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On Zero-Free Regions for Solutions of n^{th} Order Linear Differential Equations

by

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Abstract

In this paper we determine zero-free regions for solutions of broad classes of n^{th} order linear differential equations whose coefficients are analytic and possess asymptotic expansions in sectors of the plane. In the special case where the coefficients are rational functions, we obtain a generalization to higher order equations of the classical result of Hille-Nevanlinna-Wittich for second order equations.

1. Introduction

For a second-order equation of the form,

(1.1) $$w''+A(z)w=0 .$$

where $A(z)=a_m z^m+\cdots$, is a nonconstant polynomial of degree m, there is a classical result due jointly to E. Hille, R. Nevanlinna, and H. Wittich (see [14; p. 282]) which determines the possible location of the zeros of solutions of (1.1). The theorem states that for any solution $f\not\equiv 0$ of (1.1) and any $\varepsilon>0$, all but finitely many zeros of f lie in the union for $j=0, 1,\cdots, m+1$, of the sectors,

(1.2) $$W_j(\varepsilon):\ |\arg z-\theta_j|<\varepsilon ,$$

where,

(1.3) $$\theta_j=(2\pi j-\arg a_m)/(m+2) , \qquad j=0, 1, \cdots, m+1 .$$

The Hille-Nevanlinna-Wittich result was proved by using a method of asymptotic integration (see [8; Chapter 7, §4], [9; p. 345] or [7; pp. 6–10]) to construct in each of the closed sectors J which lies between two adjacent sectors in (1.2) a fundamental set $\{f_1,f_2\}$ of solutions of (1.1) each having only finitely many zeros in J, and having the property that $f_1/f_2\to\infty$ as $z\to\infty$ in J. Then clearly any nontrivial linear combination of f_1 and f_2 can have only finitely many zeros in J. (In fact, the

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solutions f_1, f_2 which are constructed satisfy the asymptotic relation $\log f_j \sim \pm(2i(a_m)^{1/2}/(m+2))z^{(m+2)/2}$ as $z\to\infty$ in J.)

In the present paper, we obtain extensions of the Hille-Nevanlinna-Wittich result to broad classes of higher-order linear differential equations. In order to best illustrate our main result, we will first show how it applies in a special case, namely to the class of equations treated in [5; Theorem 2]. We will prove:

THEOREM 1. *Given the differential equation,*

$$w^{(n)}+a_{n-1}(z)w^{(n-1)}+\cdots+a_1(z)w'+a_0(z)w=0\,, \tag{1.4}$$

where the coefficients $a_j(z)$ are polynomials, say $a_j(z)$ is of degree α_j so that,

$$a_j(z)=A_jz^{\alpha_j}+O(z^{\alpha_j-1}) \quad \text{as} \quad z\to\infty \quad \text{if} \quad a_j\not\equiv 0\,, \tag{1.5}$$

and where the following conditions are satisfied:

$$\alpha_0\geq 1 \quad \text{and} \quad \alpha_j\leq((n-j)/n)\alpha_0 \quad \text{for} \quad 1\leq j\leq n-1\,, \tag{1.6}$$

and the polynomial,

$$h(t)=t^n+\sum_{j=0}^{n-1}A_j^*t^j\,, \tag{1.7}$$

where $A_j^*=A_j$ if $\alpha_j=((n-j)/n)\alpha_0$, and $A_j^*=0$ otherwise, has only simple roots $b_1,\cdots,b_n$. Define a real constant β by the relation,

$$\beta=\pi+(n/(\alpha_0+n))((\arg(b_2-b_1)-(\pi/2))\,, \tag{1.8}$$

for any choice of $\arg(b_2-b_1)$. For each pair (k,j) of positive integers with $1\leq k<j\leq n$, let D_{kj} denote the set of zeros on $(-\pi,\pi)$ of the function of φ given by,

$$\operatorname{Sin}(((\alpha_0+n)/n)(\varphi-\pi)+\arg((b_j-b_k)/(b_2-b_1)))\,, \tag{1.9}$$

and let $D=\{\sigma_1,\sigma_2,\cdots,\sigma_q\}$, where $-\pi<\sigma_1<\cdots<\sigma_q<\pi$, be the union of the sets D_{kj} for all (k,j). Set $\sigma_{q+1}=\pi$. Then, for any solution $f\not\equiv 0$ of (1.4) and any $\varepsilon>0$, all but finitely many zeros of f lie in the union for $j=1,2,\cdots,q+1$ of the sectors,

