COMMENTARII MATHEMATICI
UNIVERSITATIS SANCTI PAULI
Vol. 35, No. 2, 1986

ed. RIKKYO UNIV/MATH
IKEBUKURO TOKYO
171 JAPAN

(m, n)-Two-sided Pure Semigroups

by

Stojan BOGDANOVIĆ and Todor MALINOVIĆ

(Received December 16, 1985)

In this paper we shall describe a semigroup S in which S^{m+n+1} is a semilattice of groups (Theorem 1). This result is a generalization of the main result of N. Kuroki, [5]. By Theorem 2 we characterize a nilpotent semigroup.

A subsemigroup A of a semigroup S is a bi-ideal of S if $ASA \subseteq A$, [1]. A bi-ideal A of a semigroup S is called two-sided pure if $A \cap xSy = xAy$ holds for every $x, y \in S$. A semigroup S is called T^*-pure if every bi-ideal of it is two-sided pure, [5].

Throughout this paper, Z^+ will denote the set of all positive integers.

For undefined notions and notations we refer to [1] and [8].

DEFINITION 1. A subset B of a semigroup S is a (m, n)-*two-sided* pure if

$$B \cap x_1 \cdots x_m S y_1 \cdots y_n = x_1 \cdots x_m B y_1 \cdots y_n$$

holds for every $x_1, \cdots, x_m, y_1, \cdots, y_n \in S$, where $m, n \in Z^+$. A semigroup S is (m, n)-*two-sided pure* if every bi-ideal of it is an (m, n)-two-sided pure subset of S.

Remark. $(1, 1)$-two-sided pure bi-ideal is two-sided pure, [5].

LEMMA 1. *Let S be an (m, n)-two-sided pure semigroup. Then*

$$x_1 \cdots x_m S y_1 \cdots y_n = (x_1 \cdots x_m)^2 S (y_1 \cdots y_n)^2$$

for every $x_1, \cdots, x_m, y_1, \cdots, y_n \in S$.

Proof. Since $x_1 \cdots x_m S y_1 \cdots y_n$ is a bi-ideal of S we have

$$\begin{aligned} x_1 \cdots x_m S y_1 \cdots y_n &= x_1 \cdots x_m S y_1 \cdots y_n \cap x_1 \cdots x_m S y_1 \cdots y_n \\ &= x_1 \cdots x_m (x_1 \cdots x_m S y_1 \cdots y_n) y_1 \cdots y_n \\ &= (x_1 \cdots x_m)^2 S (y_1 \cdots y_n)^2 . \quad \blacksquare \end{aligned}$$

COROLLARY 1. *An (m, n)-two-sided pure semigroup S is completely π-regular.*

Proof. By Lemma 1 we obtain that

$$a^{m+n+1} \in a^m S a^n = a^{2m} S a^{2n} = \cdots = a^{m2^{m+n+1}} S a^{n2^{m+n+1}} \subseteq a^{2(m+n+1)} S a^{2(m+n+1)}$$

holds for every $a \in S$, so by Theorem IV 2 [1] we have that S is completely π-regular.

A semigroup S is *weakly commutative* if for every $a, b \in S$ there exists $k \in Z^+$ such that $(ab)^k \in bSa$, [8]. $\blacksquare$

LEMMA 2. *An (m, n)-two-sided pure semigroup S is weakly commutative.*

Proof. By Definition 1 we have that

$$y_1 \cdots y_n(x_1 \cdots x_m S y_1 \cdots y_n) x_1 \cdots x_m = x_1 \cdots x_m S y_1 \cdots y_n \cap y_1 \cdots y_n S x_1 \cdots x_m$$

holds for every $x_1, \cdots, x_m, y_1, \cdots, y_n \in S$ (since $x_1 \cdots x_m S y_1 \cdots y_n$ is a bi-ideal of S). From this it follows

$$\underbrace{ab \cdots ab}_{n-1} \cdot a \cdot b \cdot \underbrace{ab \cdots ab}_{m-1}(ab)\underbrace{ab \cdots ab}_{n-1} \cdot a \cdot b \cdot \underbrace{ab \cdots ab}_{m-1} \in b \cdot ab \cdots abSab \cdots ab \cdot a \subseteq bSa$$

for every $a, b \in S$. Thus

$$(ab)^{2m+2n-1} \in bSa$$

i.e. S is weakly commutative. ■

LEMMA 3 ([6], S. Lajos). *A semigroup S is a semi-lattice of groups if and only if the set of all bi-ideals of S is a semilattice under the multiplication of subsets.* ■

