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The Functional Structure of the ω-Type-Iteration of Ordinal Diagrams

Dedicated to Professor Gaisi Takeuti on his sixtieth birthday

by

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We consider Od(ω), the self-iterating scheme of ordinal diagrams developed in [4] with the restriction that Ω be ω (the set of natural numbers). Let $\{\boldsymbol{O}_n\}_n$ be defined as follows.

$$\boldsymbol{O}_1=\boldsymbol{O}(\boldsymbol{I}, \boldsymbol{A})\,, \qquad \boldsymbol{O}_{n+1}=\boldsymbol{O}(\boldsymbol{O}_n, \boldsymbol{O}_n)\,,$$

where in $\boldsymbol{O}_{n+1}$ the order of $\boldsymbol{O}_n$ is assumed to be $<_{\circ}$, $\circ$ being the first element of $\boldsymbol{I}$. Theorem 2 in [3] claims that

$$|\mathrm{Od}(\omega)|=\lim|\boldsymbol{O}_n|$$

all with the order $<_{\circ}$, and the value be a fixed point of the theory $\boldsymbol{O}$. For the sake of simplicity, we deal with the case where $\boldsymbol{I}$ be a singleton and the second basic sets be also singletons, say $\{\circ\}$. The systems can then be written as:

$$\boldsymbol{O}_1=\boldsymbol{O}(\{\circ\})\,, \qquad \boldsymbol{O}_{n+1}=\boldsymbol{O}(\boldsymbol{O}_n)\,.$$

We shall here present the universe of "methods" that ensures the accessibility of the system Od(ω) (in the simplified version) combining the techniques in [5], [7] and [9] as described below.

1°. A system $\mathscr{S}(n)$ is defined for each n in which the accessibility proof of $\boldsymbol{O}_n$ can be carried out. The treatment is similar to that of [5], but here $\mathscr{S}(n)$ is based on intuitionistic logic.

2°. The HP (denoted as n-HP) in [9] be applied to $\mathscr{S}(n)$. As a consequence, the accessibility of $\boldsymbol{O}_n$ becomes valid. That is, a modulus of finiteness function for decreasing sequences exists in n-HP. The functional interpretation of $\mathscr{S}(n)$ must be included in n-HP as the step to $\mathscr{S}(n+1)$.

3°. By virtue of the uniformity of 1° and 2° with respect to n, a consistent algorithm can be defined for Od(ω) to obtain a modulus of finiteness function. This is in its essence the CP in [7].

The entire process of 1°$\sim$3° will be called ω-HP, and this represents the functional structure of Od(ω). It is a theory which combines syntax and semantics, and we propose that such an approach be admitted in the world of proof-theory.

For the reader's convenience, we repeat some definitions in some references. We assume, however, the knowledge in the theory of ordinal diagrams as well as in the material of our recently published articles.

§ 1. Preliminaries and the system

Definition 1.1. (See Definition 1.1 in [4].)

1) Let ω be the set of natural numbers with the natural order. An element of ω will be a stage indicator, and will be denoted by $r, s, \cdots$. We shall define the elements of $\mathrm{Od}(\omega)$, which we call ω-diagrams, and the function $\mathrm{stg}(\alpha)$, the "stage of construction" of α, for α ω-diagrams, simultaneously.

(1) $\circ$ is a connected ω-diagram; $\mathrm{stg}(\circ)=-1\ (<0)$.

(2) If $r\in\omega$, $i\in\mathrm{Od}(\omega)$ with $\mathrm{stg}(i)<r$, and $\alpha\in\mathrm{Od}(\omega)$ with $\mathrm{stg}(\alpha)\leq r$, then $\beta\equiv(r; i, \alpha)$ is a connected ω-diagram; $\mathrm{stg}(\beta)=r$.

(3) The non-connected elements and their components are defined as usual.

$$\mathrm{stg}(\alpha_1\#\cdots\#\alpha_m)=\max(\mathrm{stg}(\alpha_1), \cdots, \mathrm{stg}(\alpha_m)) .$$

2) $O(r)=\{\alpha\in\mathrm{Od}(\omega); \mathrm{stg}(\alpha)\leq r\}$. An element of $O(r)$ will be called an r-diagram. Obviously $O(s)$, $s=0, 1, 2, \cdots$, are increasing as sets.

