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On the Additive Groups of Chain Rings

by

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1. Introduction

By a chain ring we mean a (not necessarily associative or unital) ring R whose two-sided ideals are totally ordered by inclusion. Shalom Feigelstock considered the problem of describing those abelian groups A supporting a chain ring, i.e. $A \simeq R^+$ for some chain ring R. For the case that A is periodic a complete characterization was given in [6]; for the non-periodic case Feigelstock obtained the following result. (A group H is called q-local, q a prime, if $pH=H$ for all primes $p \neq q$.)

THEOREM 1.1 (Feigelstock). *If the non-periodic group A supports a chain ring then*

$$A=\bigoplus_{\alpha} Z(p^{\infty})\oplus\bigoplus_{\beta} Q\oplus H$$

where p is a prime, α and β are cardinals, and H is a reduced torsion-free q-local abelian group for some prime q.

The purpose of this note is to complete Feigelstock's theorem for the case that the reduced part of A has finite torsion-free rank. Throughout, $d(A)$ denotes the maximal divisible subgroup and $t(A)$ the torsion subgroup of A. All groups considered are abelian. The word "rank" is used to mean torsion-free rank. An abelian group H is said to be E-uniserial if the lattice of fully invariant subgroups of H forms a chain. Our main result is

THEOREM 1.2. *Let A be a non-periodic abelian group such that $A/d(A)$ has finite rank. Then A supports a chain ring if and only if*

$$A=\bigoplus_{\alpha} Z(p^{\infty})\oplus\bigoplus_{\beta} Q\oplus H$$

where: (1) *p is a prime*; (2) *α and β are cardinals with α at most countable*; (3) *H is a torsion-free reduced E-uniserial group*; *and* (4) *if $\alpha \neq 0$ and $pA=A$, then $\alpha=1$ and $A/t(A)$ has rank at least two.*

Thus, if $Z_{(q)}$ denotes the group of rationals with denominator relatively prime to

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q and $p \neq q$, then the group $A = Z(p^\infty) \oplus Z_{(q)}$ does not support a chain ring while the group $B = Z(p^\infty) \oplus Z_{(p)}$ does. This shows that, in order to characterize the additive groups of chain rings, the usual restriction to the reduced torsion-free case (cf. [7; 3.2.1, 3.2.8, 3.2.10]) is not possible. We do, however, have

THEOREM 1.3. *A torsion-free abelian group G supports a chain ring if and only if $G/d(G)$ does.*

It is easy to see that every group supporting a chain ring is E-uniserial. Torsion-free E-uniserial groups have been investigated in [9]. In particular, these groups are local. Combining results from [8] and [9] we have

THEOREM 1.4. *Let H be a reduced torsion-free group of finite rank. Then the following conditions are equivalent.*

(i) *H is E-uniserial.*

(ii) *H supports a chain ring.*

(iii) *$H = \bigoplus_\gamma G$ where γ is an integer and G is a (strongly) indecomposable E-uniserial group.*

Since finite rank indecomposable E-uniserial groups are strongly indecomposable [9], Theorem 1.4 holds with and without the parenthesis in (iii).

We use the term "discrete valuation ring" in the sense of Kaplansky [10]. In particular, such a ring is a principal ideal domain in the usual sense. The ring R is said to be an E-ring if every endomorphism of R^+ can be achieved by left multiplication with a suitable ring element in R [11]. Being isomorphic to an endomorphism ring, E-rings are associative and have an identity; they are also commutative [11]. By a discrete valuation E-ring we mean a discrete valuation ring which simultaneously is an E-ring. Results of Bowshell and Schultz [2] together with [9] imply

THEOREM 1.5. *A finite rank reduced torsion-free abelian group G is (strongly) indecomposable and E-uniserial if and only if $G \cong R^+$ for some discrete valuation E-ring R.*

Thus, for groups of finite rank, we have come full circle: the description of the additive groups of (not necessarily associative or unital or commutative) chain rings reduces to the description of the additive groups of discrete valuation E-rings. These are precisely the strongly indecomposable integrally closed local subrings of algebraic number fields [1].

