ and $A(z)$ be the properties of z described below.

$$R(z)\colon \lg(z) \geq L(z).$$

$$A(z)\colon \forall \mathbf{d} = (\mathbf{b}, S/S_0, x/x_0)(S_0\restriction \lg(z) = z \wedge x_0 \geq \lg(z) \vdash$$

$$\text{“}I(\Pi(\mathscr{B};\ \Xi, x, S), \mathbf{d}, \mathbf{a}) \text{ is well-defined”})\ .$$

If we can show that Hyp1 $\sim$ Hyp4 in $BI(R, A)$ hold (see Definition 1.3 of Part I), then $A(\langle\ \ \rangle)$ can be concluded; that is,

$$\forall \mathbf{d} \quad \text{as above}, \quad I(\Pi(\mathscr{B};\ \Xi, x, S), \mathbf{d}, \mathbf{a}) \quad \text{is well-defined},$$

which establishes the well-definedness of the interpretation under consideration.

Hyp1. $$\forall S \forall l(\lg(S\restriction l) \geq L(S\restriction l) \vdash \forall m > l(\lg(S\restriction m) \geq L(S\restriction m)))$$

Suppose $\lg(S\restriction l) \geq L(S\restriction l)$. If $m > l$, then obviously $\lg(S\restriction m) \geq L(S\restriction l)$. $(S\restriction m)\restriction L(S\restriction l) = (S\restriction l)\restriction L(S\restriction l)$. So, by CNPR$(L, S\restriction l)$, $L(S\restriction l) = L(S\restriction m)$, and hence $\lg(S\restriction m) \geq L(S\restriction m)$. (We have treated the formal letters such as L, S as if they were the objects of the hyper-universe. This kind of abuse of notation will be practiced occasionally.

Hyp2. $$\forall S \exists l(\lg(S\restriction l) \geq L(S\restriction l)).$$

Put $l = L(S)$. Then by CNPR$(L, S\restriction l)$, $L(S\restriction l) = L(S)$. So

$$\lg(S\restriction l) = L(S) = L(S\restriction l).$$

Hyp3. $$\begin{aligned} &\forall z(\text{sqn}(z) \wedge \lg(z) \geq L(z) \\ &\vdash \forall S_0 \forall x_0(S_0\restriction \lg(z) = z \wedge x_0 \geq \lg(z) \\ &\vdash I(\Pi(\mathscr{B};\ \Xi, x, S), \mathbf{d}, \mathbf{a}) \text{ is well-defined})) \end{aligned}$$

Put $\lg(z) = n$, and assume $n \geq L(z)$, $S\restriction n = z$ and $x \geq n$. By CNPR(L, z), $L(S) = L(z)$. $x \geq n$ implies $x \geq L(z) = L(S)$. So (1°) holds for x, L and S, and hence $I(\Pi(\mathscr{B};\ \Xi, x, S), \mathbf{d}, \mathbf{a})$ is directly defined.

Hyp4. $$\begin{aligned} &\forall z(\text{sqn}(z) \wedge \forall s \forall S' \forall x' \\ &(S'\restriction \lg(z * s) = z * s \wedge x' \geq \lg(z * s) \\ &\vdash \text{“}I(\Pi(\mathscr{B};\ \Xi, y, S), (\mathbf{b}, S/S', y/x'), \mathbf{a}) \text{ is well-defined”})) \\ &\vdash \forall S_0 \forall x_0(S_0\restriction \lg(z) = z \wedge x_0 \geq \lg(z) \\ &\vdash \text{“}I(\Pi(\mathscr{B};\ \Xi, x, S), \mathbf{d}, \mathbf{a}) \text{ is well-defined”})) \end{aligned}$$

Assume sqn(z), $\lg(z) = n$,

$$\begin{aligned} &\forall s \forall S' \forall x'(S'\restriction n+1 = z * s \wedge x' \geq n+1 \\ &\vdash \text{“}I(\Pi(\mathscr{B};\ \Xi, y, S), (\mathbf{b}, S/S', y/x'), \mathbf{a}) \text{ is well-defined”}) \end{aligned}$$

and $S_0\restriction n = z \wedge x_0 \geq n$.

Suppose $x_0 \geq n+1$, and put $s = S_0(n)$, $S' = S_0$ and $x' = x_0$. Then

$$S'\restriction n+1 = S_0\restriction n+1 = S_0\restriction n * S_0(n) = z * s$$

and $x' \geq n+1$. So by the hypothesis $I(\Pi(\mathscr{B};\ \Xi, x, S), \mathbf{d}, \mathbf{a})$ is well-defined.

Suppose $x_0 = n$. For each s, put $S' = z * s$, and $y_0 = x_0 + 1$. Then $S'\restriction n+1 = z * s$

and $y_0 \geq n+1$. So by the hypothesis

$$I(\Pi(\mathscr{B};\ \Xi, y, S), (\mathbf{b}, S/(S'\upharpoonright x_0+1)*s), \mathbf{a})$$

is well-defined, and hence

$$I(\lambda s\Pi(\mathscr{B};\ \Xi, y+1, (S\upharpoonright y+1)*s), (\mathbf{b}, S/S', y/x_0), \mathbf{a})$$

is well-defined. So by (2°)

$$I(\Pi(\mathscr{B};\ \Xi, x, S), \mathbf{d}, \mathbf{a})$$

is well-defined.

PROPOSITION 3.2. 1) *Computable functionals of finite types are special cases of the interpretations of hyperfunctionals.*

2) $I\Pi(\mathrm{TPL}(m; n, \Phi(n)); l) = I\mathscr{C}[l<m; \Phi(l), \mathrm{ept}]$

$$= \begin{cases} I\Phi(l) & \text{if } Il < \mathrm{Im}\,, \\ \mathrm{ept} & \text{otherwise}\,. \end{cases}$$

That is $\Pi(\mathrm{TPL}(m; n, \Phi(n)); l)$ functions just as the projection of Φ(if $l<m$). We may use TPL as if it were a primitive symbol.

§ 4. Formula-forms and hyper-formulas

DEFINITION 4.1. 1) Let $\mathscr{L}_0$ be the language we are to consider. $\mathscr{L}_0$ contains the language of $\mathscr{L}$-recursive formulas (see Definition 1.1 of Part I) except that the function symbol $\mathscr{I}$ is replaced by a new function symbol $\mathscr{I}_0$ and the predicate symbol lss is replaced by a new predicate symbol Σ; $\mathscr{L}_0$ also contains the languages $\mathscr{L}_{tp}$ and $\mathscr{L}_{tm}$ (see Sections 1 and 2), as well as the new predicate constants Δ and Θ. The logical connectives accepted are $\wedge$, $\vdash$ and $\forall$.

Note. 1) $\mathscr{L}_0$ contains all the primitive recursive functions and predicates which are necessary to formulate the elementary theory of ordinal diagrams. For example, "σ is an i-section of α" can be formulated in $\mathscr{L}_0$.

2) $=$ is allowed in $\mathscr{L}_0$ only for the type N_0.

3) Let us emphasize that $\vee$ and $\exists$ are not accepted. $\neg A$ will be used to abbreviate $A \vdash 0=1$. The $\mathscr{L}$-recursive formulas can be expressed in terms of $\wedge$ and $\vdash$ (and $\neg$).

4) $\rightarrow$may be used for $\vdash$ (see Definitions 1.1 and 1.2 of Part I).

2) The formula-forms of $\mathscr{L}_0$ are defined as follows. The associated type-forms, free and bound variables and variable-forms in a formula-form and the associated variables (in type-forms) are defined in a manner similar to those in a term-form (Definition 2.1); so we do not repeat them.

(1) The $\mathscr{L}$-recursive formulas (modified as above) are formula-forms.

(2) For any pair of term-forms Φ and Ψ of type N_0, $\Phi=\Psi$ is an atomic

formula-form.

(3) $\Delta(i, \Phi, \alpha)$ is an atomic formula-form, where i and α are at-type $\mathscr{L}$-terms and Φ is a term-form.

(4) $\Theta(i, \Psi, \gamma)$ is an atomic formula-form, where i and γ are at-type $\mathscr{L}$-terms and Ψ is a term-form.

(5) $\Sigma(i; j, \gamma, \delta)$ is an atomic formula-form.

(6) The class of formula-forms are closed with respect to $\wedge$ and $\vdash$.

(7) If A is a formula-form and X is a variable (-form) which is not bound in A or in the associated type-forms, then $\forall XA$ is a formula-form. X becomes bound in the new formula-form.

(8) Formula-forms are defined only by (1)$\sim$(7).

3) A formula-form which does not have associated free variables will be called a hyper-formula.

§5. The axiom set

DEFINITION 5.1. The axiom set $\mathscr{A}_0$ of the language $\mathscr{L}_0$ consists of ($\mathscr{A}_0$-1)$\sim$($\mathscr{A}_0$-7) below.

($\mathscr{A}_0$-1) The reduction rules of type-forms in Definition 1.1.

($\mathscr{A}_0$-2) The axiom on Δ. It is of the form $\Delta_1 \wedge \Delta_2$, where Δ_1 and Δ_2 are defined subsequently. We first introduce some abbreviated notations.

Recall that $\mathbf{I}(j)$, $\mathbf{O}(\alpha)$ etc. are primitive recursive (see Section 1 of Part I). We shall agree that $i, j, k, \cdots$ are used for the elements of $\mathbf{I}$, and $\alpha, \beta, \cdots$ are used for the elements of $\mathbf{O}$. So, for example, $\forall jB(j)$ in fact stands for $\forall j(\mathbf{I}(j) \vdash B(j))$ and $\forall\alpha C(\alpha)$ stands for $\forall\alpha(\mathbf{O}(\alpha) \vdash C(\alpha))$. Now let P_1 and P_2 express the following.

$$P_1(j, \sigma, \alpha)\text{: } \sigma \text{ is a } j\text{-section of } \alpha\,.$$

$$P_2(i, \alpha, S)\text{: } S \text{ is an } i\text{-decreasing sequence (of ordinal diagrams) led by } \alpha\,.$$

Assume $[X]=T\langle i\rangle$ and $[Y]=T_2(i)$.

$$\Delta_1\text{: } \forall i\forall\alpha\forall X\{\Delta(i, X, \alpha)\rightarrow\forall j<i\forall\sigma(P_1(j, \sigma, \alpha)\vdash \Delta(j, \Pi(X; 0, j, \sigma), \sigma)) \wedge \forall S\forall Y(P_2(i, \alpha, S) \wedge \forall n\forall j<i\forall\sigma(P_1(j, \sigma, \Pi(S; n)) \vdash\Delta(j, \Pi(Y; n, j, \sigma), \sigma) \vdash\forall n\geq\Pi(X; 1, S, Y)(\Pi(S; n)=\mathrm{ept}))\}$$

Notice that $\Pi(X; 0, j)$ is of type-form

$$\mathscr{C}[j<i; N_0\rightarrow T\langle j\rangle, \mathrm{ept}]\,,$$

and hence $\Pi(X; 0, j, \sigma)$ is of type-form $T\langle j\rangle$ under the assumption of $j<i$. Notice also that $T_2(i)=N_0\rightarrow T_1\langle i\rangle$, and hence $\Pi(Y; n, j, \sigma)$ is of type-form $T\langle j\rangle$ under the

assumption of $j<i$.