3) $\mathrm{stg}(\alpha_1, \cdots, \alpha_m)=\max(\mathrm{stg}(\alpha_1), \cdots, \mathrm{stg}(\alpha_m))$.

4) $=$ is defined as usual.

5) $\mathfrak{l}((r; i, \alpha))=\mathfrak{l}(i)+\mathfrak{l}(\alpha)+1$; that is, $\mathfrak{l}$ represents the constructional complexity.

6) We write $I(r)$ for $O(r-1)$ when $r>0$. The elements of $I(r)$ considered in $O(r)$ are called r-atoms.

7) Various elementary notions of ω-diagrams with respect to $r(\in\omega)$ can be defined as in Definitions 1.1 and 1.2 of [4]. See § 1 of [4] for the consequences of the definitions also. The i-order of r-diagrams, i an r-atom, will be written as $<_{r,i}$.

8) Propositions 2.1 and 2.2 in [4] respectively claim that the theories of approximations and fundamental sequences can be developed for $\mathrm{Od}(\omega)$. Note also that $(\boldsymbol{O}_n, <_\circ)$ is isomorphic to $(O(n), <_{n,\circ})$, $n\in\omega$ (see Theorem 1 in [4]).

9) The groundedness property, projection and elevation of $O(r)$ are defined uniformly in r; see [8].

Definition 1.2. 1) The language $\mathscr{L}_\omega$ is defined similarly to the $\mathscr{L}$ in Definition 1.1, Part I of [9], where the predicate constants G, Ord and lss as well as the function constant $\mathscr{I}$ have one extra argument place (for r).

2) A formula is $\mathscr{L}_\omega$-recursive if it does not contain G, Ord or quantifiers.

3) $\mathscr{L}_\omega$-*u*-*a* (universally arithmetical)-formulas are defined as in Definition 1.1, Part I of [9], and a formula of the form $\exists xB(x)$ is allowed only when $B(x)$ is $\mathscr{L}_\omega$-*u*-*a*.

4) An $\mathscr{L}_\omega$-formula is said to be purely isolated if it does not contain G or Ord and if every $\forall f$ is isolated (essentially Π^1_1).

5) The language $\mathscr{L}_r$ is the restriction of $\mathscr{L}_\omega$ where the first arguments of G and Ord are constant stage indicator r.

Note. The elementary properties of $O(r)$, fundamental sequences and the functions such as "projection" and "elevation" can be formulated in the language $\mathscr{L}_\omega$ "uniformly in r" (see Lemma and Proposition 2.2 in [4] and § 1, Part I of [9]). For instance, the i-projection of α in $O(r)$ is denoted by $\pi(r; i, \alpha)$, where i is supposed to be an r-atom.

Definition 1.3. We are to define formal systems $\{\mathscr{S}(r); r \leq \omega\}$ similarly to the systems in Definition 1.5 of [5], so that $\mathscr{S}(r)$ is a system of the language $\mathscr{L}_r$, $r \leq \omega$.

We first list the axioms and the inference rules of $\mathscr{S}(r)$ for $r \in \omega$. Those which refer to each r individually are indicated as r-n (the nth axiom with respect to r) with r a constant, while those which are given uniformly in (independently of) r are listed without r. In the latter case, the parameter which stands for the elements of ω will be denoted by letter e. $\mathscr{S}(r)$ is a modified version of ASOD; see Definition 1.3, Part I of [9] as well as Definitions 1.1 and 1.2 there for various notations.

In the formal language, $\alpha <_{r,i} \beta$ will be written as $<(r; i, \alpha, \beta)$ and $<_{r,i}$ will be written as $<(r; i)$

r-1) The intuitionistic arithmetic is assumed as the basis where the formulas are those of $\mathscr{L}_r$ and $\neg\neg A \vdash A$ is admitted when A is $\mathscr{L}_r$-recursive. The induction axiom for A an $\mathscr{L}_r$-formula will be denoted as IND$(r; A)$.

2) The axioms for $\mathscr{L}_\omega$-terms.

3) The axioms on the constants concerning the elementary theory of Od(ω).

4) The defining equations of π_0 and ε_0. These are uniform in r; that is, $\pi_0(e; i, \alpha, \sigma)$ and $\varepsilon_0(e; i, \gamma)$ are defined so that they make sense when $e \in \omega$ and other conditions are satisfied.