2. Preliminaries

For a (not necessarily associative) ring R and $x \in R$, we denote by $(x)_R$ the principal ideal of R generated by x (i.e. $(x)_R$ is the intersection of all ideals of R containing x). The map

$$\phi : R^+ \otimes R^+ \longrightarrow R^+$$

defined by $\phi(r_1 \otimes r_2) = r_1 r_2$, $r_i \in R$, is a homomorphism which we call "the homomorphism associated with the ring structure on R". (The unadorned symbol $\otimes$ denotes the tensor product over the ring of integers Z.) Conversely, for any abelian group G, every homomorphism

$$\phi : G \otimes G \longrightarrow G$$

gives rise to a ring structure on G [3; XII]; occasionally, we may abbreviate $\phi(g_1 \otimes g_2)$ by $g_1 \cdot g_2$ or $g_1 g_2$. We write $S \leq G$ if S is a subgroup of G.

LEMMA 2.1. *Let G be an abelian group supporting a chain ring R and let $\phi : G \otimes G \to G$ be the associated homomorphism. Then*

$$G/\phi(G \otimes G) \leq Z(r^{\infty})$$

for some prime r.

Proof. Observe that every subgroup of G containing $\phi(G \otimes G)$ is an ideal of R. Hence, the subgroup lattice of $G/\phi(G \otimes G)$ is a chain. This is the case only for quasi-cyclic groups.

Of special interest will be certain chain algebras constructed in [8]: Let S be a commutative, associative chain ring with identity element, and let Γ be a non-empty set. Let

$$H = \bigoplus_{\gamma \in \Gamma} S w_\gamma$$

be the free S-module on Γ with basis $\{w_\gamma\}$. In [8], an S-linear map $\psi : H \otimes_s H \to H$ was defined such that the associated multiplication $\mu = \psi \circ \otimes_s$ made H into an S-algebra $\hat{H}$ with totally ordered ideal lattice. (We use the "$\hat{\ }$" to distinguish between the algebra and the underlying group, i.e. $(\hat{H})^+ = H$.) We shall refer to $\hat{H}$ as "the standard chain S-algebra on Γ"; the homomorphism

$$\phi : H \otimes H \longrightarrow H$$

associated with this algebra structure is $\phi = \psi \circ \eta$ where $\eta : H \otimes H \to H \otimes_s H$ is the natural map defined by $h \otimes h' \longmapsto h \otimes_s h'$.

We collect some results from [8].

LEMMA 2.2. *With definitions and notation as above, the following hold.*

(i) *$\hat{H}$ is a chain ring.*

(ii) *For all $h \in H$,*

$$(h)_{\hat{H}} = \{h \cdot y \mid y \in \hat{H}\}\ ,$$

and $(h)_{\hat{H}} = J \cdot \hat{H}$ for some ideal J of S.

(iii) *If S is a q-local discrete valuation ring and $I \neq 0$ an ideal of $\hat{H}$ then $q^m H \subseteq I$ for some non-negative integer m.*

Proof. (i) and (ii) follow from [8, pp. 326, 327]. For (iii) note that $I = J \cdot \hat{H}$ where $J \neq 0$ is an ideal of the q-local discrete valuation domain S. Hence

$$J \not\subseteq 0 = \bigcap_n q^n S$$

and S being a chain ring implies $q^m S \subseteq J$ for some $m \geq 0$. Thus

$$I = J \cdot \hat{H} \supseteq q^m S \cdot \hat{H} = q^m H .$$

The following easy result on tensor products will be needed.

LEMMA 2.3. *Let S be a commutative associative ring with $1 \neq 0$. Then*

$$S \otimes_z S = U \oplus V$$

where $U = \{a \otimes 1 \mid a \in S\}$, *and* $V = \langle a \otimes b - [(ab) \otimes 1] \mid a, b \in S \rangle$. *Moreover,* $V = \ker \tau$ *where*

$$\tau : S \otimes_z S \longrightarrow S \otimes_s S$$

is the homomorphism satisfying $\tau(a \otimes b) = a \otimes_s b$ *for all* $a, b \in S$.