$$|\arg z-(\sigma_j-\beta)|<\varepsilon\,. \tag{1.10}$$

(We remark that the constant β defined in (1.8) is introduced to avoid the possibility of introducing an extraneous sector (1.10) which may enter because of a limitation in our method of proof. This will be discussed further in this section.) In the special case of second-order equations of the form (1.1), it is easy to check that Theorem 1 gives the Hill-Nevanlinna-Wittich result discussed earlier. For the class of equations treated in Theorem 1, it follows from the Valiron-Wiman theory [13; Chapter 4], that any solution $f\not\equiv 0$ has order of growth equal to $(\alpha_0+n)/n$, and it was proved in [5; Theorem 2] that every fundamental set of solutions contains at least one solution whose zero-sequence has exponent of convergence equal to $(\alpha_0+n)/n$. (In the special case where $a_j\equiv 0$ for $1\leq j\leq n-1$ in (1.4), it was shown in [6] that at least $n-1$

of the solutions in a fundamental set have this property. However, this strong conclusion does not hold for the general class considered in Theorem 1, as shown by the example,

(1.11) $$w''' - zw'' + (1-4z^2)w' + (4z^3 - 12z)w = 0 ,$$

which possesses the solutions $\exp(\pm z^2)$.)

The key tool in proving our main result (see § 3) concerning zero-free regions for solutions, is a general existence theorem which was proved in [3] for n^{th} order linear differential equations.

(1.12) $$\sum_{j=0}^{n} R_j(z) w^{(j)} = 0 ,$$

where the $R_j(z)$ are complex functions which are defined and analytic in an unbounded sectorial region S, and which have asymptotic expansions as $z \to \infty$ in S in terms of real powers of z. (A rigorous definition of the class of equations treated in [3], and the results obtained in [3] for this class, are given in the statement of Theorem 2 below.) It was shown in [3] that associated with the equation (1.12) are a nonnegative integer $p \leq n$ (called the critical degree), and an algebraic polynomial $G(v)$ of degree $n-p$ (which we call here, the factorization polynomial) whose coefficients are of the same type as the $R_j(z)$, such that the following are true: In sectorial subregions of S, the equation possesses p linearly independent solutions $g_1, \cdots, g_p$, each g_j being asymptotically equivalent as $z \to \infty$ to a function of the form $z^{a_j}(\text{Log } z)^{b_j}$, where a_j is complex, b_j is an integer, and the pairs (a_j, b_j) are distinct for $j = 1, \cdots, p$. In addition, if the roots of $G(v)$ are asymptotically distinct, the equation possesses in sectorial subregions of S, solutions $h_1, \cdots, h_{n-p}$, where each h_j is of the form $\exp \int W_j$ for some function W_j which is asymptotically equivalent as $z \to \infty$, to a function of the form $c_j z^{-1+d_j}$, where $c_j \neq 0$ is complex, and $d_j > 0$. The pairs (c_j, d_j) are all distinct, and the solutions $g_1, \cdots, g_p, h_1, \cdots, h_{n-p}$, form a fundamental set in any region where they all exist. In attempting to use this existence theorem to determine zero-free regions for solutions of (1.12), some problems arise immediately. First, in the case where $p > 0$, the method used in [3] to construct the solutions $g_1, \cdots, g_p$, does not produce explicitely the actual regions in which these solutions exist. However, even if this drawback could be overcome by using another method to produce these solutions, a further problem can arise, namely there can be distinct pairs (a_j, b_j) and (a_k, b_k) where $b_j = b_k$ and a_j and a_k have the same real part. In this case, the ratio g_j/g_k of the corresponding solutions will have no definite behavior as $z \to \infty$, thus obscuring the location of zeros of a linear combination of the g_j (see § 8). Fortunately, these problems do not arise for the solutions $h_1, \cdots, h_{n-p}$. We remark that for a more restricted class of equations (1.12), the condition that the roots of $G(v)$ be asymptotically distinct was removed by C. Powder [10]. However, the problems of explicit domains and relative growth mentioned earlier occur again in this case. Thus, our main result (Theorem 2 in § 3) considers the class of equations where $p = 0$ and the roots of $G(v)$ are asymptotically distinct. We emphasize that this is a very broad class of equations. Even in the case of equations (1.4) having polynomial coefficients, the class of equations which can be treated is much more extensive than the class treated in Theorem 1, and an example illustrating this fact is worked out in § 7. We remark that the general existence theorem in [3], was proved by using powerful asymptotic existence theorems for linear and nonlinear differential equations developed by W. Strodt [11, 12].