5) The axioms on the fundamental methods for $I(e)$ uniform in e;

$$fm(e; I(e), <(e; \circ), \mathscr{I}(e)).$$

This is read: if $e \in \omega$, then $\mathscr{I}(e)$ is the mechanism to produce fundamental sequences for $I(e)$ with respect to $<(e; \circ)$.

r-6) The equality axioms for the $\mathscr{L}_r$-formulas.

r-7) The axioms of transfinite induction along $(I(r), <(r; \circ))$, TI$(r; I(r), A)$, for A an $\mathscr{L}_r$-formula.

r-8) The axioms of bar induction, BI$(r; R, A)$, where R is $\mathscr{L}_\omega$-recursive and A is an $\mathscr{L}_r$-formula.

r-9) Definition by transfinite induction along $I(r) \cup \{\infty\}$ $(= I(r, \infty))$, DTI$(r; I(r, \infty); G, U)$:

$$\forall i \forall \alpha (i \in I(r) \wedge \alpha \in O(r) \vdash (G(r; i, \alpha) \equiv U(r; i, \alpha, G[r; i]))),$$

where $G[r; i]$ abbreviates

$$\{j, \sigma\}(<(r; \circ, j, i) \vee G(r; j, \sigma)),$$

and U is defined as below.

$F(r; i, \alpha, X)$: $\forall j < i \forall \sigma$ ("σ is a j-section of α" $\vdash X(j, \sigma)$)

$U(r; i, \alpha, X): F(r; i, \alpha, X) \wedge$ "α is (r, i)-accessible in $\{\beta\}F(r; i, \beta, X)$"

$G(r; i, \alpha)$ is read: "α is (r, i)-grounded in $O(r)$ if i is an r-atom."

r-10) Definition by induction, DI(r; Ord, V). This can be defined similarly to the previous DI(Ord, V) in terms of the following.

$H(r; i, \alpha)$: "α is of the form $(r; k, b, \beta)$" $\wedge$ "$<(r; \circ, k, i)$ in $I(r)$" $\wedge\, G(r; k, \beta)$

$J(r; i, \gamma): H(r; i, \varepsilon_0(r; i, \gamma))$

$V(r; i, \gamma, X): J(r; i, \gamma) \wedge$ "γ is of the form $\gamma_1\#\cdots\#\gamma_m$ $\wedge\, \forall n \leq m X(i, \gamma_n)$" $\vee$ "γ is of the form $(r; j, c, \delta) \wedge \leq(r; \circ, i, j)$ in $I(r) \wedge X(i, \delta)$".

We write $J(r; i)$ for the set of γ satisfying $J(r; i, \gamma)$. Similarly for other relations. Ord$(r; i)$ represents the theory of diagrams $\boldsymbol{O}(I\langle r; i\rangle, I(r) * J(r; i))$ where $I\langle r; i\rangle = \{j; \leq(r; \circ, i, j) \wedge j \in I(r)\}$.

11) Definition by double induction, DDI(e; lss; W). This is defined as previously, and lss$(e; i; j, \gamma, \delta)$ represents the relation $<(e; j, \gamma, \delta)$ in Ord$(e; i)$. This relation is $\mathscr{L}_\omega$-recursive.

r-12) The axiom on $\mathscr{J}$, $fm(r; \mathscr{J}(r))$. That is, $\mathscr{J}(r; i)$ is the fundamental method for the system $J(r; i)^\#$ with regards to $\prec_{r,i}$ (the (r, i)-order for $J(r)$) for i an r-atom.

r-13) the $\forall$-elimination of function-type with respect to $\mathscr{L}_r$-formulas.

r-14) The $\forall$-introduction of function-type with respect to $\mathscr{L}_r$-formulas.

Now the systems. Assume $r \in \omega$.

[0] $\mathscr{S}(0)$ is the system with 0-1)$\sim$0-14).

Suppose $r > 0$.

[r] $\mathscr{S}(r)$ is the system $\mathscr{S}(r-1)$ augmented by r-1), r-6)$\sim r$-10) and r-12)$\sim r$-14).

[ω] Put $\mathscr{S}(\omega) = \bigcup \{\mathscr{S}(r); r \in \omega\}$.