Proof. Since, for all $a, b \in S$

$$a \otimes b = (ab \otimes 1) + [(a \otimes b) - (ab \otimes 1)]$$

and

$$\tau[(a \otimes b) - (ab \otimes 1)] = 0 ,$$

$S \otimes_z S = U + V$ and $V \subseteq \ker \tau$. Also,

$$\begin{aligned} \tau(U) &= \{a \otimes_s 1 \mid a \in S\} \\ &= S \otimes_s S \end{aligned}$$

which implies

$$S \otimes_z S = U + \ker \tau .$$

One verifies $U \cap \ker \tau = 0$ so that the sum is direct and $V = \ker \tau$.

3. The proofs

It will be convenient to write $I \lhd R$ if I is an ideal of R. The endomorphism ring of an abelian group A is denoted by $\mathrm{End}(A)$.

Proof of Theorem 1.3. Let $G = D \oplus H$ be torsion-free with D divisible and H reduced. By [8, 2.1] it suffices to show that G supports a chain ring if H does. Thus, assume H supports a chain ring and let

$$\psi : H \otimes H \longrightarrow H$$

be the associated homomorphism. Clearly we may assume $D \neq 0$ so that there exists an epimorphism

$$\eta : D \otimes H \longrightarrow D .$$

Let

$$\theta: H\otimes D \longrightarrow G$$

be the zero map and let

$$\phi: D\otimes D \longrightarrow D$$

be the homomorphism associated with the standard chain Q-algebra $\hat{D}$ on D. Since

$$G\otimes G=(D\otimes H)\oplus(H\otimes D)\oplus(D\otimes D)\oplus(H\otimes H)$$

[4, p. 255], defining

$$\sigma: G\otimes G \longrightarrow G$$

by $\sigma=\eta+\theta+\phi+\psi$ (making the usual identifications) is a homomorphism providing G with a ring structure. We claim that G is a chain ring. By a well known theorem (which holds for non-associate rings as well) it suffices to verify that the principal ideals of G are totally ordered. Since H is torsion-free and supports a chain ring, H is q-local for some prime q and

$$\bigcap_n q^n H=0 .$$

Thus, if $0\neq h\in H$, then

$$(h)_G\supseteq(h)_H\supseteq q^m H$$

for some integer $m\geq 0$. Consequently,

$$(h)_G\supseteq\sigma(D\otimes q^m H)=\eta(D\otimes H)=D$$

which implies

$$(h)_G=D\oplus(h)_H$$

since the right hand side is an ideal of G containing h. Let $0\neq d\in D$. By 2.2 (ii), $(d)_{\hat{D}}=D$ so that $(d)_G=D$. Finally, let

$$x=d+h .$$

Since $h\cdot y=0$ for all $y\in D$,

$$(x)_G\supseteq\{d\cdot y\mid y\in D\}=D$$

by 2.2 (ii). Hence $h\in(x)_G$ and

$$(x)_G=D\oplus(h)_H .$$

The fact that H is a chain ring completes the proof.

Proof of Theorem 1.5. Let G be reduced torsion-free of finite rank. Assume, firstly, that G is indecomposable and E-uniserial. By [9, Corollary 2] G is strongly indecomposable and $G\simeq[\mathrm{End}(G)]^+$ with $\mathrm{End}(G)$ a discrete valuation E-ring.

Conversely, suppose $G \simeq S^+$, S a discrete valuation E-ring. Being a torsion-free, finite rank integral domain, S is a full subring of an algebraic number field. By [2, 3.14], $S^+ \simeq G$ is strongly indecomposable.

Proof of Theorem 1.4. Let H be reduced torsion-free of finite rank. The equivalence of (i) and (iii) is contained in [9, Theorem 1]; trivially, (ii) implies (i). Assume the validity of (iii). Then, by 1.5, $H = \bigoplus_{\gamma} S^+$ where S is a discrete valuation ring. Apply [8, 2.3].

Proof of Theorem 1.2. Let A be a non-periodic abelian group with $A/d(A)$ of finite torsion-free rank.