LEMMA 4. *Let S^{m+n+1} be a semilattice of groups. Then*

(i) $x_1 \cdots x_m S y_1 \cdots y_n = y_1 \cdots y_n S x_1 \cdots x_m$

(ii) $x_1 \cdots x_m S y_1 \cdots y_n = (y_1 \cdots y_n)^2 S x_1 \cdots x_m$

(iii) $x_1 \cdots x_m S y_1 \cdots y_n = x_1 \cdots x_m y_1 \cdots y_n S$

(iv) $x_1 \cdots x_m S y_1 \cdots y_n = S x_1 \cdots x_m y_1 \cdots y_n$

(v) $x_1 \cdots x_m S y_1 \cdots y_n = (x_1 \cdots x_m y_1 \cdots y_n)^{m+n+1} S.$

Proof. (i) By Lemma 3, we have

$$\begin{aligned}
x_1 \cdots x_m S y_1 \cdots y_n
&= (x_1 \cdots x_m S y_1 \cdots y_n)^2 \\
&= (x_1 \cdots x_m S y_1 \cdots y_n x_1 \cdots x_m S)(y_1 \cdots y_n x_1 \cdots x_m S y_1 \cdots y_n) \\
&= (y_1 \cdots y_n x_1 \cdots x_m S y_1 \cdots y_n)(x_1 \cdots x_m S y_1 \cdots y_n x_1 \cdots x_m S) \\
&= y_1 \cdots y_n (x_1 \cdots x_m S y_1 \cdots y_n x_1 \cdots y_m)(S y_1 \cdots y_n x_1 \cdots x_m S) \\
&= y_1 \cdots y_n (S y_1 \cdots y_n x_1 \cdots x_m S)(x_1 \cdots x_m S y_1 \cdots y_n x_1 \cdots x_m) \\
&\subseteq y_1 \cdots y_n S x_1 \cdots x_m
\end{aligned}$$

and since the opposite inclusion also holds we have (i).

(ii) By Lemma 3 and by (i) we obtain that

$$
\begin{aligned}
&x_1 \cdots x_m S y_1 \cdots y_n \\
&\quad = (x_1 \cdots x_m S y_1 \cdots y_n)^3 \\
&\quad = (x_1 \cdots x_m S y_1 \cdots y_n)(y_1 \cdots y_n S x_1 \cdots x_m)(x_1 \cdots x_m S y_1 \cdots y_n) \\
&\quad = (x_1 \cdots x_m S (y_1 \cdots y_n)^2) S ((x_1 \cdots x_m)^2 S y_1 \cdots y_n) \\
&\quad = ((y_1 \cdots y_n)^2 S x_1 \cdots x_m) S (y_1 \cdots y_n S (x_1 \cdots x_m)^2) \\
&\quad \subseteq (y_1 \cdots y_n)^2 S x_1 \cdots x_m \\
&\quad \subseteq y_1 \cdots y_2 S x_1 \cdots x_m \\
&\quad = x_1 \cdots x_m S y_1 \cdots y_n .
\end{aligned}
$$

Hence, (ii) holds.

(iii) By Lemma 3, we have

$$
\begin{aligned}
y_1 \cdots y_n S x_1 \cdots x_m &= (y_1 \cdots y_n S x_1 \cdots x_m)^3 \\
&\subseteq S^{m+n+1} x_1 \cdots x_m y_1 \cdots y_n S x_1 \cdots x_m \\
&\subseteq x_1 \cdots x_m y_1 \cdots y_n S x_1 \cdots x_m S^{m+n+1} \\
&\subseteq x_1 \cdots x_m y_1 \cdots y_n S .
\end{aligned}
$$

From this and from (i) it follows that

$$x_1 \cdots x_m S y_1 \cdots y_n \subseteq x_1 \cdots x_m y_1 \cdots y_n S .$$

Further,

$$
\begin{aligned}
x_1 \cdots x_m y_1 \cdots y_n S &= (x_1 \cdots x_m y_1 \cdots y_n S)^3 \\
&\subseteq x_1 \cdots x_m y_1 \cdots y_n S x_1 \cdots x_m y_1 \cdots y_n S^{m+n+1} \\
&= x_1 \cdots x_m y_1 \cdots y_n S^{m+n+1} S x_1 \cdots x_m y_1 \cdots y_n \\
&\subseteq x_1 \cdots x_m S y_1 \cdots y_n .
\end{aligned}
$$

Thus, the conditions (iii) holds. In a similar way it can be proved that (iv) holds.