Suppose $[Z]=T_1\langle i\rangle$, $[V]=T_2\langle i\rangle$ and $[U]=T_2(i)$.

$$\Delta_2\text{: } \forall i\forall\alpha\forall Z\forall V\{\forall j<i\forall\sigma(P_1(j,\sigma,\alpha)\vdash\Delta(j,\Pi(Z;j,\sigma),\sigma))$$
$$\wedge\forall S\forall U(P_2(i,\alpha,S)\wedge\forall n\forall j<i\forall\sigma(P_1(j,\sigma,\Pi(S;n))$$
$$\vdash\Delta(j,\Pi(U;n,j,\sigma),\sigma))\vdash\forall n\geq\Pi(V;S,U)(\Pi(S;n)=\text{ept}))$$
$$\rightarrow\Delta(i,(Z,V),\alpha)\}\,,$$

where (Z, V) denotes the pairing of Z and V (see 8) of Definition 2.1).

($\mathscr{A}_0$-3) The axiom on Θ. It is of the form $\Theta_1\wedge\Theta_2$, where Θ_1 and Θ_2 are defined subsequently. We first introduce some abbreviations.

$A_1(i,\gamma)$: $\varepsilon(i,\gamma)$ is of the form (k_0, b_0, β_0) where $k_0<i$.

$A_2(i,\gamma)$: γ is of the form $\gamma_0\#\cdots\#\gamma_{m-1}$ (arranged in the i-non-increasing order) where $m\geq 2$.

$A_3(i,\gamma)$: γ is of the form (j_0, c_0, δ_0) where $j_0\geq i$.

As was agreed in (3) of Definition 2.1, we shall use the notations below.

$$k_0(i,\gamma)=k_0\,,\qquad b_0(i,\gamma)=b_0\,,\qquad \beta_0(i,\gamma)=\beta_0\,,$$
$$m_0(i,\gamma)=m_0\,,\qquad \gamma_l=\gamma_0(i,\gamma,l)\,,$$
$$j_0(i,\gamma)=j_0,\qquad c_0(i,\gamma)=c_0\,,\qquad \delta_0(i,\gamma)=\delta_0\,.$$

We shall next define some type-forms. $\mathfrak{s}=\mathfrak{t}$ will indicate that $\mathfrak{s}$ abbreviates $\mathfrak{t}$.

$$\mathfrak{s}=\mathfrak{s}(i,\gamma)=T\langle i\rangle$$
$$\mathfrak{r}(i,\gamma,\Xi_1)=\text{TPL}(m_0;l,\Xi_1)\,,$$

where Ξ_1 is a parameter. Let Ξ_2 be another parameter.

$$\mathfrak{t}=\mathfrak{t}(i,\gamma,\Xi_1,\Xi_2)=\mathscr{C}[A_1(i,\gamma),A_2(i,\gamma),A_3(i,\gamma)\,;\ \mathfrak{s}(i,\gamma),\mathfrak{r}(i,\gamma,\Xi_1),\Xi_2,\text{ept}]\,,$$
$$v_1(x_1)=v_1(i,\gamma,l)=\#(\gamma_0(i,\gamma,l))\,,$$
$$v_2(x_2)=v_2(i,\gamma)=\#(\delta_0(i,\gamma))\,,$$

where $\#(\gamma)$ denotes the constructional complexity of γ. We shall abbreviate v_1, v_2 to v and i, γ, l to x, and define $\mathbf{M}$ and $\mathbf{N}$ as follows.

$$\mathbf{M}(\mathfrak{s},\mathfrak{t},v,x;0)=\mathfrak{s}$$
$$\mathbf{N}_p(\mathfrak{s},\mathfrak{t},v,x;n)=\mathbf{M}(\mathfrak{s},\mathfrak{t},v,x;v_p^*(n,x_p))\,,\qquad p=1,2$$
$$\mathbf{M}(\mathfrak{s},\mathfrak{t},v,x;n)=\mathfrak{t}(i,\gamma,\mathbf{N}(1,n),\mathbf{N}(2,n))$$

(See (8) of Definition 1.1)

If we can show that

(*) $\mathbf{M}(\mathfrak{s}, \mathrm{t}, v, x; n)$ (abbreviated to $\mathbf{M}(n)$) is a type-form for each n,

then by definition $\mathscr{R}[\mathfrak{s}, \mathrm{t}, v, x; n]$ is a type-form, and

$$\mathscr{R}[\mathfrak{s}, \mathrm{t}, v, x; n] \Rightarrow \mathbf{M}(\mathfrak{s}, \mathrm{t}, v, x; n) .$$

Put $\rho_0(i, \gamma) = \mathscr{R}[\mathfrak{s}, \mathrm{t}, v, x; \#(\gamma)]$.

Assuming (*), we now present (assuming $[X] = \rho_0(i, \gamma)$)

$$\begin{aligned} \Theta_1\colon \forall i \forall \gamma \forall X\{\Theta(i, X, \gamma) \rightarrow &(A_1(i, \gamma) \vdash \Delta(k_0(i, \gamma), \Pi(X; 0, k_0(i, \gamma), \beta_0(i, \gamma)))) \\ &\wedge (A_2(i, \gamma) \vdash \forall l(0 \leq l < m_0 \vdash \Theta(i, \Pi(X; l), \gamma_0(i, \gamma, l)) \\ &\wedge (A_3(i, \gamma) \vdash \Theta(i, X, \delta_0(i, \gamma)))\} , \end{aligned}$$

where $\forall i$ is specified by $\forall i(\mathbf{I}(i) \vdash \cdots)$, while $\forall \gamma$ is unspecified.

We shall show (*) by induction on n.

$n = 0$. $\mathbf{M}(0) = \mathfrak{s}$, which is a type-form .

$n > 0$. Suppose $\mathbf{M}(m)$ has been proven to be a type-form for every $m < n$. So, $\mathbf{N}_p(n) = \mathbf{M}(v_p^*(n))$ is a type-form since $v_p^*(n) < n$, $p = 1, 2$.

$$\begin{aligned} \mathbf{M}(n) &= \mathrm{t}(\mathbf{N}_1(n), \mathbf{N}_2(n)) \\ &= \mathscr{C}[A_1(i, \gamma), A_2(i, \gamma), A_3(i, \gamma); \mathfrak{s}(i, \gamma), \mathfrak{r}(i, \gamma, \mathbf{N}_1(n)), \mathbf{N}_2(n), \mathrm{ept}] \end{aligned}$$

If $A_1(i, \gamma)$, then

$$\mathbf{M}(n) = \mathfrak{s} ,$$

which is a type-form. If $A_2(i, \gamma)$, then

$$\mathbf{M}(n) = \mathfrak{r}(i, \gamma, \mathbf{N}_1(n)) = \mathrm{TPL}(m_0; l, \mathbf{N}_1(n)) ,$$

where $\mathbf{N}_1(n) = \mathbf{M}(v_1^*(n, i, \gamma, l))$, which is a type-form involving l. So $\mathrm{TPL}(m_0; l, \mathbf{N}_1(n))$ is a type-form. If $A_3(i, \gamma)$, then

$$\mathbf{M}(n) = \mathbf{N}_2(n) ,$$

which is a type-form by the induction hypothesis. This proves (*).

We shall next show the following.

If $A_1(i, \gamma)$, then

$$\rho_0(i, \gamma) \Rightarrow T\langle i \rangle ,$$

where $k_0 = k_0(i, \gamma) < i$;

if $A_2(i, \gamma)$ and $0 \leq l < m_0$, then $\Pi(\rho_0(i, \gamma); l) \simeq \rho_0(i, \gamma_l)$;

if $A_3(i, \gamma)$, then $\rho_0(i, \gamma) = \rho_0(i, \delta_0)$.

For, suppose $A_1(i, \gamma)$. Then $\varepsilon(i, \gamma) = (k_0, b_0, \beta_0)$ where $k_0 < i$, and

$$\rho_0(i, \gamma) \Rightarrow \mathbf{M}(0) = T\langle i \rangle .$$

If $A_2(i, \gamma)$ and $0 \leq l < m_0$, then

$$\begin{aligned} \rho_0(i, \gamma) &\Rightarrow \mathbf{M}(\#(\gamma)) = \mathrm{t}(\mathbf{N}(1, \#(\gamma)), \mathbf{N}(2, \#(\gamma))) \\ &\Rightarrow \mathrm{TPL}(m_0; l, \mathbf{N}(1, \#(\gamma))) , \end{aligned}$$

and so

$$\begin{aligned} \Pi(\rho_0(i, \gamma); l) &\Rightarrow \mathbf{M}(v_1^*(\#(\gamma), i, \gamma, l)) \\ &= \mathbf{M}(\#(\gamma_l)) \simeq \rho_0(i, \gamma_l) . \end{aligned}$$

If $A_3(i, \gamma)$, then

$$\begin{aligned} \rho_0(i, \gamma) &\Rightarrow \mathbf{N}(2, \#(\gamma)) = \mathbf{M}(v_2^*(n, i, \gamma)) \\ &= \mathbf{M}(\#(\delta_0)) \simeq \rho_0(i, \delta_0) . \end{aligned}$$

From these, we can see that in Θ_1 the type-form of Y in $\Theta(i, Y, \sigma)$ is $\rho_0(i, \sigma)$ and the type-form of X in $\Delta(k_0, X, \beta_0)$ is $T\langle k_0 \rangle$.