As a corollary of our main result (see § 3), we show that for the equations in our class having rational functions for coefficients, all but finitely many zeros of a meromorphic solution $f \not\equiv 0$ on the plane, must lie in the union of a finite number of sectors of the form $|\arg z - \lambda_j| < \varepsilon$ where $\varepsilon > 0$ is arbitrary. The constants λ_j can be calculated in advance from the equation, and, except possibly for one λ_j, they are the zeros on $(-\pi, \pi]$ of certain "indicial" functions (see § 2). The exceptional λ_j occurs when π is not a zero of any of the indicial functions. In this case, π must then be adjoined as a λ_j for the following reason: The situation concerning zeros around the ray $\arg z = \pi$ is obscured by the fact that the Strodt theory does not produce any solution whose domain includes this ray. (This is natural even in the case of rational coefficients, since such equations can possess solutions which only exist in the slit plane (e.g. Bessel's equation)). However, this problem of a possible extraneous sector is easily circumvented by determining an angle β so that under the rotation $\zeta = e^{i\beta}z$, the equation (1.12) is transformed into an equation where at least one of the corresponding indicial functions vanishes at π, and thus π appears in a legitimate way as some λ_j. This is the role of β in Theorem 1. We also illustrate this device in the example in § 7.

2. Concepts from the Strodt theory

(a) [11; § 94]: *The neighborhood system $F(a, b)$.* Let $-\pi \leq a < b \leq \pi$. For each nonnegative real-valued function g on $(0, (b-a)/2)$, let $V(g)$ be the union (over all $\delta \in (0, (b-a)/2)$) of all sectors, $a + \delta < \arg(z - h(\delta)) < b - \delta$, where $h(\delta) = g(\delta)e^{i(a+b)/2}$. The set of all $V(g)$ (for all choices of g) is denoted $F(a, b)$, and is a filter base which converges to ∞. Each $V(g)$ is a simply-connected region (see [11; § 93]), and we require the following simple fact:

LEMMA A. *Let V be an element of $F(a, b)$, and let $\varepsilon > 0$ be arbitrary. Then there is a constant $R_0(\varepsilon) > 0$ such that V contains the set, $a + \varepsilon \leq \arg z \leq b - \varepsilon$, $|z| \geq R_0(\varepsilon)$.*

Proof. Let g be a nonnegative real-valued function on $(0, (b-a)/2)$ for which $V = V(g)$. Then $V(g)$ contains a sector of the form, $a + \delta < \arg(z - \zeta_0) < b - \delta$, where $0 < \delta < \varepsilon$ and $\arg \zeta_0 = (a+b)/2$, and it is obvious by geometric reasoning that the sides of this sector must intersect the corresponding sides of the sector $a + \varepsilon \leq \arg z \leq b - \varepsilon$. The lemma now follows immediately.

(b) [11: § 13]: *The relation of asymptotic equivalence.* If $f(z)$ is an analytic function on some element of $F(a, b)$, then $f(z)$ is called *admissible* in $F(a, b)$. If c is a complex number, then the statement $f \rightarrow c$ in $F(a, b)$ means (as is customary) that for any $\varepsilon>0$, there exists an element V of $F(a, b)$ such that $|f(z)-c|<\varepsilon$ for all $z\in V$. The statement $f\ll 1$ in $F(a, b)$, means that in addition to $f\to 0$, all the functions $\theta_j^k f\to 0$ in $F(a, b)$, where θ_j denotes the operator $\theta_j f=z(\text{Log } z)\cdots(\text{Log}_{j-1}z)f'(z)$, and where (for $k\geq 0$), θ_j^k is the k^{th} iterate of θ_j. The statements $f_1\ll f_2$ and $f_1\sim f_2$ in $F(a, b)$ mean respectively $f_1/f_2\ll 1$ and $f_1-f_2\ll f_2$. (This strong relation of asymptotic equivalence is designed to ensure that if M is a non-constant *logarithmic monomial of rank* $\leq p$ (i.e. a function of the form,