Remark 1. $\mathscr{S}(0)$ corresponds to the system of diagrams whose order type is ε_0, and the functional structure of such a system has been determined in [7] in terms of the "construction principle", CP, which is an extension of primitive recursive functionals of finite type and hence a modulus of finiteness function can be found in CP for $(I(1), <(1; \circ))$.

2. The constants in 3)$\sim$5) and 11) above can be defined in Heyting arithmetic, and hence these axioms are valid.

Definition 1.4. An $\mathscr{L}_\omega$-formula $C(e)$ with e a designated parameter is said to be $\mathscr{S}(\omega)$-provable uniformly with respect to e if there is a primitive recursive method $M(e)$ such that $M(r)$ provides a proof of $C(r)$ in $\mathscr{S}(r)$ (and hence $C(r)$ is an $\mathscr{L}_r$-formula) for every $r \in \omega$.

Definition 1.5. 1) acc$(e; i, \alpha)$ will express "α is (e, i)-accessible in $O(e)$", that is, $\forall f m f(e; i; \alpha, f)$ (see Definition 1.2, Part I of [9]).

2) $\mathrm{acc}(\omega; i, \alpha)$ will express "α is i-accessible in $\mathrm{Od}(\omega)$."

3) $\mathrm{acc}(I(r))$ will express "every r-atom is $(r, \circ)$-accessible in $I(r)$."

PROPOSITION 1.1. *Suppose* $\mathrm{acc}(I(r))$ *is* $\mathscr{S}(r-1)$*-provable. Then* $\mathrm{TI}(r; I(r), A)$ *is* $\mathscr{S}(r)$*-provable for every* A *an* $\mathscr{L}_r$*-formula, and hence r-7) would be redundant.*

Proof. We can follow the proof of Proposition 1.1, Part III of [9].

Definition 1.6. Let $B(e) \vdash A$ be an $\mathscr{L}_\omega$-formula where $B(e)$ is $\mathscr{L}_\omega$-recursive with parameter e, A is purely isolated (see Definition 1.2) and is free of e. If this is $\mathscr{S}(\omega)$-provable uniformly with respect to e, then we introduce a special infinite reasoning, called the ω-reasoning. That is, from the assumption of uniform provability of $B(e) \vdash A$, we allow that

$$(1) \qquad \exists e \in \omega B(e) \vdash A$$

be inferred, and finitely many cuts with the theorems of $\mathscr{S}(0)$ are allowed to be applied to (1).

A formula which is provable in this way will be said to be $\mathscr{S}(\omega)$-provable with the ω-reasoning.

§ 2. The accessibility statement and the proof-procedure

THEOREM 1. *The formula in* (1) *below is* $\mathscr{S}(\omega)$*-provable uniformly with respect to e.*

$$(1) \qquad B(e; i, \alpha) \vdash \mathrm{acc}(e; i, \alpha)\,,$$

where $B(e; i, \alpha)$ *abbreviates* $\alpha \in O(e) \wedge i \in I(e; \infty)$.

Conclusion. (2) $\alpha \in \mathrm{Od}(\omega) \wedge i \in \mathrm{Od}(\omega) \vdash \mathrm{acc}(\omega; i, \alpha)$ is $\mathscr{S}(\omega)$-provable with the ω-reasoning (see Definition 1.6).

For the derivation of the Conclusion from Theorem 1, we can follow the argument in § 2 of [5]; the inferences there can be modified to intuitionistic ones. That is, from (1) follows

$$(3) \qquad B(e; i, \alpha) \vdash \mathrm{acc}(\omega; i, \alpha)$$

uniformly. Then we obtain

$$(4) \qquad \exists e \in B(e; i, \alpha) \vdash \mathrm{acc}(\omega; i, \alpha)$$

by the ω-reasoning, since B is $\mathscr{L}_\omega$-recursive and $\mathrm{acc}(\omega; i, \alpha)$ is purely isolated. On the other hand,

$$(5) \qquad \alpha \in \mathrm{Od}(\omega) \wedge i \in \mathrm{Od}(\omega) \vdash \exists e \in \omega B(e; i, \alpha)$$

is a theorem of $\mathscr{S}(0)$. A cut applied to (4) and (5) then yields (2).