Let us next determine Θ_2. First define the following.

$$\rho_1(i, \gamma) = \mathscr{C}[A_1(i, \gamma); \mathfrak{s}, \mathrm{ept}]$$

$$\rho_2(i, \gamma) = \mathscr{C}[A_2(i, \gamma); \mathrm{TPL}(m_0; l, \rho_0(i, \gamma_l)), \mathrm{ept}]$$

$$\rho_3(i, \gamma) = \mathscr{C}[A_3(i, \gamma); \rho_0(i, \gamma), \mathrm{ept}]$$

$$\rho_4(i, \gamma) = N_0, \rho_1(i, \gamma), N_0, \rho_2(i, \gamma), \rho_3(i, \gamma) \rightarrow \rho_0(i, \gamma)$$

Suppose $[X_1] = \rho_1(i, \gamma)$, $[X_2] = \rho_2(i, \gamma)$ and $[X_3] = \rho_3(i, \gamma)$.

$$\begin{aligned} W_0(i, \gamma) = \lambda u X_1 z X_2 X_3 \\ &\mathscr{C}[u = 0 \wedge A_1(i, \gamma), u > 0 \wedge z = 0 \wedge A_2(i, \gamma) , \\ &u > 0 \wedge z > 0 \wedge A_3(i, \gamma); X_1, \Phi_0(i, \gamma, X_2), X_3, \mathrm{ept}] , \end{aligned}$$

where

$$\Phi_0(i, \gamma, X_2) = \mathrm{TPL}(m_0; l, \Pi(X_2; l)) .$$

$$[W_0] = \rho_4(i, \gamma)$$

$$\begin{aligned} \Theta_2\colon & \forall i \forall \gamma \forall u X_1 \forall z \forall X_2 \forall X_3 [\{u = 0 \vdash (A_1(i, \gamma) \wedge \Delta(k_0, X_1, \beta_0))\} \\ & \wedge \{u > 0 \vdash (z = 0 \vdash A_2(i, \gamma) \wedge \forall l (0 \leq l < m_0 \vdash \Theta(i, \Pi(X_2; l), \gamma)) \\ & \wedge (z > 0 \vdash A_3(i, \gamma) \wedge \Theta(i, X_3, \delta_0))\} \rightarrow \Theta(i, \Pi(W_0(i, \gamma); u, X_1, z, X_2, X_3), \gamma)] \end{aligned}$$

We shall investigate the type-forms of some term-forms in Δ and Θ.

X_1 is of type-form $\rho_1(i, \gamma)$. Under the premise $A_1(i, \gamma)$,

$$\rho_1(i, \gamma) \Rightarrow \mathfrak{s}(i, \gamma) = T\langle k_0 \rangle .$$

X_2 is of type-form $\rho_2(i, \gamma)$, and, under the premise $A_2(i, \gamma)$,

$$\rho_2(i, \gamma) \Rightarrow \mathrm{TPL}(m_0;\, l, \rho_0(i, \gamma_l)) .$$

So, if $0 \le l < m_0$, the $\Pi(X_2; l)$ is of type-form $\rho_0(i, \gamma_l)$. X_3 is of type-form $\rho_3(i, \gamma)$, and, under the premise $A_3(i, \gamma)$,

$$\rho_3(i, \gamma) \Rightarrow \rho_0(i, \gamma) \simeq \rho_0(i, \delta_0)$$

(which was established earlier). Finally $W_0(i, \gamma)$ is of type-form $\rho_4(i, \gamma)$. So, $\Pi(W_0(i, \gamma);\, u, X_1, z, X_2, X_3)$ is of type-form $\rho_0(i, \gamma)$.

($\mathscr{A}_0$-4) The axiom on Σ. This is in fact identical with the axiom on lss (see 11) of Definition 1.3 in Part I), but we shall write it down.

$$\Sigma\colon \forall i \forall j \ge i \forall \gamma \forall \delta ((\Sigma(i; j, \gamma, \delta) \vdash W(i; j, \gamma, \delta, \Sigma(i)[j, \gamma, \delta]))$$
$$\wedge (W(i; j, \gamma, \delta, \Sigma(i)[j, \gamma, \delta]) \vdash \Sigma(i; j, \gamma, \delta))) ,$$

where $\Sigma(i)[j, \gamma, \delta]$ abbreviates

$$\{k, \kappa, \lambda\}(\iota(k, \kappa, \lambda) < \iota(j, \gamma, \delta) \wedge \Sigma(i; k, \kappa, \lambda)) ,$$

and W is essentially $\mathscr{L}$-recursive.

($\mathscr{A}_0$-5) The axiom on $(\mathscr{J}_0, \eta_0, \zeta_0)$. First define some predicates.

$P_3(i, l, \gamma)$: $\varepsilon(i, \gamma_l)$ is of the form (k, b, β) where $k < i$.

$P_4(\mathscr{J}_0; i, \gamma)$: $\forall n(\Pi(\mathscr{J}_0; i, \gamma, n) \prec_i \Pi(\mathscr{J}_0; i, \gamma, n+1) \prec_i \gamma)$

$P_5(\mathscr{J}_0; i, \gamma, \delta, n)$: $\delta \prec_i \Pi(\mathscr{J}_0; i, \gamma, n)$

Suppose $[X] = \mathfrak{s}(i, \gamma)$ and $[Y] = T_4(i, \delta)$.

$$\mathrm{fm}(\mathscr{J}_0, \eta_0, \zeta_0)\colon \forall i \forall \gamma \forall X\{\Delta(k_0(i, \gamma), X, \beta_0(i, \gamma))$$
$$\vdash \forall n \forall l(0 \le l < m_0(i, \Pi(\mathscr{J}_0; i, \gamma, n)) \vdash [P_3(i, l, \Pi(\mathscr{J}_0; i, \gamma, n))$$
$$\wedge \Delta(k_0(i, \gamma_0(i, \Pi(\mathscr{J}_0; i, \gamma, n), l)), \Pi(\eta_0; i, \Pi(\mathscr{J}_0; i, \gamma, n), X, n, l) ,$$
$$\beta_0(i, \gamma_0(i, \Pi(\mathscr{J}_0; i, \gamma, n), l))) \wedge P_4(\mathscr{J}_0; i, \gamma)]$$
$$\wedge \forall \delta \forall Y[\delta \prec_i \gamma \wedge \forall q(0 \le q < m_0(i, \delta) \vdash P_3(i, q, \delta)$$
$$\wedge \Delta(k_0(i, \delta_q), \Pi(Y; q), \beta_0(i, \delta_q))) \vdash P_5(\mathscr{J}_0; i, \gamma, \delta, \Pi(\zeta_0; i, \gamma, X, \delta, Y))]\} .$$

See (3) and (4) of Definition 2.1 for η_0 and ζ_0 and their type-forms.

We shall figure out the type-forms of some term-forms.

X is of type-form $\mathfrak{s}(i, \gamma)$, which is $T\langle k_0(i, \gamma)\rangle$.

$$T_4(i, \gamma) = \Lambda l T_3(i, l, \gamma)$$

and

$$\mathrm{t}_\eta = \Lambda i \Lambda \gamma(\mathfrak{s}(i, \delta) \rightarrow T_4(i, \gamma)),$$

and hence

$$\Pi(\eta_0; i, \Pi(\mathscr{J}_0; i, \gamma, n), X, l)$$

is of type-form $T_3(i, \Pi(\mathscr{J}_0; i, \gamma, n), l)$, which becomes $T\langle k_0(i, \gamma_0(i, \Pi(\mathscr{J}_0; i, \gamma, n), l))\rangle$. Y is of type-form $T_4(i, \delta)$, and hence $\Pi(Y; q)$ is of type-form $T_3(i, \delta, q)$, which becomes $T\langle k_0(i, \delta_q)\rangle$ when $0 \leq q < m_0(i, \delta)$. ζ_0 is of type-form

$$\mathrm{t}_\zeta = \Lambda i \Lambda \gamma(\mathfrak{s}(i, \gamma) \rightarrow \Lambda \delta(T_4(i, \delta) \rightarrow N_0)),$$

and hence $\Pi(\zeta_0; i, \gamma, X, \delta, Y)$ is of type-form N_0.

($\mathscr{A}_0$-6) The axiom on μ_0.

$$\mathrm{mf}(\mu_0): \forall S(\mathrm{dcr}(\mathbf{I}; S) \vdash \mathrm{mf}(\Pi(\mu_0; S), S)),$$

where $\mathrm{dcr}(\mathbf{I}; S)$ abbreviates $\mathrm{dcr}(\mathbf{I}, <_{\mathbf{I}}; S)$ (see 8) of Definition 1.2 in Part I).

($\mathscr{A}_0$-7) $\forall L \forall S \forall S'(S' \upharpoonright \Pi(L; S) = S \upharpoonright \Pi(L; S) \vdash \Pi(L; S') = \Pi(L; S))$

See 7) of Definition 2.1 for $\upharpoonright$. L is of type $\mathrm{Sq} \rightarrow N_0$, and S and S' are of type Sq.

§ 6. Semantics of formula-forms

In addition to CNPR in Section 3, we shall henceforth assume the two principles FSPR and MFPR explained subsequently.

FSPR: The hyper-methods $\mathscr{J}^*$, η^* and ζ^* in Section 3 satisfy ($\mathscr{A}_0$-5) in Section 5.

MFPR: The hyper-method μ^* in Section 3 satisfies ($\mathscr{A}_0$-6) in Section 5.

FSPR asserts in its essence that $\mathscr{J}^*(i, \gamma)$ is the fundamental sequence for (i, γ) in the system $J_i^\#$ relative to η^* and ζ^*. MFPR asserts that μ^* serves as the modulus of finiteness for $<_{\mathbf{I}}$.

DEFINITION 6.1. 1) An assignment $\mathbf{a}$ of closed $\mathscr{L}$-terms to the associated free variables of a formula-form A is defined as in 1) of Definition 3.1.

2) An assignment $\mathbf{b}$ of hyper-methods to the free variable-forms in A is defined as in 2) of Definition 3.1.

3) Completenesses of $\mathbf{a}$ and $\mathbf{b}$ are defined as before (relative to A). When $\mathbf{ba}$ is complete for A, we say that $\mathbf{ba}$ determines an instantiation of A. Let tr denote the "truth" and let fs denote the "falsehood". Then $\mathrm{inst}(A, \mathbf{b}, \mathbf{a})$ will denote one of tr or fs

(the truth value) of A determined by **ba**. tr and fs are interpreted classically in the sense that the truth value is fs if and only if it is not tr, and vice versa.

4) The value of inst(A, **b**, **a**), or inst A for short, is defined as follows.

(1) If A is $\mathscr{L}$-recursive, then inst A is defined naturally (classically).

(2) When A is $T_1=T_2$, then inst A=tr if and only if $IT_1=IT_2$ (as natural numbers.)

(3)$\sim$(5) will be dealt with later.

(6) $\wedge$ and $\vdash$ are interpreted classically; inst($B\vdash C$, **b**, **a**)=tr if and only if inst(B, **b**, **a**)=fs or inst(C, **b**, **a**)=tr.

(7) inst($\forall XF$, **b**, **a**)=tr if and only if inst(F, (**b**, X/f), **a**)=tr for every f a hyper-method of appropriate hyper-type.

Now let us work on (3)$\sim$(5).

(3) For the economy of notations, we shall identify the formal expressions and their interpretations. Consider $\Delta(i, X, \alpha)$. If it is not the case that i be an indicator, α be an ordinal diagram and X be a hyper-method of $T\langle i\rangle$, then define $\Delta(i, X, \alpha)$ to be fs. Assume next the affirmative situation, and define the truth value of $\Delta(i, X, \alpha)$ to be that of the succedent of $\rightarrow$ in Δ_1 in ($\mathscr{A}_0$-2). The well-definedness of $\Delta(i, X, \alpha)$ can be established by transfinite induction on i for arbitrary (and appropriate) X and α. Concerning the induction hypotheses, notice that $\Pi(X; 0, j, \sigma)$ and $\Pi(Y; n, j, \sigma)$ are of hyper-type $T\langle j\rangle$ if Y is of hyper-method of $T_2(i)$, n is a natural number, $j<_{\mathrm{I}} i$ and σ is an ordinal diagram.