$$M(z)=Kz^{a_0}(\text{Log } z)^{a_1}\cdots(\text{Log}_p z)^{a_p}, \tag{2.1}$$

for real a_j, and complex $K\neq 0$), then $f\sim M$ implies $f'\sim M'$ in $F(a, b)$ (see [11; § 28]). As usual, z^α and $\text{Log } z$ will denote the principal branches of these functions on $|\arg z|<\pi$.) We will write $f_1\approx f_2$ to mean $f_1\sim cf_2$ for some nonzero constant c. An admissible function $f(z)$ in $F(a, b)$ is called *trivial* in $F(a, b)$ if $f\ll z^{-\alpha}$ in $F(a, b)$ for every $\alpha>0$. If $f\sim cz^{-1+d}$ in $F(a, b)$, where $c\neq 0$ and $d>0$, then the indicial function of f is the function,

$$IF(f, \varphi)=\text{Cos}(d\varphi+\arg c) \qquad \text{for} \quad a<\varphi<b. \tag{2.2}$$

If g is any admissible function in $F(a, b)$, we will denote by $\int g$, any primitive of g in an element of $F(a, b)$. We will require the following fact which is proved in [1; § 10]:

Lemma B. *Let $f\sim cz^{-1+d}$ in $F(a, b)$, where $c\neq 0$ and $d>0$. If (a_1, b_1) is any subinterval of (a, b) on which $IF(f, \varphi)<0$ (respectively, $IF(f, \varphi)>0$), then for all real α, $\exp\int f\ll z^\alpha$ (respectively, $\exp\int f\gg z^\alpha$) in $F(a_1, b_1)$.*

(c) The operator θ_1 defined by $\theta_1 f=zf'$, will be denoted simply θ. It is easy to prove by induction that for each $n=1, 2, \cdots$,

$$f^{(n)}=z^{-n}\left(\sum_{j=1}^{n} b_{jn}\theta^j f\right), \tag{2.3}$$

where the b_{jn} are integers, and $b_{nn}=1$. (In fact, it is easy to see that as polynomials in x,

$$\sum_{j=1}^{n} b_{jn}x^j=x(x-1)\cdots(x-(n-1)).) \tag{2.4}$$

(d) [11; § 49]. A logarithmic domain of rank zero (briefly, an LD_0) over $F(a, b)$ is a complex vector space L of admissible functions in $F(a, b)$, which contains the constants, and such that any finite linear combination of elements of L, with coefficients which are logarithmic monomials of rank $\leq p$ for some $p\geq 0$, is either trivial in $F(a, b)$ or is $\sim$ to a logarithmic monomial of rank $\leq p$ in $F(a, b)$. (The simplest examples of such sets L (where we can take (a, b) to be any open subinterval of $(-\pi, \pi)$) are the set of all polynomials, the set of all rational functions, and the set of all rational combinations of logarithmic monomials of rank ≤ 0. More extensive examples can be found in [11; §§ 128, 53].)

(e) [2; § 3]. If $G(v)$ is a polynomial in v, whose coefficients belong to an LD_0

over $F(a, b)$, then a logarithmic monomial M is called a *critical monomial* of G if there exists an admissible function $h \sim M$ in $F(a, b)$ such that $G(h)$ is not $\sim G(M)$ in $F(a, b)$. The set of critical monomials of G can be produced by using the algorithm in [2; § 26] which is based on a Newton polygon construction. This algorithm shows that the critical monomials are of rank ≤ 0. (In the special case where the coefficients of $G(v)$ are rational functions, the critical monomials are precisely the functions cz^{α} which form the first term of one of the expansions around $z=\infty$ of the algebraic function defined by $G(v)=0$.) The algorithm also associates with each critical monomial, a positive integer called the *multiplicity*. For our purposes here, it suffices to know (see [2; § 29(b)]) that if D and d represent the maximum and minimum degrees respectively of the nontrivial terms in $G(v)$ (as a polynomial in v), and if $G(v)$ possesses $D-d$ distinct critical monomials, then each has multiplicity equal to 1 (i.e. each is *simple*).