NECESSITY: Suppose A supports a chain ring. By Feigelstock's (1.1)

$$A = \bigoplus_{\alpha} Z(p^\infty) \oplus \bigoplus_{\beta} Q \oplus H$$

with H reduced and q-local. By [8, 2.1], H supports a chain ring, hence H is E-uniserial. Suppose $\alpha \neq 0$. Let $Z(p^\infty) \simeq P \leq A$. Then, for all integers n, the ideal $(P)_A$ generated by P is not contained in $t(A)[p^n]$. Hence, $t(A)[p^n] \subseteq (P)_A \subseteq t(A)$ for all n proving $(P)_A = t(A)$. But, since $t(A) \otimes d(A) = 0$, $(P)_A$ is the sum of all subgroups of A which can be obtained from P by a finite number of multiplications (from either side) with elements in H. Since H is countable, only countably many such subgroups exist so that $(P)_A = t(A)$ is countable. Suppose $\alpha \neq 0$ and $pA = A$. Then, $t(A) \otimes A = 0$, and every subgroup of $t(A)$ is an ideal of A which implies $\alpha = 1$. Assume, by way of contradiction, that $\alpha = 1$ and $A/t(A)$ has rank at most 1. Since $A \neq t(A)$ this implies $A = Z(p^\infty) \oplus B$ where $B \leq Q$ is q-local. Hence

$$A \otimes A = B \otimes B \simeq B\,.$$

By 2.1, $A/\phi(A \otimes A)$ is torsion, thus $\phi(A \otimes A)$ must be torsion-free. It follows that $t(A)$ and $\phi(A \otimes A)$ are two incomparable ideals contradicting the fact that A is a chain ring.

SUFFICIENCY: Suppose that A has the stated form. In view of [8, 3.2] we may assume $H \neq 0$ and, in view of (1.3) and (1.4), that $\alpha \neq 0$. By (1.4) and (1.5), H is a free S-module for some discrete valuation E-ring S. Thus,

$$A = T \oplus D \oplus H$$

with

$$T = \bigoplus_{i \in I} P_i\,, \qquad P_i \simeq Z(p^\infty)\,, \qquad i \in I\,, \quad |I| = \alpha \geq 1\,,$$

$$D = \bigoplus_{\lambda \in \Lambda} Qu_\lambda\,, \qquad Qu_\lambda \simeq Q\,, \qquad \lambda \in \Lambda\,, \quad |\Lambda| = \beta\,,$$

$$H = \bigoplus_{\mu \in M} Sv_\mu\,, \qquad Sv_\mu \simeq S\,, \qquad \mu \in M \neq \phi\,.$$

For convenience, assume $0 \in I$ and $0 \in M$, with $v_0 = 1 \in S$ in case $M = \{0\}$. Let

$$\phi: D\otimes D \longrightarrow D\,, \quad \psi: H\otimes H \longrightarrow H$$

be the homomorphisms associated with the standard chain algebra structures on D and H, respectively.

Note that $H=S$ if $M=\{0\}$. Pick homomorphisms

$$\eta: D\otimes H \longrightarrow P_0\,,$$
$$\theta: H\otimes D \longrightarrow D$$

such that η and θ are epic if $D\neq 0$. We need to distinguish cases.

Case 1. $pA\neq A$. Then we may assume that either $I=Z$, the integers, or $I=\{0, 1, \cdots, m\}$ for some integer $m\geq 0$. For $i\in I$, let

$$P_i=\langle a_i^0, a_i^1, \cdots\rangle\,, \quad o(a_i^0)=p\,, \quad pa_i^k=a_i^{k-1}\,, \quad k\geq 1\,.$$

For all i, there exists a homomorphism

$$\psi_i: P_i \longrightarrow P_{i+1}$$

such that

$$\psi_i(a_i^k)=a_{i+1}^k \qquad \text{for all} \quad k\geq 0$$

(If $|I|=m+1$, we do arithmetic modulo m, i.e. $\psi_m: P_m\to P_1$). Since

$$\mathrm{Hom}(P_i, P_{i+1})\simeq J_p$$

and the group J_p of p-adic integers is pure injective [4, 38.1, 39.4] and p-local, there exist homomorphisms