(4) The truth value of $\Theta(i, X, \gamma)$ is defined in a manner similar to that of $\Delta(i, X, \alpha)$ by induction on the complexity of γ (although γ is not specified as a diagram here). Recall the consistency of hyper-types of Y in $\Theta(i, Y, \sigma)$ under certain conditions.

(5) The truth-value of $\Sigma(i; j, \gamma, \delta)$ can be determined in accordance with ($\mathscr{A}_0$-4) by induction on $\iota(j, \gamma, \delta)$.

5) A formula-form A is said to be valid if inst(A, **b**, **a**)=tr for every **b** and **a**. A hyper-formula A is valid if inst(A, **b**)=tr for every **b**.

PROPOSITION 6.1. *The axioms* ($\mathscr{A}_0$-2)$\sim$($\mathscr{A}_0$-7) *are valid* (*under the principles* CNPR, FSPR *and* MFPR; *see the beginnings of Sections* 3 *and* 6).

Proof. ($\mathscr{A}_0$-2) Since the truth value of Δ was defined so that the necessary condition in Δ_1 is also sufficient, we must show that such an interpretation is consistent with Δ_2. Assume the antecedent of $\rightarrow$ in Δ_2. Then in Δ_1 put $X=(Z, V)$, and then the sufficiency of the condition implies $\Delta(i,(Z, V), \alpha)$.

($\mathscr{A}_0$-3) The reasoning goes similarly to that of ($\mathscr{A}_0$-2). To see the consistency, assume the antecedent of $\rightarrow$ in Θ_2. If we assign $M=I\Pi(W_0(i, \gamma); u, X_1, z, X_2, X_3)$ to the X in Θ_1, we can easily show that the necessary condition holds, and hence, by its sufficiency,

$$\Theta(i, \Pi(W_0(i, \gamma); u, X_1, z, X_2, X_3), \gamma)$$

becomes true.

($\mathscr{A}_0$-5) is valid by FSPR, ($\mathscr{A}_0$-6) is valid by MFPR and ($\mathscr{A}_0$-7) by CNPR.

§ 7. Hyper-principle

The theory of type-forms (including the reduction and the definition of hyper-methods), the theory of term-forms and the theory of formula-forms (including the semantics for the latter two, the axiom sets, three principles and the validity of the axioms) will be called the "hyper-principle", which will be symbolized by HP.

We assume HP as the basis of the foundations of the ordinal diagrams.

If a formula-form A is valid (constantly true) according to the semantics in Section 6, we say that A is HP-valid.

Part III. The Functional Interpretation of Ordinal Diagrams

§ 1. A modified version of ASOD

The system we are to interpret is a modified version of the system ASOD, which was defined in Definition 1.3 of Part I.

DEFINITION 1.1. Let $\mathscr{S}_0$ be the system

$$\text{ASOD} - \{\text{TI}(\mathbf{I}), \text{TI}(\mathbf{A}), \text{IND}\} + \{\text{WF}(\mathbf{I}), \text{WF}(\mathbf{A})\},$$

where WF(**I**) stands for

$$\forall f(\text{dcr}(\mathbf{I}, <_{\mathbf{I}}; f) \vdash \exists x \, \text{mf}(x, f))$$

(see Definitions 1.2 and 1.3 of Part I).

Notice that $\mathscr{S}_0$ is based on the language $\mathscr{L}$. We shall abbreviate dcr(**I**, $<_{\mathbf{I}}; f$) to dcr(**I**; f).

PROPOSITION 1.1. TI(**I**), TI(**A**) *and* IND *are provable in* $\mathscr{S}_0$.

Proof. We deal with TI(**I**; A) as an example. We must show

$$\forall i(\forall j <_{\mathbf{I}} i A(j) \vdash A(i)) \rightarrow \forall i A(i).$$

Put

$$R(z)\colon \exists n < \text{lg}(z) \dot{-} 1 (z_n \leq_{\mathbf{I}} z_{n+1})$$

and

$$Q(z)\colon \forall j(\text{dcr}(\mathbf{I}; z * j) \vdash A(j)).$$

Then $Q(\langle \ \rangle)$ is equivalent to $\forall i A(i)$. It thus suffices to show

(1) $\text{Prg}(A) \rightarrow \text{Hyp}(R, Q)$,

where Prg(A) stands for

$$\forall i(\forall j <_{\mathbf{I}} i A(j) \vdash A(i))$$

and Hyp(R, Q) stands for Hyp1 ~ Hyp4 of BI(R, Q). Notice that R is $\mathscr{L}$-recursive. For (1), assume Prg(A) and prove Hyp(R, Q).

Hyp1 trivially follows.

Hyp2: $\forall S \exists l R(S \upharpoonright l)$

Define f by:

$$f(0) = S(0)\,,$$

$$f(n+1) = \begin{cases} \text{ept} & \text{if } f(n) = \text{ept} \quad \text{or} \quad f(n) = S(n) \leq_{\mathbf{I}} S(n+1)\,, \\ S(n+1) & \text{otherwise}\,. \end{cases}$$

f is $\mathscr{L}$-recursive (in S). The definition immediately implies dcr(**I**;f). So, by WF(**I**), $\exists x\,\mathrm{mf}(x, f)$, that is, $\exists x \forall m \geq x(f(m) = \text{ept})$. The definition of f then implies that

$$\exists x \exists n \leq x(S(n) \leq_{\mathbf{I}} S(n+1))\,,$$

from which follows $\exists l R(S \upharpoonright l)$.

Hyp3: $\forall z(\mathrm{sqn}(z) \wedge R(z) \vdash Q(z))$

$$v(z) = \mathbf{min}(n, n < \mathrm{lg}(z) \dot{-} 1 \wedge z_n \leq_{\mathbf{I}} z_{n+1})$$

is primitive recursive, and the premise $\mathrm{sqn}(n) \wedge R(z)$ implies

$$z_{v(z)} \leq_{\mathbf{I}} z_{v(z)+1} \wedge v(z) < \mathrm{lg}(z) \dot{-} 1\,.$$

Suppose dcr(**I**; $z * j$) holds; that is,

$$\forall l < \mathrm{lg}(z) + 1(u_{l+1} <_{\mathbf{I}} u_l)\,,$$

where $u_l = z_l$ if $l \leq \mathrm{lg}(z)$ and $u_l = j$ if $l \doteq \mathrm{lg}(z) + 1$. So, under the premise as above,

$$\begin{aligned} \mathrm{dcr}(\mathbf{I};\, z * j) \rightarrow & (z_{v(z)} \leq_{\mathbf{I}} z_{v(z)+1}) \\ & \wedge (z_{v(z)+1} <_{\mathbf{I}} z_{v(z)}) \\ & \rightarrow\,, \end{aligned}$$

and hence dcr(**I**; $z * j$) $\rightarrow A(j)$ trivially follows, which implies Q.

Hyp4: $\forall z(\mathrm{sqn}(z) \wedge \forall s Q(z * s) \vdash Q(z))$,

that is,

$$\mathrm{sqn}(z) \wedge \forall s \forall j(\mathrm{dcr}(\mathbf{I};\, z * s * j) \vdash A(j)) \vdash \forall k(\mathrm{dcr}(\mathbf{I};\, z * k) \vdash A(k))\,.$$

Assume sqn(z). We have, in succession,

$$\mathrm{dcr}(\mathbf{I}; z*k)\,,\quad j<_{\mathbf{I}}k\rightarrow\mathrm{dcr}(\mathbf{I}; z*k*j)\,,$$

$$\mathrm{dcr}(\mathbf{I}; z*k)\,,\quad j<_{\mathbf{I}}k\,,\quad \mathrm{dcr}(\mathbf{I}; z*k*j)\vdash A(j)\rightarrow A(j)\,,$$

$$\mathrm{dcr}(\mathbf{I}; z*k)\,,\quad \forall s\forall j(\mathrm{dcr}(\mathbf{I}; z*k*j)\vdash A(j))\rightarrow j<_{\mathbf{I}}k\vdash A(j)\,,$$

$$\mathrm{dcr}(\mathbf{I}; z*k)\,,\quad \forall sQ(z*s)\rightarrow\forall j<_{\mathbf{I}}kA(j)\,,$$

$$\mathrm{Prg}(A)\,,\quad \forall j<_{\mathbf{I}}kA(j)\rightarrow A(k)\,,$$

$$\mathrm{dcr}(\mathbf{I}; z*k)\,,\quad \mathrm{Prg}(A)\,,\quad \forall sQ(z*s)\rightarrow A(k)\,,$$

$$\mathrm{Prg}(A)\,,\quad \forall sQ(z*s)\rightarrow\mathrm{dcr}(\mathbf{I}; z*k)\vdash A(k)\,,$$

$$\mathrm{Prg}(A)\,,\quad \forall sQ(z*s)\rightarrow Q(z)\,.$$

Proposition 1.1 and the Conclusion in Section 2 of Part I lead us to the conclusion that $\mathscr{S}_0$ suffices for the development of the theory of ordinal diagrams.

§ 2. mr-translation

We transform the formulas of $\mathscr{S}_0$ into the "generalized formulas", which are generalizations of the formula-forms of the language $\mathscr{L}_0$ defined in Section 4 of Part II. Technically, the transformation we are to employ is in its essence the modified realizability (mr), which is seen in [8] and [13]. It will be of the form $\exists\mathfrak{X}A(\mathfrak{X})$, where $\exists\mathfrak{X}$ abbreviates $\exists X_1\cdots\exists X_n$ and A does not contain $\exists$. The crucial point of the mr-translation lies in that of $\vdash$. That is, if $\exists XA(X)$ is read as "there exists a method X to certify A", then $\exists XA(X)\vdash\exists YB(Y)$ will mean that there exists a uniform method Z such that, for any method X which certifies A, Z produces a method Y, depending on X, which certifies B. So the transformation of this becomes

$$\exists Z\forall X(A(X)\vdash B(Z(X)))\,.$$

We do not employ the mr-translation as a technically convenient means for the functional interpretation of $\mathscr{S}_0$, but rather we have been naturally led to it by the considerations over the "methods" of determining the modulus of finiteness for a decreasing sequence of ordinal diagrams. This will become clear when the translation of the predicate G is worked out.

DEFINITION 2.1. 1) Let $\exists\mathfrak{X}$ abbreviate $\exists X_1\cdots\exists X_n$, $n\geq 0$, where each X_i is a variable-form (see Definition 2.1 of Part II). A generalized formula of $\mathscr{L}_0$ is an expression $\exists\mathfrak{X}A(\mathfrak{X})$, where $A(\mathfrak{X})$ is a formula-form and the variable-forms of $\mathfrak{X}$ are not bound in A (see Definition 4.1 of Part II). $\exists$ and $\vee$ do not occur in A.