Now Theorem 1 here corresponds to the Conclusion in § 2, Part I of [9]; that is, for each $r \in \omega$,

(1-r) $$B(r; i, \alpha) \vdash \mathrm{acc}(r; i, \alpha).$$

We can follow the proof in Part I of [9] uniformly in r, and it can be easily checked that the entire proof for r can be carried out within $\mathscr{S}(r)$. So (1) above is $\mathscr{S}(\omega)$-provable uniformly with respect to e.

§3. A progress of hyper-principles and the functional interpretations of the systems

We are to determine the hyper-principle for $\mathscr{S}(r)$, $r \in \omega$, based on that for $\mathscr{S}(r-1)$, which here includes the functional interpretation of $\mathscr{S}(r-1)$, "progressively" in r.

[0] We start out with $\mathscr{S}(0)$, in which, as was remarked in §1, the transfinite induction along ε_0 is provable and it admits the functional interpretation in terms of CP. Since $(I(1), <(1; \circ))$ is isomorphic to ε_0, this means that a modulus of finiteness function be found in CP for $(I(1), <(1; \circ))$. We shall call CP the 0th hyper-principle, 0-HP.

[r] Assume $(r-1)$-HP the $(r-1)$th hyper-principle for $\mathscr{S}(r-1)$. This means that the "accessibility of $O(r-1)$" (which is a theorem of $\mathscr{S}(r-1)$) is valid (in $(r-1)$-HP). This implies that a modulus of finiteness for $I(r)$, say $\mu(r)$, can be found in the universe of $(r-1)$-HP, and hence our $\mathscr{S}(r)$ corresponds to the system $\mathscr{S}_0$ in Definition 1.1, Part III of [9], but without the axiom WF($\boldsymbol{I}$). One should also recall that, since acc($I(r)$) is $\mathscr{S}(r-1)$-provable, Proposition 1.1 claims that r-7) (the transfinite induction) is redundant in $\mathscr{S}(r)$. In other words, r-7) can be applied in $\mathscr{S}(r)$ as needed, but it does not require interpretation.

We shall implicate the property of $\mu(r)$ as a principle:

(r-mf) $$\mathrm{mf}(I(r); \mu(r)).$$

Let us now develop the theory of r-hyper-principle in correspondence with that in Part II of [9].

Definition 3.1. 1) The language $\mathscr{L}_{tp}(r)$ of r-type-forms consists of the language of $\mathscr{L}_\omega$-terms and $\mathscr{L}_\omega$-recursive formulas in §1 as well as the new symbols for $\mathscr{L}_{tp}$ (see Part II of [9]); that is,

$$N_0,\ \mathrm{ept},\ \Lambda,\ \{\ \},\ \rightarrow,\ \mathscr{C},\ [\ ],\ \Rightarrow,\ \Pi,\ T_r,\ \langle\ \rangle,\ \mathscr{R}_r.$$

2) The definition of r-type-forms is parallel to Definition 1.1, Part II of [9], and hence let us confine ourselves to some new features.

(5) $T_r\langle i\rangle$ is an r-type-form if i is an $\mathscr{L}_\omega$-term of *at*-type. We shall write $T\langle r; i\rangle$ for $T_r\langle i\rangle$. This has the reduction rule:

$$T\langle r; i\rangle \Rightarrow \Lambda \mathrm{l}\mathscr{C}[\mathrm{l}=0, \mathrm{l}=1; T_1\langle r; i\rangle, T_2\langle r; i\rangle, \mathrm{ept}],$$

where the right hand side will be abbreviated to $T'(r; i)$. $T_1\langle r; i\rangle$ abbreviates

$$\Lambda j\mathscr{C}[<(r; \circ, j, i); N_0 \to T\langle r; j\rangle, \text{ept}]$$

and $T_2\langle r; i\rangle$ abbreviates

$$S_q \longrightarrow ((N_0 \to T_1\langle r; i\rangle) \to N_0)\,.$$

$N_0 \to T_1\langle r; i\rangle$ will be abbreviated to $T_2(r; i)$. Notice that $<(r; \circ, j, i)$ makes sense only when j and i are r-atoms, and hence $\mathscr{C}[<(r; \circ, j, i); N_0 \to T\langle r; j\rangle, \text{ept}]$ is reduced to ept otherwise.