3. Statements of Main Results

(The proofs will be given in §§ 4, 5)

THEOREM 2. *Let n be a positive integer, and let $\{R_0(z), \cdots, R_n(z)\}$ be contained in an LD_0 over $F(a, b)$ for some (a, b) with $-\pi \leq a < b \leq \pi$, and assume that $R_n(z)$ is nontrivial (see § 2(b)) in $F(a, b)$. Using (2.3), rewrite the equation,*

$$R_n(z)w^{(n)} + R_{n-1}(z)w^{(n-1)} + \cdots + R_0(z)w = 0\,, \tag{3.1}$$

in the form,

$$\sum_{j=0}^{n} B_j(z)\theta^j w = 0\,, \qquad where \quad \theta^0 w = w\,. \tag{3.2}$$

Assume that,

$$B_j \ll B_0 \qquad in \quad F(a, b) \qquad for\ all \quad j \geq 1\,. \tag{3.3}$$

Define a sequence of integers $0 = t(0) < t(1) < \cdots < t(\sigma) = n$ as follows: $t(0)=0$, and if $t(j)$ has been defined and is less than n, let $t(j+1)$ be the largest integer k, such that $t(j) < k \leq n$, and such that for each i satisfying $t(j) < i \leq n$, either $B_i \ll B_k$ or $B_i \approx B_k$ in $F(a, b)$. Let,

$$G(v) = \sum_{j=0}^{\sigma} z^{t(j)} B_{t(j)}(z) v^{t(j)}\,. \tag{3.4}$$

Assume that $G(v)$ has n distinct critical monomials $N_1, \cdots, N_n$, and let them be arranged so that, for each j, either $N_j \ll N_{j+1}$ or $N_j \approx N_{j+1}$. Then:

(a) *For each j, N_j is of the form $c_j z^{-1+d_j}$, where $d_j > 0$.*

(b) *For each pair (k, j) of integers with $1 \leq k < j \leq n$, we have $N_j - N_k \approx N_j$ in $F(a, b)$.*

(c) *For each pair (k, j) of integers with $1 \leq k < j \leq n$, let E_{kj} denote the set of zeros on (a, b) of the function $IF(N_j - N_k, \varphi)$ (see (1.14)), and let $E = \{\lambda_1, \cdots, \lambda_p\}$, where*

$\lambda_1 < \lambda_2 < \cdots < \lambda_p$, *denote the union (over all* (k, j)*) of the sets* E_{kj}. *Set* $\lambda_0 = a$ *and* $\lambda_{p+1} = b$, *and let* j *be any element of* $\{0, 1, \cdots, p\}$. *Then*:

(i) *There exist admissible functions* $V_1, \cdots, V_n$ *in* $F(\lambda_j, \lambda_{j+1})$ *such that* $V_k \sim N_k$ *in* $F(\lambda_j, \lambda_{j+1})$ *for* $k = 1, \cdots, n$, *and such that if* f_k *is a function of the form* $\exp \int V_k$ *for* $k = 1, \cdots, n$, *then the set* $\{f_1, \cdots, f_n\}$ *is a fundamental set of solutions of* (3.1).

(ii) *If* $f(z) \not\equiv 0$ *is any solution of* (3.1) *which is admissible in* $F(\lambda_j, \lambda_{j+1})$, *then there is an element* T *of* $F(\lambda_j, \lambda_{j+1})$ *on which* f *has no zeros.*

DEFINITION. Let the equation (3.1) have the form (3.2) and assume (3.3) holds. We will call the polynomial $G(v)$ which is constructed in (3.4), the *factorization polynomial* for (3.1).

COROLLARY. *Let* n *be a positive integer, and let* $R_0(z), \cdots, R_n(z)$ *be rational functions with* $R_n \not\equiv 0$, *and consider the equation,*

$$R_n(z)w^{(n)} + R_{n-1}(z)w^{(n-1)} + \cdots + R_0(z)w = 0\,. \tag{3.5}$$

Using (2.3), *rewrite* (3.5) *in the form,*

$$\sum_{j=0}^{n} B_j(z)\theta^j w = 0\,, \qquad \text{where} \quad \theta^0 w = w\,, \tag{3.6}$$

and assume that $B_j \ll B_0$ *in* $F(-\pi, \pi)$ *for each* $j \geq 1$. *Let* $G(v)$ *be the factorization polynomial for* (3.5), *and assume that* $G(v)$ *possesses* n *distinct critical monomials* $N_1, \cdots, N_n$, *arranged so that for each* j, *either* $N_j \ll N_{j+1}$ *or* $N_j \approx N_{j+1}$. *For each pair* (k, j) *of integers with* $1 \leq k < j \leq n$, *let* E_{kj} *denote the set of zeros on* $(-\pi, \pi)$ *of* $IF(N_j - N_k, \varphi)$, *and let* $E = \{\lambda_1, \cdots, \lambda_p\}$, *where* $\lambda_1 < \lambda_2 < \cdots < \lambda_p$, *denote the union (over all* (k, j)*) of the sets* E_{kj}. *Set* $\lambda_{p+1} = \pi$. *Then, for any solution* $f \not\equiv 0$ *of* (3.5) *which is meromorphic on the plane, and any* $\varepsilon > 0$, *all but finitely many zeros of* f *lie in the union for* $j = 1, 2, \cdots, p+1$, *of the sectors,*