2) For each formula C of the language of $\mathscr{S}_0$, we define mr(C) to be a generalized fromula with the same free variables as C (hence of at-type and fn-type). As before, $[X]=\mathfrak{s}$ will claim that X is of type-form $\mathfrak{s}$.

If mr(C) is $\exists\mathfrak{X}A(\mathfrak{X})$, then we call $A(\mathfrak{X})$ the kernel of C or of $\exists\mathfrak{X}A(\mathfrak{X})$.

(i) If C is $\mathscr{L}$-recursive (see 2) of 3 in Definition 1.1 of Part I), then

$$\mathrm{mr}(C) = C',$$

where C' is C save that $\mathscr{J}$ is replaced by $\mathscr{J}_0$ and lss is replaced by Σ. We shall not, however, make explicit that $\mathscr{J}$ be replaced by $\mathscr{J}_0$ in most cases, and hence $\mathscr{L}$-terms and formulas thus altered will be denoted by the same letters as the original ones.

Suppose $\mathrm{mr}(C) \equiv \exists \mathscr{X} A(\mathscr{X})$ and $\mathrm{mr}(D) \equiv \exists \mathfrak{y} B(\mathfrak{y})$ have been defined.

(ii) $\mathrm{mr}(C \wedge D) \equiv \exists \mathscr{X} \exists \mathscr{Y}(A(\mathscr{X}) \wedge B(\mathscr{Y}))$

(iii) $\mathrm{mr}(C \vee D) \equiv \exists z \exists \mathscr{X} \exists \mathscr{Y}((z=0 \vdash A(\mathscr{X})) \wedge (z>0 \vdash B(\mathscr{Y})))$

(iv) $\mathrm{mr}(C \vdash D) \equiv \exists \mathscr{Z} \forall \mathscr{X}(A(\mathscr{X}) \vdash B(\Pi(\mathscr{Z}; \mathscr{X})))$,

where $\Pi(\mathscr{Z}; \mathscr{X})$ abbreviates a sequence of term-forms.

(v) $\mathrm{mr}(\forall X C(X)) \equiv \exists \mathscr{Z} \forall X A(X, \Pi(\mathscr{Z}; X))$,

where $\mathrm{mr}(C(X)) \equiv \exists \mathscr{Y} A(X, \mathscr{Y})$. Notice that X is either of at-type or fn-type. Recall that $C(x)$ is $\mathscr{L}$*-u-a* (see 3) of 3 in Definition 1.1 of Part I).

(vi) $\mathrm{mr}(\exists x C(x)) \equiv \exists x C(x)$

(vii) $\mathrm{mr}(G(i, \alpha)) \equiv \exists X \Delta(i, X, \alpha)$,

where i and α are at-type $\mathscr{L}$-terms, and $[X] = T\langle i \rangle$. (See (5) of Definition 1.1 in Part II for $T\langle i \rangle$.)

(viii) $\mathrm{mr}(\mathrm{Ord}(i, \gamma)) \equiv \exists X \Theta(i, X, \gamma)$,

where $[X] = \rho_0(i, \gamma)$, which was defined in ($\mathscr{A}_0$-3) of Definition 5.1 in Part II.

(ix) $\mathrm{mr}(\mathrm{lss}(i; j, \gamma, \delta)) \equiv \Sigma(i; j, \gamma, \delta)$

PROPOSITION 2.1. 1) *In* (iii) *z does not occur in $A(\mathscr{X})$ or $B(\mathscr{Y})$.*

2) *It is easy to see that the definienda are generalized formulas, and that they have exactly the free variables of the original formulas.*

(3) *In* (iv), *$[\mathscr{Z}] = [\mathscr{X}] \to [\mathscr{Y}]$ (in the simplified notation). With* (v), *there are two cases. If the type-form of $\mathscr{Y}$ does not depend on X, then $[\mathscr{Z}] = [X] \to [\mathscr{Y}]$. If the type-form of $\mathscr{Y}$ is $\tau(X)$ depending on X, then the type-form of $\mathscr{Z}$ is $\Lambda Y \tau(Y)$, Y being of the same type as X.*

§3. mr-translation of $\mathscr{S}_0$

We shall present the mr-translations of some of the axioms and the rules of inference of $\mathscr{S}_0$. For the part of intuitionistic arithmetic (of our language), we employ Kleene's system in [2]. For the sake of notational simplicity, we write single variable-forms instead of finite sequences of them unless otherwise desired. Let A^*, B^*, $\cdots$ denote formulas of $\mathscr{S}_0$, and let $\exists X A(X)$, $\exists Y B(Y)$, $\cdots$ denote the corresponding mr-translations.

1b. $\mathrm{mr}((A^* \vdash B^*) \vdash [(A^* \vdash (B^* \vdash C^*)) \vdash (A^* \vdash C^*)])$

$$\equiv \exists U \forall Y\{\forall X(A(X) \vdash B(\Pi(Y; X)) \vdash \forall Z[\forall X(A(X) \vdash \forall V(B(V) \vdash C(\Pi(Z; X, V)))) \vdash \forall X(A(X) \vdash C(\Pi(U; Y, Z, X)))]\}$$

2. $\dfrac{A^*, \quad A^* \vdash B^*}{B^*}$

$$\mathrm{mr}(A^* \vdash B^*) \equiv \exists Z \forall X(A(X) \vdash B(\Pi(Z; X))) .$$

5b. $\mathrm{mr}(B^* \vdash A^* \vee B^*) \equiv \exists Z \exists U \exists V \forall Y(B(Y) \vdash [(\Pi(Z; Y) = 0 \vdash A(\Pi(U; Y))) \wedge (\Pi(Z; Y) > 0 \vdash B(\Pi(V; Y)))])$

12. $\dfrac{A^*(a) \vdash C^*}{\exists x A^*(x) \vdash C^*}$

$$\mathrm{mr}(A^*(a) \vdash C^*) \equiv \exists Z(A(a) \vdash C(Z))$$

$$\mathrm{mr}(\exists x A^*(x) \vdash C^*) \equiv \exists U \forall x(A(x) \vdash C(\Pi(U; x)))$$

We shall now work out the mr-translations of BI, DDTI, DI, DDI, fm($\mathscr{J}$), fnc(A^*, t), fncv(A^*) and WF(I) (see Definition 1.3 of Part I and Definition 1.1 here).

BI(R, A) Recall that R is $\mathscr{L}$-recursive.

$$\mathrm{mr}(R(z)) \equiv R(z)$$

Assume

$$\mathrm{mr}(A^*(z)) \equiv \exists X A(X, z) ,$$

where $[X] = \mathrm{t}(z)$. (The case where $\mathrm{t}(z)$ is z-free can be dealt with similarly.)

Hyp1 remains invariant under the translation.

$$\mathrm{mr}(\mathrm{Hyp2}) \equiv \exists L \forall S R(S \upharpoonright L(S))$$

$$\mathrm{mr}(\mathrm{Hyp3}) \equiv \exists X_1 \forall z(\mathrm{sqn}(z) \wedge R(z) \vdash A(\Pi(X_1; z), z)) ,$$

where $[X_1] = \Lambda z\, \mathrm{t}(z)$.

$$\mathrm{mr}(\mathrm{Hyp4}) \equiv \exists X_2 \forall z \forall Z(\mathrm{sqn}(z) \wedge \forall s A(\Pi(Z; s), z * s) \vdash A(\Pi(X_2; z, Z), z)) ,$$

where $[X_2] = \Lambda z(\Lambda s\, \mathrm{t}(z * s) \to \mathrm{t}(z))$ and $[Z] = \Lambda s\, \mathrm{t}(z * s)$.

$$\mathrm{mr}(\mathrm{Conclusion}) \equiv \exists X A(X, \langle\ \rangle) ,$$

where $[X] = \mathrm{t}(\langle\ \rangle)$.

We can now write down mr(BI).

$$\mathrm{mr}(\mathrm{BI}) \equiv \exists X_0 \forall L \forall X_1 \forall X_2 \{\forall S \forall l(R(S \upharpoonright l) \vdash \forall m > l R(S \upharpoonright m)) \wedge \forall S R(S \upharpoonright L(S)) \wedge \forall z(\mathrm{sqn}(z) \wedge R(z) \vdash A(\Pi(X_1; z), z))$$

$$\wedge \forall z \forall Z(\mathrm{sqn}(z) \wedge \forall s A(\Pi(Z; s), z * s) \vdash A(\Pi(X_2; z, Z), z))$$

$$\vdash A(\Pi(X_0; L, X_1, X_2), \langle\ \ \rangle)\},$$

where $[L] = \mathrm{sq} \to N_0$, $[X_1] = \Lambda z \mathrm{t}(z)$, $[X_2] = \Lambda z(\Lambda s \mathrm{t}(z * s) \to \mathrm{t}(z))$, $[Z] = \Lambda s \mathrm{t}(z * s)$ and $[X_0] = [L], [X_1], [X_2] \to \mathrm{t}(\langle\ \ \rangle)$.

DTI(I_∞; G, U)

Recall that

$$\mathrm{mr}(G(i, \alpha)) \equiv \exists X \Delta(i, X, \alpha).$$

Let $P_1(j, \sigma, \alpha)$ and $P_2(i, \alpha, f)$ be as in Definition 5.1 of Part II, and let $P_0(x, f)$ be defined as follows.

$$P_0(x, f)\colon \forall n \geq x(f(n) = \mathrm{ept})$$

$P_0 \sim P_2$ are $\mathscr{L}$-*u*-*a* (in f), and hence invariant under the mr-translation. Now $U(i, \alpha, \Xi)$ (see 9) of Definition 1.3 in Part I) can be written as follows.

$$\forall j < i \forall \sigma(P_1(j, \sigma, \alpha) \vdash \Xi(j, \sigma))$$

$$\wedge \forall f(P_2(i, \alpha, f) \wedge \forall n \forall j < i \forall \sigma(P_1(j, \sigma, f(n)) \vdash \Xi(j, \sigma)) \vdash \exists x P_0(x, f)).$$

Put

$$G^*[i]\colon \{j, \sigma\}(j < i \wedge \exists X \Delta(j, X, \sigma)),$$

where $X = T\langle j \rangle$. We are to consider the mr-translation of $U(i, \alpha, G^*[i])$. We shall write successive translations, relating them by $\sim$.

$$U(i, \alpha, G[i]) \sim U(i, \alpha, G^*[i]) \sim (\mathrm{a}) \wedge (\mathrm{b}),$$

where

$$(\mathrm{a}) \sim \forall j < i \forall \sigma(P_1(j, \sigma, \alpha) \vdash \exists X \Delta(j, X, \sigma)),$$

$$(\mathrm{b}) \sim \forall f(P_2(i, \alpha, f) \wedge (\mathrm{c}) \vdash \exists x P_0(x, f))$$

and

$$(\mathrm{c}) \sim \forall n \forall j < i \forall \sigma(P_1(J, \sigma, f(n)) \vdash \exists X \Delta(j, X, \sigma)).$$

$$(\mathrm{a}) \sim \exists Y \forall j < i \forall \sigma(P_1(j, \sigma, \alpha) \vdash \Delta(j, \Pi(Y; j, \sigma), \sigma)) \sim \exists Y R_1(i, \alpha, Y)$$

(abbreviated by R_1). $[X] = T\langle j \rangle$ and $[Y] = T_1\langle i \rangle$.

$$(\mathrm{c}) \sim \exists Z\, \forall n \forall j < i \forall \sigma(P_1(j, \sigma, f(n)) \vdash \Delta(j, \Pi(Z; n, j, \sigma), \sigma))$$

$$\sim \exists Z R_2(i, Z)$$

$$(\mathrm{b}) \sim \forall f \exists U \forall Z(P_2(i, \alpha, f) \wedge R_2(i, Z) \vdash P_0(\Pi(U\colon Z), f))$$

$$\sim \exists V \forall f \forall Z(P_2(i, \alpha, f) \wedge R_2(i, Z) \vdash P_0(\Pi(V; f, Z), f))$$

$\sim \exists V R_4(i, \alpha, V)$

$$[Z]=T_2(i), \quad [U]=T_2(i)\to N_0, \quad [V]=T_2\langle i\rangle .$$

See (5) of Definition 1.1 in Part II for $T\langle j\rangle$, $T_1\langle i\rangle$, $T_2\langle i\rangle$ and $T_2(i)$.