(8) $\mathscr{R}_r[\mathfrak{s}, \mathfrak{t}, v, x; \mathfrak{l}]$ (written as $\mathscr{R}[r; \mathfrak{s}, \mathfrak{t}, v, x; \mathfrak{l}]$ henceforth) is an r-type-form if certain conditions are satisfied, where $(r-1)$-type-forms are allowed for $\mathfrak{s}$ and $\mathfrak{t}$ (but $\mathscr{R}_r$ is not allowed in them).

Proposition 1.1 of Part II in [9] holds if read in the present context. The notions of reducts and normality in Definition 1.2 there can also be referred to.

Definition 3.2. 1) For $\mathfrak{s}$ an r-type-form, we define $\text{stg}(\mathfrak{s})(\leq r)$, the stage of $\mathfrak{s}$ to be the first s such that $\mathfrak{s}$ is an s-type-form.

2) The constructional complexity of $\mathfrak{s}$ an r-type-form, written as $*(\mathfrak{s})$ is defined as follows. We write $\omega\,(m; n)$ for $\omega^m \cdot n$. If $\text{stg}(\mathfrak{s}) \leq r-1$, then $*(\mathfrak{s}) < \omega(2r; 1)$ is assumed to have been defined; see below otherwise. (See 3) of Definition 1.2, Part II of [9]; here $\#$ will denote the natural sum.)

$*(\Lambda x\mathfrak{t}) = *(\mathfrak{t}) \# \omega(2r; 1)$ or $*(\mathfrak{t}) \# \omega(2r; 2)$ as the case may be.
$*(\mathfrak{s} \to \mathfrak{t}) = *(\mathfrak{s}) \# *(\mathfrak{t})$
$*(\mathscr{C}[(\mathscr{A}); (\mathfrak{t})] = *(\mathfrak{t}_1) \# \cdots \# *(\mathfrak{t}_{m+1})$
$*(\Pi(\mathfrak{s}; \phi)) = *(\mathfrak{s}) \# \omega(2r; 1)$
$*(T\langle r; i\rangle) = \omega(2r; 2)$
$*(\mathscr{R}[r; \mathfrak{s}, \mathfrak{t}, v, x; \mathfrak{l}]) = \omega(2r+1; 1)$

Proposition 1.2 and the Corollary of it, Part II of [9] holds with the modification that

$$*(\mathfrak{s}) < \omega(2r+1; 1)$$

if $\mathfrak{s}$ does not contain $\mathscr{R}_r$.

Definition 3.3. For the "objects" of r-hyper-types (called r-methods), see Definition 1.3, Part II of [9] (under the assumption of the $(r-1)$-methods). The collection of r-methods will be called the r-universe.

PROPOSITION 3.1. *The definition of r-methods is consistent and complete.*

Proof. This is proved in the same manner as Proposition 1.4, Part II of [9] by induction on $*(\mathfrak{s})$ for any $\mathfrak{s}$ an r-hyper-type. For (5), that is, the definition of a $T\langle r; i\rangle$-object for $i \in I(r)$, we can rely on the (informal) transfinite induction along $(I(r), <(r; \circ))$, since the accessibility of this system is assumed to be valid (in $(r-1)$-HP).

Definition 3.4. (See Definition 2.1, Part II of [9].)

1) The language $\mathscr{L}_{tm}(r)$ of r-term-forms consists of $\mathscr{L}_{tp}(r)$, the "variable-forms of the associated r-type-forms" and the symbols

$$\mathscr{I}_0, \eta_0, \zeta_0, \mathscr{B}, \lambda, \Pi, \mathscr{C}, [\ ]\,.$$

Note. Among the primitive symbols, unlike the original case, the symbol μ_0 is missing. The reason is that μ_0 represented a modulus of finiteness function for $\boldsymbol{I}$ the basic set, while here such a function for $I(r)$, the basic set for $O(r)$, is assumed to exist in the $(r-1)$-universe.

2) The r-term-form of a certain r-type-form and some related notions can be defined as before. We write $\Phi(r; \Xi)$ for an expression of the form $\Phi_r(\Xi)$.

3) The semantics of r-term-forms can be developed as in § 3, Part II of [9].