$$\psi_i': H \longrightarrow \mathrm{Hom}(P_i, P_{i+1})$$

such that $\psi_i'(v_0)=\psi_i$. By [4, p. 256(J)], there exist homomorphisms

$$\delta_i: H\otimes P_i \longrightarrow P_{i+1}$$

such that, for all $k\geq 0$,

$$\delta_i(v_0\otimes a_i^k)=a_{i+1}^k\,.$$

Similarly, for each $i\in I$, there exists a homomorphism

$$\delta_i^*: P_i\otimes H \longrightarrow T$$

such that, for all $k\geq 0$,

$$\delta_i^*(a_i^k\otimes v_0)=\begin{cases} a_0^k & \text{if} \quad i=0 \\ 0 & \text{if either} \quad i<0 \quad \text{or if} \quad i>0 \quad \text{and} \quad |I|<\infty \\ a_{-1}^k & \text{if} \quad i>0 \quad \text{and} \quad |I|=\infty\,. \end{cases}$$

Since

$$A\otimes A\simeq\bigoplus_i(P_i\otimes H)\oplus\bigoplus_i(H\otimes P_i)\oplus(D\otimes H)\oplus(H\otimes D)\oplus(D\otimes D)\oplus(H\otimes H),$$

the map

$$\sigma=\sum_i\delta_i^*+\sum_i\delta_i+\eta+\theta+\phi+\psi$$

is a homomorphism from $A\otimes A$ to A defining a multiplication "$\cdot$" on A. In order to verify A is a chain ring let

$$0\neq x=t+d+h\in A$$

with $t\in T$, $d\in D$ and $h\in H$. Note that

$$x\in T\oplus D\oplus(h)_{\hat{H}}\lhd A\,.$$

If $h\neq 0$ then, by 2.2 (ii),

$$0\neq(h)_{\hat{H}}=\{h\cdot y\mid y\in H\}\supseteq p^nH$$

for some integer n. Hence, for all $y\in H$ and all $a\in T$,

$$\begin{aligned}(x\cdot y)\cdot a&=(t\cdot y)\cdot a+(d\cdot y)a+(h\cdot y)a\\&=(h\cdot y)\cdot a\in(x)_A\end{aligned}$$

so that

$$\sigma(p^nH\otimes T)=\sigma(H\otimes T)=\sigma\left(\bigoplus_i(H\otimes P_i)\right)=\bigoplus_i P_{i+1}=T\subseteq(x)_A\,.$$

Hence $x'=d+h\in(x)_A$ and $\sigma(D\otimes H)\subseteq T\subseteq(x)_A$ implies

$$p^nH\subseteq(h)_{\hat{H}}\subseteq(x)_A\,.$$

Thus, $\sigma(p^mH\otimes D)=\sigma(H\otimes D)=D\subseteq(x)_A$ proving $(x)_A=T\oplus D\oplus(h)_{\hat{H}}$. If $x=t+d$, $d\neq 0$, then

$$\{d\cdot z\mid z\in D\}=(d)_{\hat{D}}=D\subseteq(x)_A$$

which implies $P_0=\sigma(D\otimes H)\subseteq(x)_A$. Using the homomorphisms δ_i and δ_i^*, it follows that $T\subseteq(x)_A$ so that $(x)_A=T\oplus D$. Finally, assume

$$0\neq x=t=\sum_{j=k}^{l}n_ja_j^{m_j}\,,\qquad p\nmid n_j\in Z\,,$$

and let $k\leq s\leq l$ be such that

$$o(x)=p^r=o(n_sa_s^{m_s})\,.$$

Then $x\in T[p^r]\lhd A$. In order to show that $(x)_A=T[p^r]$ it suffices to verify $n_sa_0^{m_s}\in(x)_A$. Since

$$v_0 \cdot x = \sum_{j=k}^{l} n_j a_{j+1}^{m_j}$$

and

$$x \cdot v_0 = \begin{cases} n_0 a_0^{m_0} & \text{if } |I| < \infty \\ \sum_{j \geq 0} n_j a_{-j}^{m_j} & \text{if } |I| = \infty , \end{cases}$$

we may restrict ourselves to the case $I = Z$. Multiplying x suitably many times by v_0 from the left and then from the right will result in an element $y \in (x)_A$ whose P_0-coordinate is $n_s a_0{}^{m_s}$. Hence

$$(y \cdot v_0)v_0 = n_s a_0{}^{m_s} \in (x)_A$$

as desired.