We thus have

$$U(i, \alpha, G[i]) \sim \exists Y \exists V(R_1(i, \alpha, Y) \wedge R_4(I, \alpha, V)) \sim \exists Y \exists V R_0(i, \alpha, Y, V) .$$

$$[Y]=T_1\langle i\rangle , \quad [V]=T_2\langle i\rangle .$$

Nest define (d), (e), F_1 and F_2.

$$(\mathrm{d}) \sim \exists X \Delta(i, X, \alpha) \vdash \exists Y \exists V R_0(i, \alpha, Y, V)$$

$$\sim \exists W_1 \exists W_2 \forall X(\Delta(i, X, \alpha) \vdash R_0(i, \alpha, \Pi(W_1; X), \Pi(W_2; X)))$$

$$\sim \exists W_1 \exists W_2 F_1(i, \alpha, W_1, W_2)$$

$$[X]=T\langle i\rangle , \quad [Y]=T_1\langle i\rangle ,$$

$$[V]=T_2\langle i\rangle , \quad [W_1]=\mathrm{t}_1(i)=T\langle i\rangle \to T_1\langle i\rangle ,$$

$$[W_2]=\mathrm{t}_2(i)=T\langle i\rangle \to T_2\langle i\rangle .$$

$$(\mathrm{e}) \sim \exists W \forall Y \forall V(R_0(i, \alpha, Y, V) \vdash \Delta(i, \Pi(W; Y, V), \alpha)) \sim \exists W F_2(i, \alpha, W)$$

$$[Y]=T_1\langle i\rangle , \quad [V]=T_2\langle i\rangle ,$$

$$[W]=\mathrm{t}(i)=T_1\langle i\rangle , \quad T_2\langle i\rangle \to T\langle i\rangle$$

$D(I_\infty; G, U)$ with i and α free, which we write $\mathrm{DTI}(i, \alpha)$ is translated as follows.

$$\mathrm{DTI}(i, \alpha) \sim \exists X \Delta(i, X, \alpha) \sim \exists Y \exists V R_0(i, \alpha, Y, V) \sim (\mathrm{d}) \wedge (\mathrm{e})$$

$$\sim \exists W_1 \exists W_2 \exists W(F_1(i, \alpha, W_1, W_2) \wedge F_2(i, \alpha, W)) ,$$

where type-forms of the variable-forms are as in (d) and (e).

DI(Ord, V)

First consult ($\mathscr{A}_0$-3) in Definition 5.1 of Part II for $A_1(i, \gamma)$, $A_2(i, \gamma)$, $A_3(i, \gamma)$ and other notations. Then

$$V(i, \gamma, X)\colon (A_1(i, \gamma) \wedge G(k_0(i, \gamma), \beta_0(i, \gamma))) \vee (A_2(i, \gamma) \wedge \forall l(0 \leq l < m_0(i, \gamma) \vdash X(i, \gamma_0(i, \gamma, l)))) \vee (A_3(i, \gamma) \wedge X(i, \delta_0(i, \gamma))) ,$$

which we shall also write

$$H_1(i, \gamma) \vee H_2(i, \gamma, X) \vee H_3(i, \gamma, X) .$$

$$H_1(i, \gamma) \sim A_1(i, \gamma) \wedge \exists X \Delta(k_0, X, \beta_0) ,$$

where $k_0=k_0(i, \gamma)$, $\beta_0=\beta_0(i, \gamma)$ and $[X]=T\langle k_0\rangle$.

Recall that

$$\mathrm{mr}(\mathrm{Ord}(i, \gamma)) \equiv \exists X \Theta(i, X, \gamma)\,,$$

where $[X] = \rho_0(i, \gamma)$. So,

$$H_2(i, \gamma, \mathrm{Ord}(i)[\gamma]) \sim A_2(i, \gamma) \wedge \forall l(0 \leq l < m_0(i, \gamma) \vdash \mathrm{Ord}(i, \gamma_l))$$

$$\sim A_2(i, \gamma) \wedge \forall l(0 \leq l < m_0 \vdash \exists Y_1 \Theta(i, Y_1, \gamma_l))$$

$$\sim \exists Y(A_2(i, \gamma) \wedge Al(0 \leq l < m_0 \vdash \Theta(i, \Pi(Y; l), \gamma_l)))\,.$$

$$[Y_1] = \rho_0(i, \gamma_l)\,, \quad [Y] = \Lambda l \rho_0(i, \gamma_l)$$

$$H_3(i, \gamma, \mathrm{Ord}(i)[\gamma]) \sim A_3(i, \gamma) \wedge \mathrm{Ord}(i, \delta_0)$$

$$\sim A_3(i, \gamma) \wedge \exists Z \Theta(i, Z, \delta_0) \sim \exists Z(A_3(i, \gamma) \wedge \Theta(i, Z, \delta_0))$$

$$[Z] = \rho_0(i, \delta_0)$$

$$V(i, \gamma, \mathrm{Ord}(i)[\gamma]) \sim \exists u \exists X \exists z \exists Y \exists Z$$

$$\{(u = 0 \vdash [A_1(i, \gamma) \wedge \Delta(k_0, X, \beta_0)])$$

$$\wedge (u > 0 \vdash [z = 0 \vdash A_2(i, \gamma) \wedge \forall l(0 \leq l < m_0 \vdash \Theta(i, \Pi(Y; l), \gamma_l))$$

$$\wedge [z > 0 \vdash A_3(i, \gamma) \wedge \Theta(i, Z, \delta_0)])\}$$

$$[X] = T\langle k_0 \rangle\,, \quad [Y] = \Lambda l \rho_0(i, \gamma_l)\,, \quad [Z] = \rho_0(i, \delta_0)$$

$$\mathrm{Ord}(i, \gamma) \vdash V(i, \gamma, \mathrm{Ord}(i)[\gamma]) \sim \mathrm{DI}_1(i, \gamma):$$

$$\exists U \exists X_0 \exists V \exists Y_0 Z_0 \forall W[\Theta(i, W, \gamma) \vdash$$

$$\{\Pi(U; W) = 0 \vdash [A_1(i, \gamma) \wedge \Delta(k_0; \Pi(X_0; W), \beta_0)]\}$$

$$\wedge \{\Pi(U; W) > 0 \vdash [\Pi(V; W) = 0 \vdash A_2(i, \gamma)$$

$$\wedge \forall l(0 \leq l < m_0 \vdash \Theta(i, \Pi(Y_0; W, l), \gamma_l))]$$

$$\wedge [\Pi(V; W) > 0 \vdash A_3(i, \gamma) \wedge \Theta(i, \Pi(Z_0; W), \delta_0)]\}]\,,$$

where

$$[W] = \rho_0(i, \gamma)\,, \quad [U] = [W] \to N_0\,, \quad [X_0] = [W] \to T\langle k_0 \rangle\,,$$

$$[V] = [W] \to N_0\,, \quad [Y_0] = [W] \to \Lambda l \rho_0(i, \gamma_l)\,,$$

$$[Z_0] = [W] \to \rho_0(i, \delta_0)\,.$$

As was noted in ($\mathscr{A}_0$-3),

$$A_1(i, \gamma) \quad \text{implies} \quad \rho_0(i, \gamma) \equiv T\langle i \rangle\,,$$

$$A_2(i, \gamma) \quad \text{and} \quad 0 \leq l < m_0 \quad \text{imply} \quad \Pi(\rho_0(i, \gamma); l) = \rho_0(i, \gamma_l)\,,$$

$$A_3(i, \gamma) \quad \text{implies} \quad \rho_0(i, \gamma) = \rho_0(i, \delta_0)\,.$$

So, the type-forms of the following term-forms can be determined like this:

$[\Pi(X_0; W)] = T\langle k_0\rangle$,

$[\Pi(Y_0; W, l)] = \Pi(\rho_0(i, \gamma); l)$,

$[\Pi(Z_0; W)] = \rho_0(i, \gamma)$.

$V(i, \gamma, \mathrm{Ord}(i)[\gamma]) \vdash \mathrm{Ord}(i, \gamma) \sim \mathrm{DI}_2(i, \gamma)$:

$\exists W_0 \forall u \forall X \forall z \forall Y \forall Z[\{u = 0 \vdash A_1(i, \gamma) \wedge \Delta(k_0, X, \beta_0)\}$

$\wedge \{u > 0 \vdash [z = 0 \vdash A_2(i, \gamma) \wedge \forall l(0 \leq l < m_0$

$\vdash \Theta(i, \Pi(Y; l), \gamma_l))] \wedge [z > 0 \vdash A_3(i, \gamma) \wedge \Theta(i, Z, \delta_0)]\}$

$\vdash \Theta(i, \Pi(W_0; u, X, z, Y, Z), \gamma)]$

$[X] = T\langle k_0\rangle$, $[Y] = \Lambda l \rho_0(i, \gamma_l)$, $[Z] = \rho_0(i, \delta_0)$,

$[W_0] = N_0$, $T\langle k_0\rangle$, N_0, $\Lambda l \rho_0(i, \gamma_l)$, $\rho_0(i, \delta_0) \to \rho_0(i, \gamma)$.

Now,

$$\mathrm{DI}(i, \gamma) \sim \mathrm{DI}_1(i, \gamma) \wedge \mathrm{DI}_2(i, \gamma)$$
$$\sim \exists U \exists X_0 \exists V \exists Y_0 \exists Z_0 \exists W_0(\mathrm{DI}_1'(i, \gamma) \wedge \mathrm{DI}_2'(i, \gamma)),$$

where DI_n' stands for the matrix of DI_n, $n = 1, 2$.