Definition 3.5. 1) The language $\mathscr{L}(r)$ containing $\mathscr{L}_{tp}(r)$ and $\mathscr{L}_{tm}(r)$ can be defined as the language $\mathscr{L}_0$ in Definition 4.1, Part II of [9]. In particular it contains the predicate constants Δ_r, Θ_r and Σ_r.

2) The r-formula-forms are defined as those of $\mathscr{L}_0$. We write $\Delta(r; i, \Phi, \alpha)$ for $\Delta_r(i, \Phi, \alpha)$. Similarly for other symbols.

Definition 3.6. 1) The axiom set $\mathscr{A}(r)$ of the language $\mathscr{L}(r)$ is similar to the axiom set $\mathscr{A}_0$ in Definition 5.1, Part II of [9], but ($\mathscr{A}_0$-6) has no equivalent here, since it asserts that μ_0 be a modulus of finiteness function for the basic set $\boldsymbol{I}$, that is, mf($\boldsymbol{I}$, μ_0), while here mf($I(r)$, $\mu(r)$) is a fact by the induction hypothesis (see (r-mf) above).

2) The assumption (r-FSPR) for $\mathscr{L}(r)$ asserts that ($\mathscr{I}^*_r$, η^*_r, ζ^*_r) satisfy ($\mathscr{A}(r)$-5). MFPR is replaced by (r-mf).

3) Assignments and the related notions can be defined as in Definition 6.1, Part II of [9]; the well-definedness for the case of $\Delta(r; i, X, \alpha)$ can be established by the transfinite induction along ($I(r)$, $<(r; \circ)$).

PROPOSITION 3.2. *The axioms in* $\mathscr{A}(r)$ *are valid under the principles* CNPR, (r-FSPR) *and* (r-mf). (*See Proposition* 6.1, *Part II in* [9].)

Remark. As we stated at the beginning, the functional interpretation of the system under consideration is a part of the hyper-principle, and hence r-HP can be defined only after the functional interpretation of $\mathscr{S}(r)$ is completed.

Definition 3.7. The mr-translation of the formula of $\mathscr{S}(r)$ can be defined as in Definition 2.1, Part III of [9]. For instance,

$$\text{mr}(G(r; i, \alpha)) \equiv \exists X \Delta(r; i, X, \alpha)\,,$$

where $[X] = T\langle r; i\rangle$.

THEOREM 2. *If A is a theorem of* $\mathscr{S}(r)$ *and*

$$\text{mr}(A) \equiv \exists X B(\boldsymbol{X}, \boldsymbol{Q})\,,$$

where Q stands for the free variables in A and $\exists X$ stands for all the existential quantifiers in the translation, then there is a sequence of r-hyper-terms Φ^* (with parameters Q) such that $B(\Phi^*, Q)$ is valid in the semantics of the r-formula-forms (see Definition 3.6). *Furthermore, there is a primitive recursive scheme $\mathscr{P}(e, x)$ such that for any r and any P a proof in $\mathscr{S}(r)$, $\mathscr{P}(r, \ulcorner P \urcorner)$ provides such Φ^* for the theorem proven by P, where $\ulcorner P \urcorner$ is a coding of P.*

Definition 3.8. The $(r-1)$-HP, the theories (both syntactical and semantical) of r-type-forms, r-terms-forms, r-formula-forms, the axiom set, the mr-translation and the result in Theorem 2 (which is the functional interpretation of $\mathscr{S}(r)$) all put in one will be called the r-hyper-principle, r-HP.

COROLLARY. *The accessibility of $O(r)$ with respect to the (r, i)-order for any i an r-atom has a sound functional interpretation in the r-universe; in other words, it is r-*HP-*valid. As a special case, the accessibility of $I(r+1)$ with respect to the $(r+1, \circ)$-order is r*-HP-*valid.*

§ 4. The functional interpretation of Od(ω)

Definition 4.1. 1) Let $\phi(e)$ be a primitive recursive scheme with parameter e such that, for each $r \in \omega$, $\phi(r)$ provides an r-hyper-term with the same free variable (for all r). Then ϕ will be called an ω-scheme. Note that if an assignment is determined for these free variables, then $\phi(r)$ becomes an r-method.

2) Let ϕ be an ω-scheme and let ρ be a primitive recursive function (with *at*-type arguments). Then $\phi \circ \rho$ (the composition of ϕ and ρ) will be called an ω-method and the collection of ω-methods will be called the ω-universe.