(v) By (iii) we have that

$$
\begin{aligned}
&x_1 \cdots x_m S y_1 \cdots y_n \\
&\quad = x_1 \cdots x_m y_1 \cdots y_n S \\
&\quad = (x_1 \cdots x_m y_1 \cdots y_n S)^{m+n+1} \quad \text{(by Lemma 3)} \\
&\quad = x_1 \cdots x_m y_1 \cdots y_n S x_1 \cdots x_m y_1 \cdots y_n S (x_1 \cdots x_m y_1 \cdots y_n S)^{m+n-1} \\
&\quad = (x_1 \cdots x_m y_1 \cdots y_n)^2 S^2 (x_1 \cdots x_m y_1 \cdots y_n S)^{m+n-1} \\
&\quad \vdots \\
&\quad = (x_1 \cdots x_m y_1 \cdots y_n)^{m+n+1} S^{m+n+1} \\
&\quad \subseteq x_1 \cdots x_m y_1 \cdots y_n S .
\end{aligned}
$$

Thus (v) holds.

THEOREM 1. *For a semigroup S the following conditions are equivalent*:

(i) *S is (m, n)-two-sided pure*;

(ii) *S^{m+n+1} is a semilattice of groups*;

(iii) $x_1 \cdots x_m S y_1 \cdots y_n = (y_1 \cdots y_n)^2 S x_1 \cdots x_m$ *for every* $x_1, \cdots, x_m, y_1, \cdots, y_n \in S$;

(iv) *S is a semilattice Y of semigroups S_α, $\alpha \in Y$, where $S_\alpha^{m+n+1} = G_\alpha (\alpha \in Y)$ is a group and*

$$x_1 \cdots x_{m+n+1} \in G_{\alpha_1 \cdots \alpha_{m+n+1}} \quad \text{for} \quad x_i \in S_{\alpha_i}$$
$$i = 1, \cdots, m+n+1, \ \alpha_i \in Y.$$

Proof. (i) ⇒ (ii). Let S be an (m, n)-two-sided pure semigroup and $a \in S^{m+n+1}$. Then by Lemma 1, we have

$$\begin{aligned} a \in x_1 \cdots x_m S y_1 \cdots y_n &= (x_1 \cdots x_m)^2 S (y_1 \cdots y_n)^2 \\ &\vdots \\ &= (x_1 \cdots x_m)^{m+n+1} S (y_1 \cdots y_n)^{m+n+1}. \end{aligned}$$

The elements $(x_1 \cdots x_m)^{m+n+1}$, $(y_1 \cdots y_n)^{m+n+1}$ are completely regular (see Corollary 1), i.e.

$$(x_1 \cdots x_m)^{m+n+1} \in G_e, \qquad (y_1 \cdots y_n)^{m+n+1} \in G_f, \qquad \text{for some} \quad e, f \in E(S).$$

Now

$$a \in e(x_1 \cdots x_m)^{m+n+1} S (y_1 \cdots y_n)^{m+n+1}$$

so

$$a = eu = vf$$

for some $u, v \in S$. Furthermore,

$$a = eu = e \cdots eu \in e \cdots eSe \cdots eu = eS(e \cdots eu)^2 = eS(eu)^2 = eSa^2 \subseteq Sa^2.$$

Analogously, $a \in a^2 S$. So by Lemma I.5.1, [1], $a \in G_r(S)$. Thus

$$S^{m+n+1} = G_r(S).$$

Since the ideal S^{m+n+1} of S is weakly commutative (since S is weakly commutative, Lemma 2) and a union of groups we have that S^{m+n+1} is a semilattice of groups (see [2], [3], [9]).

(ii) ⇒ (iii). This implication follows by Lemma 4 (ii).

(iii) ⇒ (ii). Let $s \in S^{m+n+1}$. Then

$$s = a_1 \cdots a_m b c_1 \cdots c_n \in a_1 \cdots a_m S c_1 \cdots c_n = (a_1 \cdots a_m)^k S c_1 \cdots c_n$$

for every $k \geqslant 2$. Since $(a_1 \cdots a_m)^k \in G_e$ for some $k \in Z^+$ and $e \in E(S)$ we have that

$$s=e(a_1 \cdots a_m)^k u(c_1 \cdots c_n)$$

i.e. $s=ey$ for some $y\in S$. From this it follows that

$$s=e\cdots ey\in eSs=s^2Se\subseteq s^2S\,.$$

Similarly, $s\in Ss^2$ and therefore $s\in G_r(S)$. Thus $S^{m+n+1}=G_r(S)$. Since from (iii) we have that S is weakly commutative it follows that S^{m+n+1} is a semilattice of groups (see [9]).