$\mathrm{DDI}(\mathrm{lss}; W)$

The condition on W is $\mathscr{L}$-recursive. So

$$\mathrm{mr}(\mathrm{DDI}(i, j, \gamma, \delta)) \sim \mathrm{DDI}(\Sigma; W, i, j, \gamma, \delta).$$

$\mathrm{fm}(\mathscr{I})$

We shall consider the first part,

$$\mathrm{fm}(i, \gamma) \colon J(i, \gamma) \vdash \mathrm{fs}(J(i)^{\#}, \prec_i; \gamma, \mathscr{I}(i, \gamma)).$$

The second part can be dealt with similarly. By 10) of Definition 1.3 in Part I and (vii) of 2) in Definition 2.1 we obtain

$$J(i, \gamma) \leftrightarrow H(i, \varepsilon(i, \gamma)) \sim \exists X \Delta(k_0(i, \gamma), X, \beta_0(i, \gamma)),$$

where $[X] = T\langle k_0\rangle$. Let P_3 and P_4 be the formulas defined in ($\mathscr{A}_0$-5) of Definition 5.1 in Part II.

$$\mathrm{fs}(J(i)^{\#}, \prec_i; \gamma, \mathscr{I}(i, \gamma))$$
$$\leftrightarrow \forall n \forall l(0 \leq l < m_0(i, \mathscr{I}(i, \gamma, n)) \vdash [P_3(i, l, \mathscr{I}(i, \gamma, n))$$
$$\wedge J(i, \gamma_0(i, \mathscr{I}(i, \gamma, n), l))]) \wedge P_4(\mathscr{I}; i, \gamma)$$
$$\wedge \forall \delta(J(i, \delta)^{\#} \wedge \delta \prec_i \gamma \vdash \exists n(\delta \prec_i \mathscr{I}(i, \gamma, n)))$$
$$\leftrightarrow G_1 \wedge P_4 \wedge G_2$$

$$J(i, \gamma_0(i, \mathscr{J}(i, \gamma, n), l)) \sim \exists X \Delta(k, X, \beta),$$

where $k = k_0(\varepsilon(i, \gamma_0(i, \mathscr{J}(i, \gamma, n), l)))$, $[X] = T\langle k \rangle$, and $\beta = \beta_0(\varepsilon(i, \gamma_0(i, \mathscr{J}(i, \gamma, n), l)))$.

$$G_1 \sim \exists Y \forall n \forall l (0 \leq l < m_0(i, \mathscr{J}(i, \gamma, n)) \vdash [P_3(i, l, \mathscr{J}(i, \gamma, n))$$
$$\wedge \Delta(k, \Pi(Y; n, l), \beta)])$$
$$[Y] = \mathrm{t}(i, \gamma) = \Lambda n \Lambda l T\langle k \rangle$$
$$G_2 \sim \exists Z \forall \delta \forall U[\delta \prec_i \gamma \wedge 0 \leq \forall l' < m_0(i, \delta)(P_3(i, l', \delta)$$
$$\wedge \Delta(k_0(i, \delta_{l'}), \Pi(U; l'), \beta_0(i, \delta_{l'})) \vdash P_5(\mathscr{J}; i, \gamma, \delta, \Pi(Z; \delta, U))],$$

where P_5 is $\mathscr{L}$-recursive.

$$[Z] = \mathfrak{s}(i, \gamma) = \Lambda \delta(\mathfrak{r}(i, \gamma, \delta) \to N_0), \ \mathfrak{r}(i, \gamma, \theta) = \Lambda q T\langle k_0(i, \delta_q) \rangle = [U]$$
$$\mathrm{fm}(i, \gamma) \sim \exists Y_0 \exists Z_0 \forall X \{\Delta(k_0, X, \beta_0) \vdash \forall n \forall l (0 \leq l < m_0(i, \mathscr{J}(i, \gamma, n))$$
$$\vdash P_3(i, l, \mathscr{J}(i, \gamma, n)) \wedge \Delta(k, \Pi(Y_0; X, n, l), \beta)) \wedge P_4(\mathscr{J}; i, \gamma)$$
$$\wedge \forall \delta \forall U[\delta \prec_i \gamma \wedge 0 \leq \forall l' < m_0(i, \delta)(P_3(i, l', \delta)$$
$$\wedge \Delta(k_0(i, \delta_{l'}), \Pi(U; l'), \beta_0(i, \delta_{l'}))$$
$$\vdash P_5(\mathscr{J}; i, \gamma, \delta, \Pi(Z_0; X, \delta, U))\}$$
$$[X] = T\langle k_0 \rangle, \quad [Y_0] = \mathfrak{s}_1(i, \gamma) = T\langle k_0 \rangle \to \Lambda n \Lambda l T\langle k \rangle,$$
$$[Z_0] = \mathfrak{s}_2(i, \gamma) = T\langle k_0 \rangle \to \Lambda \delta(\mathfrak{r}(i, \gamma, \delta) \to N_0).$$

Note. The translation above expresses that there are methods Y_0 and Z_0 such that, if γ has a method to confirm $\gamma \in J(i)$, then, Y_0 confirms $\mathscr{J}(i, \gamma, m) \in J(i)^{\#}$ for every m, and, for every $\delta \prec_i \gamma$ which has a method to confirm $\delta \in J(i)^{\#}$, Z_0 produces an n so that $\delta \prec_i \mathscr{J}(i, \gamma, n)$. So the condition that $\mathscr{J}(i, \gamma)$ be the fundamental sequence for (i, γ) is invariant under the translation.

$\mathrm{fnc}(A^*, t)$

Suppose $\mathrm{mr}(A^*(f)) \equiv \exists X A(f, X)$, where $[X] = \mathfrak{s}(f)$.

$\mathrm{fnc}(A^*, t) \sim \exists U \forall Z (\forall f A(f, \Pi(Z; f)) \vdash A(h, \Pi(U; Z)))$

$[Z] = \Lambda f \mathfrak{s}(f), \quad [U] = \Lambda f \mathfrak{s}(f) \to \mathfrak{s}(h)$

$\mathrm{fncv}(A^*)$

Suppose $\mathrm{mr}(B^*) \equiv \exists Y B(Y)$.

The premise of $\mathrm{fncv}(A^*) \sim \exists Z \forall Y (B(Y) \vdash A(f, \Pi(Z; Y)))$

The conclusion $\sim \exists V \forall Y (B(Y) \vdash \forall f A(f, \Pi(V; Y, f)))$

$[V]=[Y]\to \Lambda f\mathfrak{s}(f)$

$$\mathrm{WF}(\mathbf{I})\sim\exists X\forall f(\mathrm{dcr}(\mathbf{I};f)\vdash \mathrm{mf}(\Pi(X;f),f))$$

$$[X]=\mathrm{Sq}\to N_0$$

§ 4. Functional interpretation

THEOREM. *If A is a closed theorem of $\mathscr{S}_0$ (see Definition* 1.1) *and*

$$\mathrm{mr}(A)\equiv\exists\mathscr{X}C(\mathscr{X}),$$

then there is a finite sequence of hyper-terms Φ^* such that $C(\Phi^*)$ is HP-*valid in the sense of Section 7 in Part II.*

COROLLARY. *The system* ASOD *has a functional interpretation based on* HP.

Proof. By the theorem above and Proposition 1.1.

We have finally reached our objective; that is, the

CONCLUSION. The functional structure of ordinal diagrams is represented by HP.

Proof. By the Conclusion in Section 2 of Part I, Proposition 1.1 and the Corollary above.

This conclues our entire program.

Proof of the Theorem. We shall prove the theorem in a generalized form:

(*) If A is any theorem of $\mathscr{S}_0$, closed or not, and if

$$\mathrm{mr}(A)\equiv\exists\mathscr{X}C(\mathscr{X},\mathscr{Q}),$$

where $\mathscr{Q}$ stands for the free variables in A, then there is a sequence of term-forms $\Psi^*(\mathscr{Q})$ with the free parameters $\mathscr{Q}$, such that

$$C(\Psi^*(\mathscr{Q}),\mathscr{Q})$$

is HP-valid.

Obviously, the theorem is a special case of (*) where $\mathscr{Q}$ is empty.

Proof of (*). It suffices to give concretely the term-forms for the existential variable-forms in the mr-translations which were worked out in Section 3. See Definition 6.1 in Part II for the validity of formula-forms.

1b. $U_0=\lambda Y\lambda Z\lambda X\Pi(Z;X,\Pi(Y;X))$

2. Suppose X_0 and Z_0 have been determined for the premises. Then define

$$Y_0=\Pi(Z_0;X_0).$$

5b. $Z_0 = \lambda Y1$, $U_0 = \lambda Y \mathrm{ept}$, $V_0 = \lambda YY$

BI(R, A). We shall determine a term-form for the X_0 in mr(BI) in terms of a bar-constant, and justify it by an informal use of the bar induction BI(R^*, A^*), where R^* is $\mathscr{L}$-recursive and A^* is $\exists$-free.

Let $\mathscr{B}$ be the bar constant of type-form

$$[L], [X_1], [X_2] \to \Lambda x \Lambda S \mathrm{t}(S \restriction x)$$

(see mr(BI) in Section 3). Define R^* and A^* as follows.

$$R^*(z)\colon \mathrm{lg}(z) \geq L(z)$$

$$A^*(z)\colon \forall S \forall x(S \restriction \mathrm{lg}(z) = z \wedge x \geq \mathrm{lg}(z) \vdash A(\Pi(\mathscr{B}, L, X_1, X_2, x, S), S \restriction x))$$

Assume the premises in { } of mr(BI), given (the assignments to) $\mathscr{X} \equiv L, X_1, X_2$. Hyp1 ~ 4 for R^* and A^* will be established below. So by BI(R^*, A^*), $A^*(\langle\ \rangle)$ holds, and hence putting $x=0$ and $S=\phi$ (the empty sequence), we obtain

$$A(\Pi(\mathscr{B}; \mathscr{X}, 0, \phi), \langle\ \rangle)\,.$$

Now put

$$X_0 = \lambda L X_1 X_2 \Pi(\mathscr{B}; \mathscr{X}, 0, \phi)\,.$$

The type-form of $\Pi(\mathscr{B}; \mathscr{X}, 0, \phi)$ is $\mathrm{t}(\phi \restriction 0) = \mathrm{t}(\langle\ \rangle)$, which is consistent with $A(\Pi(X_0; \mathscr{X}, \langle\ \rangle))$.

It is now left for us to show Hyp1 ~ 4 for R^* and A^*.