Case 2. $pA = A$. Then

$$A \otimes A = (D \otimes H) \oplus (H \otimes D) \oplus (D \otimes D) \oplus (H \otimes H) ,$$

and, since $\alpha = 1$,

$$T = P_0 = P \simeq Z(p^\infty) .$$

In addition to η, θ, ϕ and ψ above, we define a homomorphism

$$\psi' : H \otimes H \longrightarrow T$$

as follows: if $D \neq 0$, let $\psi' = 0$; if $D = 0$ and $|M| \geq 2$, there exist two distinct subscripts (for simplicity denoted by) 0 and 1 in M such that

$$\psi(Sv_0 \otimes Sv_1) = 0$$

[8, p. 327]. Since $Sv_0 \otimes Sv_1 \simeq S \otimes S$ is q-local and torsion-free, $T = Z(p^\infty)$ is an epimorphic image. Let ψ' be any homomorphism such that

$$\psi'(Sv_\mu \otimes Sv_\nu) = \begin{cases} 0 & \text{if } (\mu, \nu) \neq (0, 1) \\ T & \text{if } (\mu, \nu) = (0, 1) . \end{cases}$$

Finally, if $D = 0$ and $|M| = 1$, then $H = S$. Note that in this case $\psi : H \otimes H \rightarrow H$ is just the plain ring multiplication:

$$\psi(s_1 \otimes s_2) = s_1 s_2 , \quad s_i \in S .$$

By 2.3,

$$S \otimes_z S = U \oplus V$$

where $V = \ker \tau$. Since $S \otimes_s S \simeq S$, and $H \simeq S^+$ has rank at least 2, we have $V \neq 0$. As above, $T = Z(p^\infty)$ is an epimorphic image of V. Select a homomorphism

$$\psi' : H \otimes H \longrightarrow T$$

such that

$$\psi'(U)=0\,, \quad \psi'(V)=T\,.$$

Note that, if $D=0$, then

$$(\psi+\psi')(H\otimes H)=T\oplus H\,.$$

Now define

$$\sigma : A\otimes A \longrightarrow A$$

by

$$\sigma=\eta+\theta+\phi+\psi+\psi'\,.$$

Then σ is a homomorphism inducing a ring structure on A. Again, let

$$x=t+d+h\in A$$

with $t\in T$, $d\in D$, $h\in H$. If $h\neq 0$,

$$\begin{aligned}(x)_A &\supseteq \{x\cdot y \mid y\in H\}=\{d\cdot y+h\cdot y \mid y\in H\}\\ &=\{\eta(d\otimes y)+\psi(h\otimes y)+\psi'(h\otimes y) \mid y\in H\}\,.\end{aligned}$$

If $n\in \boldsymbol{Z}$ such that $q^n H\subseteq (h)_{\hat{H}}$ then, by 2.2 (ii), for each $w\in H$ there exists $a\in T$ such that

(3.1) $$a+q^m w\in (x)_A\,.$$

If $D\neq 0$, for each $z\in D$ and $w\in H$,

$$(a+q^m w)\cdot z=q^m w\cdot z\in (x)_A$$

which implies

$$\sigma(q^m H\otimes D)=\sigma(H\otimes D)=D\subseteq (x)_A$$

and

$$T=\sigma(D\otimes H)\subseteq (x)_A\,.$$

Hence,

$$(x)_A=T\oplus D\oplus (h)_{\hat{H}}\,.$$

Suppose $D=0$. Then, by (3.1),

$$\begin{aligned}(\psi+\psi')(q^m H\otimes H)&=q^m(\psi+\psi')(H\otimes H)\\ &=q^m[T\oplus H]=T\oplus q^m H\subseteq (x)_A\end{aligned}$$

and it follows that

$$(x)_A=T\oplus (h)_{\hat{H}}\,.$$

Clearly, if $x=t\in T$, $(x)_A=\langle t\rangle \lhd G$; if $x=t+d$ with $0\neq d\in D$, then

$$D=(d)_{\hat{H}}=\{x\cdot z \mid z\in D\}\subseteq (x)_A$$

and $(x)_A = T\oplus D$ since $\sigma(D\otimes H)=T$. The proof is completed.

Remark. Note that, for the proof of sufficiency, the finiteness of γ was not used.

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