$$W_j(\varepsilon) : |\arg z - \lambda_j| < \varepsilon\,. \tag{3.7}$$

4. Proof of Theorem 2

We will require the following fact:

LEMMA C. *Le* n *be a positive integer. Let* $\{f_1, \cdots, f_n\}$ *be a set of admissible functions in some* $F(a, b)$, *with the following property: For any two distinct elements* k *and* j *in* $\{1, \cdots, n\}$, *either* $f_k \ll f_j$ *or* $f_j \ll f_k$ *in* $F(a, b)$. *Then, there is an element* m *in* $\{1, \cdots, n\}$ *such that* $f_j \ll f_m$ *for each element* j *in* $\{1, \cdots, n\}$, *distinct from* m.

Proof. The proof is by induction on n, being trivial for $n = 1$. Assuming the statement for n, assume $\{f_1, \cdots, f_{n+1}\}$ satisfies the hypothesis of the lemma. Then there exists m in $\{1, \cdots, n\}$ such that $f_j \ll f_m$ for $1 \leq j \leq n, j \neq m$. If $f_m \ll f_{n+1}$, then f_{n+1} is the desired element. If $f_{n+1} \ll f_m$, then f_m is the desired element, and the proof is complete.

Proof of Theorem 2. Parts (a), (b) and the first part of (c) are contained in [3; §§ 3, 7, 9]. We remark that Part (i) of (c) is proved by showing that in each $F(\lambda_j, \lambda_{j+1})$, the operator on the left side of (3.1) factors exactly into a product (i.e. composition) of first-order differential operators (see [3; § 7(b), p. 97]). The proof is then completed by successive integrations of the factored operator (see [3; p. 99]). Thus it remains to prove (ii) of Part (c). If $f \not\equiv 0$ is an admissible solution of (3.1) in some $F(\lambda_j, \lambda_{j+1})$, then on some element of $F(\lambda_j, \lambda_{j+1})$ we can write $f = \sum_{k=1}^{n} c_k f_k$, where the f_k are the solutions in Part (i), and where the c_k are constants, not all zero. If k and q are two distinct elements of $\{1, \cdots, n\}$, say $k < q$, then the ratio f_q/f_k is of the form $\exp \int (V_q - V_k)$, and clearly by Part (b), $V_q - V_k \sim N_q - N_k$. Since $IF(N_q - N_k, \varphi)$ is nowhere zero on $(\lambda_j, \lambda_{j+1})$ by construction, it follows from Lemma B that either $f_q \ll f_k$ or $f_k \ll f_q$ in $F(\lambda_j, \lambda_{j+1})$. Thus the set of all f_k for which $c_k \neq 0$ satisfies the hypothesis of Lemma C, and so there is an index m for which $c_m \neq 0$ and such that $f_k \ll f_m$ in $F(\lambda_j, \lambda_{j+1})$ for all $k \neq m$. Hence $f = c_m f_m(1+h)$ where $h \to 0$ in $F(\lambda_j, \lambda_{j+1})$. Since $f_m = \exp \int V_m$, we can choose an element of $F(\lambda_j, \lambda_{j+1})$ on which f_m has no zeros, and $|h| < 1/2$. Then clearly f has no zeros on this element of $F(\lambda_j, \lambda_{j+1})$ proving Theorem 2 completely.