(ii) ⇒ (i). Let S^{m+n+1} be a semilattice of groups and let A be a bi-ideal of S and $x_1, \cdots, x_m, y_1, \cdots, y_n\in S$. Assume $a\in A\cap x_1\cdots x_mSy_1\cdots y_n$. Then by Lemma 4 (v) we have that

$$a\in A\cap(x_1\cdots x_my_1\cdots y_n)^{m+n+1}S\,,\quad \text{i.e.}$$

$$a=(x_1\cdots x_my_1\cdots y_n)^{m+n+1}s\qquad (a\in A,\ s\in S)\,.$$

The element $(x_1\cdots x_my_1\cdots y_n)^{m+n+1}$ is completely regular, i.e.

$$(x_1\cdots x_my_1\cdots y_n)^{m+n+1}=(x_1\cdots x_my_1\cdots y_n)^{m+n+1}u(x_1\cdots x_my_1\cdots y_n)^{m+n+1}$$

and

$$(x_1\cdots x_my_1\cdots y_n)^{m+n+1}u=u(x_1\cdots x_my_1\cdots y_n)^{m+n+1}$$

for some $u\in S$. Since $a\in S^{m+n+1}$ we have $a=aba$, $ab=ba$ for some $b\in S$. So

$$ua=uaba=uba^2\,.$$

Furthermore,

$$\begin{aligned}
a&=(x_1\cdots x_my_1\cdots y_n)^{m+n+1}s\\
&=(x_1\cdots x_my_1\cdots y_n)^{m+n+1}u(x_1\cdots x_my_1\cdots y_n)^{m+n+1}s\\
&=(x_1\cdots x_my_1\cdots y_n)^{m+n+1}ua=(x_1\cdots x_my_1\cdots y_n)^{m+n+1}uba^2\\
&\subseteq x_1\cdots x_my_1\cdots y_nSa^2=(x_1\cdots x_my_1\cdots y_nSa^2)^2\quad \text{(Lemma 3)}\\
&=(x_1\cdots x_ma^2Sy_1\cdots y_n)(x_1\cdots x_mSa^2y_1\cdots y_n)\quad \text{(Lemma 4 (iii))}\\
&\subseteq x_1\cdots x_ma^2Sa^2y_1\cdots y_n\subseteq x_1\cdots x_mASAy_1\cdots y_n\\
&\subseteq x_1\cdots x_nAy_1\cdots y_n\,.
\end{aligned}$$

Therefore

$$(1)\qquad A\cap x_1\cdots x_mSy_1\cdots y_n\subseteq x_1\cdots x_mAy_1\cdots y_n\,.$$

By Lemma 4 (iii) and (iv) for every $a\in A$ we have that

$$\begin{aligned}
x_1\cdots x_may_1\cdots y_n\in S^maS^n&=(S^maS^n)^2=S^ma^2S^nS^mS^n=S^mS^nS^mS^na^2\\
&=(S^{2m+2n}a^2)^2=a^2S^{4m+4n}a^2\subseteq ASA\subseteq A\,.
\end{aligned}$$

Thus

(2) $$x_1 \cdots x_m A y_1 \cdots y_n \subseteq A\,.$$

From (1) and (2) it follows that

$$A \cap x_1 \cdots x_m S y_1 \cdots y_n = x_1 \cdots x_m A y_1 \cdots y_n$$

for every $x_1, \cdots, x_m, y_1, \cdots, y_n \in S$, i.e. A is an (m, n)-two-sided pure subset of S. Therefore, S is an (m, n)-two-sided pure semigroup.

(ii) $\Rightarrow$ (iv). Let S^{m+n+1} be a semilattice Y of groups G_α, $\alpha \in Y$. Then S is weakly commutative and π-regular and by Theorem 3.2 [4] we have that S is a semilattice of nil-extensions of groups S_α, $\alpha \in Y$. Let $s \in S^{m+n+1}$. Then $s = x_1 \cdots x_{m+n+1} \in G_\alpha$ for some $\alpha \in Y$. Thus $S_\alpha^{m+n+1} \subseteq G_\alpha$ $(\alpha \in Y)$. It is clear that

$$x_1 \cdots x_{m+n+1} \in G_{\alpha_1 \cdots \alpha_{m+n+1}} \quad \text{for} \quad x_i \in S_{\alpha_i} \quad (i = 1, \cdots, m+n+1)\,.$$