Hyp1. $l \geq L(S \restriction l)$, $m > l \to m \geq L(S \restriction l)$, and hence

$$(S \restriction m) \restriction L(S \restriction l) = (S \restriction l) \restriction L(S \restriction l)\,.$$

So, CNPR implies $L(S \restriction m) = L(S \restriction l)$, which in turn implies

$$\forall l(R^*(S \restriction l) \vdash \forall m > l\, R^*(S \restriction m))\,.$$

Hyp2. $l > L(S) \vdash (\mathrm{lg}(S \restriction l) = l \geq L(S \restriction l))$, and hence

$$\forall S \exists l R^*(S \restriction l)\,.$$

Hyp3. Suppose $\mathrm{lg}(z) \geq L(z)$ and $S \restriction \mathrm{lg}(z) = z \wedge x \geq \mathrm{lg}(z)$. We are to show $A(\Pi(\mathscr{B}; \mathscr{X}, x, S), S \restriction x)$, where $\mathscr{X}$ abbreviates a sequence of variables. Put $n = \mathrm{lg}(z)$.

$$L(S) = L(z) \leq n \leq x$$

by CNPR. $x \geq L(S)$ implies that one of the two cases below holds.

Case 1. $x > L(S)$. $R(S \restriction L(S)) \to \forall m > L(S) R(S \restriction m)$ and $R(S \restriction L(S))$ by the premises of mr(BI), and hence $R(S \restriction x)$. Then by the third premise with $z = S \restriction x$, we obtain $A(\Pi(X_1; S \restriction x), S \restriction x)$. But by (1°) of (11) in Definition 3.2 of Part II,

$$I\Pi(X_1; S \restriction x) = I\Pi(\mathscr{B}; \mathscr{X}, x, S)$$

when $I\Pi(L; S) < Ix$. (We have employed abbreviated notations.) So, we get

$$A(\Pi(\mathscr{B}; \mathscr{X}, x, S), S{\restriction}x).$$

Case 2. $x=L(S)$. The second premise claims $\forall SR(S\,|\,L(S))$, or $\forall SR(S\,|\,x)$. So, by the third premise

$$A(\Pi(X_1; S{\restriction}x), S{\restriction}x).$$

But (1°) of (11) in Definition 3.2 of Part II tells us that

$$I\Pi(\mathscr{B}; \mathscr{X}, x, S)=I\Pi(X_1; S{\restriction}x)$$

when $Ix=I\Pi(L; S)$, and hence follows

$$A(\Pi(\mathscr{B}; \mathscr{X}, x, S), S{\restriction}x).$$

Hyp4. Suppose $\forall sA^*(z*s)$, and show $A^*(z)$. Assuming $z=S{\restriction}\mathrm{lg}(z)$ and $x\geq \mathrm{lg}(z)=n$, we deduce $A(\Pi(\mathscr{B}; \mathscr{X}, x, S), S{\restriction}x)$. $A^*(z*s)$ claims that

(a) $\forall T\forall y(T{\restriction}n+1=z*s\wedge y\geq n+1\vdash A(\Pi(\mathscr{B}; \mathscr{X}, y, T), T{\restriction}y))$.

When $x\geq n+1$, put $s=S(n)$, $T=S$ and $y=x$.

$$T{\restriction}n+1=S{\restriction}n+1=(S{\restriction}n)*S(n)=z*s,$$

and $y\geq n+1$. So (a) above yields

$$A(\Pi(\mathscr{B}; \mathscr{X}, x, S), S{\restriction}x).$$

When $x=n$, for each s, put $T=z*s$ and $y=x+1$. Then $T{\restriction}n+1=z*s$ and $y\geq n+1$. So by (*) $A(\Pi(\mathscr{B}; \mathscr{X}, y, T), T{\restriction}y)$, or

$$A(\Pi(\mathscr{B}; \mathscr{X}, x+1, (S{\restriction}x)*s), z*s).$$

Put $z=S{\restriction}x$ and $Z=\lambda s\Pi(\mathscr{B}; \mathscr{X}, x+1, (S{\restriction}x)*s)$ in the fourth premise. Then

$$A(\Pi(X_2; z, Z), z).$$

By write of (2°) of (11) in Definition 3.2 of Part II,

$$I\Pi(\mathscr{B}; \mathscr{X}, x, S)=I\Pi(X_2; z, Z)$$

when $Ix<I\Pi(L; S)$. So, if $x<L(S)$, $A(\Pi(\mathscr{B}; \mathscr{X}, x, S), S{\restriction}x)$. When $x\geq L(S)$, one of the cases in the proof of Hyp3 holds, and hence, by the same arguments, we obtain

$$A(\Pi(\mathscr{B}; \mathscr{X}, x, S), S{\restriction}x).$$

DTI($\mathbf{I}_\infty$; G, U)

We must find W_1, W_2 and W satisfying F_1 and F_2. Define them as follows.

$W_1=\lambda X\lambda j\sigma\mathscr{C}[j<i; \Pi(\Pi(X; 0); j, \sigma), \mathrm{ept}]$,

$W_2=\lambda X\Pi(X; 1)$,

$W=\lambda Y\lambda V(Y, V)$,

where $[X]=T\langle i\rangle$, $[Y]=T_1\langle i\rangle$ and $[V]=T_2\langle i\rangle$.

We reduce the validity of F_1 to that of Δ_1 in ($\mathscr{A}_0$-2), and then apply Proposition 6.1 of Part II, which assures the validity of Δ_1. Similarly for F_2.

F_1: Suppose $\Delta(i, X, \alpha)$, and we are to show

$$R_0(i, \alpha, \Pi(W_1; X), \Pi(W_2; X)),$$

or

$$R_1(i, \alpha, \Pi(W_1; X)) \wedge R_4(i, \alpha, \Pi(W_2; X)).$$

$$R_1(i, \alpha, \Pi(W_1; X)): \forall j<i\forall\sigma(P_1(j, \sigma, \alpha)\vdash \Delta(j, \Pi(\Pi(W_1; X); j, \sigma), \sigma))$$

$$\leftrightarrow\forall j<i\forall\sigma(P_1(j, \sigma, \alpha)\vdash \Delta(j, \mathscr{C}[j<i; \Pi(\Pi(X; 0); j, \sigma), \mathrm{ept}], \sigma)$$

$$\leftrightarrow\forall j<i\forall\sigma(P_1(j, \sigma, \alpha)\vdash \Delta(j, \Pi(X; 0, j, \sigma), \sigma)),$$

which is the first conjunct of Δ_1.

$$R_4(i, \alpha, \Pi(W_2; X)): \forall f\forall Z(P_2(i, \alpha, f)\wedge R_2(i, Z)\vdash P_0(\Pi(W_2; X, f, Z), f)),$$

where $[Z]=T_2(i)$ and $\Pi(W_2; X, f, Z)=\Pi(X; 1, f, Z)$. So this is the second conjunct of Δ_1. That is,

$$R_0(i, \alpha, \Pi(W_1; X), \Pi(W_2; X))$$

is satisfied by Δ_1 of ($\mathscr{A}_0$-2).

Next consider F_2. The premise of F_2 is $R_0(i, \alpha, Y, V)$, which is the premise of Δ_2. So, by Δ_2, $\Delta(i,(Y, V), \alpha)$.

But $(Y, V)=\Pi(W; Y, V)$, and hence $F_2(i, \alpha, W)$ holds by Δ_2 of ($\mathscr{A}_0$-2).

DI(Ord, V)

We must find U, X_0, V, Y_0, Z_0, W_0 so that $\mathrm{DI}_1'(i, \gamma, U, X_0, V, Y_0, Z_0)$ and $\mathrm{DI}_2'(i, \gamma, W_0)$ hold. Define them as follows.

$$U=\lambda X\mathscr{C}[A_1(i, \gamma); 0, 1],$$

$$X_0=\lambda X\Pi(X; 0, k_0(i, \gamma), \beta_0(i, \gamma)),$$

$$V=\lambda X\mathscr{C}[A_2(i, \gamma); 0, 1],$$

$$Y_0=\lambda X\Lambda qX,$$

$$Z_0=\lambda XX,$$

$$W_0=\lambda u\lambda X'\lambda z\lambda Y\lambda Z\mathscr{C}[u=0\wedge A_1(i, \gamma), u>0\wedge z=0\wedge A_2(i, \gamma),$$

$$u>0\wedge z>0\wedge A_3(i, \gamma); X', \mathrm{TPL}(m_0(i, \gamma); l, \Pi(Y; l)),$$

$$Z, \mathrm{ept}],$$

where $[X]=\rho_0(i, \gamma)$, $[Y]=\Lambda l\rho_0(i, \gamma_l)$, $[X']\equiv T\langle k_0\rangle$ and $[Z]=\rho_0(i, \delta_0)$.

W_0 was in fact defined in ($\mathscr{A}_0$-3). With some simple computations, it can be seen that

$$\mathrm{DI}_1'(i, \gamma, U, X_0, V, Y_0, Z_0)$$

can be reduced to Θ_1. To show $\mathrm{DI}_2'(i, \gamma, W_0)$, assume the premises of it. In any of the three cases, these premises imply the same of Θ_2 in ($\mathscr{A}_0$-3), and hence by Θ_2 we obtain the conclusion of $\mathrm{DI}_2'(i, \gamma, W_0)$, that is

$$\Theta(i, \Pi(W_0; u, X, z, Y, Z), \gamma).$$

The reason why W_0 must have the form above was explained in Section 6 of Part II. As was explained in DTI, we can thus conclude that $\mathrm{DI}_2'(i, \gamma, W_0)$ is valid.

DDI By ($\mathscr{A}_0$-4).

fm($\mathscr{I}$)

We put $\mathscr{I}=\mathscr{I}_0$; that is, $\mathscr{I}$ is interpreted by $\mathscr{I}_0$. Define, with $[X]=T\langle k_0\rangle$,

$$Y_0=\mu X\lambda n\lambda l\Pi(\eta_0; i, \Pi(\mathscr{I}_0; i, \gamma, n), X, n, l),$$
$$Z_0=\lambda X\lambda\delta\lambda U\Pi(\zeta_0; i, \gamma, X, \delta, U).$$

It suffices to show

$$\Pi(Y_0; X, n, l)=\Pi(\eta_0; i, \Pi(\mathscr{I}_0, i, \gamma, n), X, n, l)$$

and

$$\Pi(Z_0; X, \delta, U)=\Pi(\zeta_0; i, \gamma, X, \delta, U),$$

for, then fm(i, γ) is satisfied by ($\mathscr{A}_0$-5). But the equations above are trivial.

fnc(A, t)

Put $U_0=\lambda Z\Pi(Z; h)$.

fncv(A)

Suppose there is a Z_0 satisfying the premise. Put

$$V_0=\lambda Y\lambda f\Pi(Z_0; Y).$$

Then

$$\Pi(V_0; Y, f)=\Pi(Z_0; Y).$$

So $B(Y)$ implies $A(f, \Pi(V_0; Y, f))$ for every f, which validates the conclusion.

WF(I)

μ_0 satisfies this (see ($\mathscr{A}_0$-6)).

This concludes the proof of (*).

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