5. Proof of the Corollary

If $f \not\equiv 0$ is a meromorphic solution of (3.5) on the plane, then f can have only finitely many poles since any pole of f must either be a zero of R_n or a pole of one of the functions $R_0, \cdots, R_{n-1}$. Setting $\lambda_0 = -\pi$, it then follows that f is admissible in each $F(\lambda_j, \lambda_{j+1})$, for $j = 0, \cdots, p$. Thus by Theorem 2, there is an element T_j of $F(\lambda_j, \lambda_{j+1})$ on which f has no zeros, for each $j = 0, \cdots, p$. Thus, if $\varepsilon > 0$ is given, then by Lemma A, there is a constant $K_j(\varepsilon) > 0$ such that f has no zeros on the set,

$$\lambda_j + \varepsilon \leq \arg z \leq \lambda_{j+1} - \varepsilon, \qquad |z| \geq K_j(\varepsilon). \tag{5.1}$$

for each $j = 0, 1, \cdots, p$. Since f can have only finitely many zeros in any bounded set, it follows that for each $j = 0, 1, \cdots, p$, the solution f has only finitely many zeros in the closed sector (including the origin),

$$\lambda_j + \varepsilon \leq \arg z \leq \lambda_{j+1} - \varepsilon. \tag{5.2}$$

The Corollary now follows by noting that the complement of the union (over $j = 1, 2, \cdots, p+1$) of the sectors $W_j(\varepsilon)$ in (3.7) is precisely the union (over $j = 0, 1, \cdots, p$) of the closed sectors in (5.2) since $\lambda_0 = -\pi$ and $\lambda_{p+1} = \pi$.

6. Proof of Theorem 1

To prove Theorem 1, we will require the following lemma:

LEMMA D. *Assume the hypothesis and notation of Theorem* 1, *and let* (1.4) *have* *the form* (3.2) *when rewritten in terms of the operator* θ (*using* (2.3)). *Then* (3.3) *holds, and the factorization polynomial* $G(v)$ *for* (1.4) *has the n distinct critical monomials* $b_j z^{\alpha_0/n}$ *for* $j=1, \cdots, n$.

The proof of Lemma D will require a close examination of the algorithm in [2; § 26], and will thus require some additional notation from [2] and [11].

Notation. (a) If $f \sim M$ in $F(a, b)$, where M is given by (2.1), we denote $a_0 = \delta_0(f)$. If f is trivial in $F(a, b)$, we write $\delta_0(f) = -\infty$.

(b) Let F denote $F(a, b)$ for some (a, b). Let F_1 denote the subset of F consisting of those elements V such that $|z| > 1$ for all z in V. Then, we will denote by $\operatorname{Log} F$, the set whose elements are the sets $\{\operatorname{Log} z : z \in V\}$ for each V in F. It is easy to verify that $\operatorname{Log} F$ is also a filter base which converges to ∞. For a function $g(u)$ which is admissible in $\operatorname{Log} F$ (i.e. analytic in some element of $\operatorname{Log} F$), the statement $g \to c$ in $\operatorname{Log} F$, means that given $\varepsilon > 0$, there is an element V_1 in $\operatorname{Log} F$ on which $|g(u) - c| < \varepsilon$. Analogously, the concepts $g \ll 1$, $g_1 \ll g_2$ $g_1 \sim g_2$ in $\operatorname{Log} F$ are defined as in § 2(b) and we have the following fact:

LEMMA E [11; § 46]. *Let* $\varepsilon > 0$ *and assume* $f \ll z^{-\varepsilon}$ *in* F. *Then,* $g(u) = f(e^u)$ *is trivial in* $\operatorname{Log} F$ (i.e. $g \ll u^{-\alpha}$ *in* $\operatorname{Log} F$ *for every* $\alpha > 0$).

(c) [2; §§ 7, 8]. Let $G(v)$ be a polynomial in v whose coefficients belong to an LD_0 over $F(a, b)$, say,

$$G(v) = \sum_{j=0}^{n} B_j(z) v^j . \tag{6.1}$$

If α is any real number, we will denote,

$$G_\alpha(y) = \sum_{j=0}^{n} B_j(e^u) e^{\alpha j u} y^j , \tag{6.2}$$

so that $G_\alpha(y)$ is obtained from $G(v)$ by the change of variables $z = e^u$, $v = y e^{\alpha u}$. If not all $B_j(z)$ are trivial in F, we denote

$$\delta^* = \max\{\alpha j + \delta_0(B_j(z)) : j = 0, \cdots, n\} , \tag{6.3}$$

and

$$[\alpha; G](y) = e^{-\delta^* u} G_\alpha(y) . \tag{6.4}$$

The following fact from [2] will be needed:

LEMMA F [2; §8]. *If not all* $B_j(z)$ *are trivial in* F, *then some coefficient of* $[\alpha; G](y)$ *is not trivial in* $\operatorname{Log} F$.