(iv) $\Rightarrow$ (ii). Assume $s \in S^{m+n+1}$. Then $s = x_1 \cdots x_{m+n+1} \in G_\delta$ for some $\delta \in Y$. Hence $S^{m+n+1} = G_r(S)$. For $e, f \in E(S)$ we have that $ef = e \cdots ef \in G_{\alpha\beta}$ and $fe = fe \cdots e \in G_{\alpha\beta}$, so $ef, fe \in G_{\alpha\beta}$, whence by Lemma 2.3 [7] we obtain $ef = fe$. Therefore S^{m+n+1} is a semilattice of groups (Lemma V 1 [7]).

Remark. The conditions $(m-n+1, n)$ are equivalent for all $1 \leqslant n \leqslant m$.

THEOREM 2. *Every subsemigroup of a semigroup S is an (m, n)-two-sided pure subset of S if and only if*

(3) $$S^{m+n+1} = \{0\}\,.$$

Proof. If every subsemigroup of S is an (m, n)-two-sided pure subset of S, then every bi-ideal of S is (m, n)-two-sided pure. From this and Theorem 1 we have that S^{m+n+1} is a semilattice of groups. Assume $e \in E(S)$. Then

$$\{e\} \cap x_1 \cdots x_m S y_1 \cdots y_n = x_1 \cdots x_m \{e\} y_1 \cdots y_n$$

for every $x_1, \cdots, x_m, y_1, \cdots, y_n \in S$, so

$$\{e\} \cap x_1 \cdots x_m S y_1 \cdots y_n = \{x_1 \cdots x_m e y_1 \cdots y_n\}$$

and $x_1 \cdots x_m e y_1 \cdots y_n = e$. Thus

$$x_1 e \cdots e \cdot e \cdot e \cdots e = e$$

i.e. $x_1 e = e$. Similarly $e x_1 = e$. Thus

$$e x_1 = x_1 e = e$$

for all $x_1 \in S$ and all $e \in E(S)$. Therefore, S has only one idempotent, the zero 0 of S. Since S^{m+n+1} is a semilattice of groups with only one idempotent which is the zero of S we have that (3) holds.

Conversely, let A be a subsemigroup of S. Then $x_1 \cdots x_m A y_1 \cdots y_n = \{0\}$ and since 0 is in every subsemigroup of S we have that $A \cap x_1 \cdots x_m S y_1 \cdots y_n = \{0\}$. So A

is an (m, n)-two-sided pure subset of S.

Remark. If S is T^*-pure anchimedean semigroup, then S is weakly commutative (Lemma 2) with idempotent (Corollary 1). So S is t-archimedean with idempotent e and by Theorem 1, we have that $S^3 = G_e$. Conversely, if S has an idempotent e and $S^3 = G_e$, then by Theorem 1, S is a T^*-pure archimedean semigroup. Therefore, Theorem 1 is a generalization of the main result of N. Kuroki, [5].

References

[1] BOGDANOVIĆ, S.; Semigroups with a system of subsemigroups, Novi Sad 1985.

[2] BOGDANOVIĆ, S.; Q_r-semigroups, *Publ. Inst. Math.*, **29**(43) (1981), 15–21.

[3] BOGDANOVIĆ, S.; O slabo komutativnoj polugrupi, *Mat. vesnik*, **5**(18) (33) (1981), 145–148.

[4] BOGDANOVIĆ, S.; Semigroups of Galbiati-Veronesi, Proceedings of the Conference "Algebra and Logic," Zagreb, 1984, 9–20.

[5] KUROKI, N.; T^*-Pure Arhimedean Semigroups, *Comment. Math. Univ. St. Pauli*, **31** (1982), 115–128.

[6] LAJOS, S.; A note on semilattices of groups, *Acta Sci. Math. Szeged*, **33** (1972), 315–317.

[7] MADISON, B. L., MUKHERJEE, T. K. and SEN, M. K.; Periodic properties of groupbound semigroups, *Semigroup Forum*, **22** (1981), 225–234.

[8] PETRICH, M.; *Introduction to Semigroups*, Merill Publ. Company, Ohio 1973.

[9] PONDĚLIČEK, B.; On weakly commutative semigroups, *Czech. Math. J.*, **25**(100) (1975), 20–23.

Institute of Mathematics
PMF Novi Sad
P.O.B. 224
21000 Novi Sad
Yugoslavia