3) Let $D(e, X_1, \cdots, X_n)$ be a formula-form-like expression without Δ, Θ or Σ, with parameters $e, X_1, \cdots, X_n$. Let $\phi_1, \cdots, \phi_n$ be ω-schemes. If, for each $r \in \omega$, $D(r, \phi_1(r), \cdots, \phi_n(r))$ becomes an r-formula-form, then we say that $(D, \phi_1, \cdots, \phi_n)$ is an ω-expression.

4) $\bigcup \{r\text{-HP}; r \in \omega\}$ augmented by the ω-universe and the ω-expressions will be called the ω-hyper-principle, ω-HP.

THEOREM 3. *The accessibility of* Od(ω),

$$i \in \mathrm{Od}(\omega) \wedge \alpha \in \mathrm{Od}(\omega) \vdash \mathrm{acc}(\omega; i, \alpha),$$

*is "valid in ω-*HP*" in the following sense. Let*

$$\exists X \forall f (S(i, \alpha, f) \vdash \mathrm{mf}(X(f), f))$$

be the mr-*translation of* acc(ω; i, α), *where S is $\mathscr{L}_\omega$-u-a and* mf(x, f) *asserts that x be a modulus of finiteness of f. Then there is an ω-method $\phi \circ \rho$ such that for every $i \in$* Od(ω), *$\alpha \in$* Od(ω) *and f satisfying $S(i, \alpha, f)$,* mf$(\phi(\rho(i, \alpha))(f), f)$ *is $\rho(i, \alpha)$-valid.*

Proof. Recall that the accessibility of Od(ω) is $\mathscr{S}(\omega)$-provable with the ω-reasoning (see the Conclusion in § 2), and the proof procedure is described in (3) $\sim$ (5)

in §2. By (3) there (which is an immediate consequence of Theorem 1), there is a primitive recursive method $M(e)$ such that, for each $r \in \omega$, $M(r)$ provides a proof of

$$B(r; i, \alpha) \vdash \mathrm{acc}(\omega; i, \alpha)$$

in $\mathscr{S}(r)$. The proof of Theorem 2 (in §3) yields that there is a primitive recursive method P with the property that $P(r, M(r))$ provides an r-hyper-term Φ_r^* so that

(a) $$B(r; i, \alpha) \vdash \forall f(S(i, \alpha, f) \vdash \mathrm{mf}(\Pi(\Phi_r^*; f), f))$$

is r-HP-valid. If we put $\phi(r) = P(r, M(r))$, then ϕ is an ω-scheme. On the other hand, the (formal) mr-translation of (4) in §3 is

(b) $$\exists Y \forall e(e \in \omega \wedge B(e; i, \alpha) \vdash \forall f(S(i, \alpha, f) \vdash \mathrm{mf}(\Pi(Y; e; f), f)))\,.$$

If we let Y be the ϕ above, then

$$\Pi(Y; r; f) = \phi(r)(f) = \Phi_r^*(f)$$

(with the natural interpretation of Π), and hence (b) with $Y = \phi$ and $e = r$ becomes r-valid by virtue of (a) for $r \in \omega$. On the other hand, (5) in §2 is a theorem of $\mathscr{S}(0)$, and hence there is a primitive recursive ρ such that

(c) $$\alpha \in \mathrm{Od}(\omega) \wedge i \in \mathrm{Od}(\omega) \vdash \rho(i, \alpha) \in \omega \wedge B(\rho(i, \alpha); i, \alpha)\,.$$

(In fact $\rho(i, \alpha) = \max(\mathrm{stg}(\alpha), \mathrm{stg}(i) + 1)$.) (b) with $e = \rho(i, \alpha)$ and $Y = \phi$ and (c) yield that for any $\alpha \in \mathrm{Od}(\omega)$ and $i \in \mathrm{Od}(\omega)$,

$$\forall f(S(i, \alpha, f) \vdash \mathrm{mf}(\phi(\rho(i, \alpha))(f), f))$$

be $\rho(i, \alpha)$-valid. This proves the theorem.

We can now conclude that the functional structure of $\mathrm{Od}(\omega)$ is supplied by the ω-HP as defined above.

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