Proof of Lemma D. Using (2.3), we easily see that when (1.4) is rewritten in the form (3.2), we have (setting $a_n \equiv 1$), (6.5) $$B_0 \equiv a_0 \quad \text{and} \quad B_k = \sum_{j=k}^{n} a_j z^{-j} b_{kj} \qquad \text{for} \quad k \geq 1 .$$

From (1.6), it is clear that (3.3) holds. We now set (taking $\alpha_n = 0$),

(6.6) $$J = \{j\colon 0 \leq j \leq n,\ \alpha_j = ((n-j)/n)\alpha_0\} ,$$

and

(6.7) $$H(v) = \sum_{k \in J} z^k B_k(z) v^k .$$

We now assert that,

(6.8) $$B_k = a_k z^{-k}(1 + \Gamma_k) \qquad \text{where} \quad \delta_0(\Gamma_k) < 0 \qquad \text{if} \quad k \in J .$$

To prove (6.8), define Γ_k by (6.8), so by (6.5), Γ_k is a sum of terms $b_{kj} e_j$ where $e_j = (a_j/a_k) z^{k-j}$ for $j > k$. If $k \in J$, then from (1.6),

(6.9) $$\delta_0(e_j) \leq (k-j)(1 + (\alpha_0/n)) < 0 \qquad \text{if} \quad k < j ,$$

proving (6.8). It now easily follows from (6.8), (1.5), and Lemma E, that

(6.10) $$H_{\alpha_0/n}(y) = e^{\alpha_0 u} \sum_{k \in J} A_k^*(1 + V_k(u)) y^k ,$$

where we are using the notation (6.2), and where A_k^* is defined in (1.7), and where

(6.11) $$V_k(u) \quad \text{is trivial in Log}\, F(-\pi, \pi) \qquad \text{for all} \quad k.$$

We now assert that in $F(-\pi, \pi)$,

(6.12) $$B_s \ll B_k \qquad \text{if} \quad k \in J \quad \text{and} \quad s > k .$$

The proof of (6.12) follows immediately from (6.5), (6.8) and (6.9).

Now let $G(v)$ be the factorization polynomial of (1.4) as defined in (3.4). It is clear from (6.12) and the construction of the sequence $t(j)$ in Theorem 2 that each $k \in J$ is some $t(j)$, and thus we may write,

(6.13) $$G(v) - H(v) = L(v) .$$

where

(6.14) $$L(v) = \sum_{m \in I} z^m B_m(z) v^m ,$$

where I is the set of integers m in $[0, n]$ for which,

(6.15) $$m \notin J \quad \text{but} \quad m = t(j) \qquad \text{for some} \quad j .$$

Now for each m in $\{0, 1, \cdots, n\}$, we have from (6.5), (1.5) and Lemma E that,

(6.16) $$B_m(e^u) = \sum_{j=m}^{n} A_j e^{(\alpha_j - j)u}(1 + W_j(u)) b_{mj} ,$$

where we take $A_j=0$ if $a_j\equiv 0$, and where,

$$W_j(u) \quad \text{is trival in } \operatorname{Log} F(-\pi,\pi) \qquad \text{for each} \quad j. \tag{6.17}$$

Thus, in the notation of (6.2), we have from (6.14) and (6.16) that,

$$L_{\alpha_0/n}(y)=e^{\alpha_0 u}\sum_{m\in I}\left(\sum_{j=m}^{n} \Delta_{mj}(u)\right)y^m, \tag{6.18}$$

where

$$\Delta_{mj}(u)=A_j b_{mj} e^{d(m,j)u}(1+W_j(u)), \tag{6.19}$$

and where,

$$d(m,j)=m-\alpha_0+(\alpha_0/n)m+\alpha_j-j \qquad \text{for} \quad j\geq m. \tag{6.20}$$

Now if $m\in I$, then $m\notin J$ by (6.15) so $\alpha_m<((n-m)/n)\alpha_0$. Hence, $d(m,m)<0$. For $j>m$, we have $\alpha_j\leq((n-j)/n)\alpha_0$ from which it follows that,

$$d(m,j)\leq(m-j)(1+\alpha_0/n))<0 \qquad \text{if} \quad j>m. \tag{6.21}$$

Thus, each $d(m,j)<0$ for $j\geq m$ if $m\in I$, and thus from Lemma E (and (6.17)), it follows from (6.19) that each $\Delta_{mj}(u)$ is trivial in Log $F(-\pi,\pi)$. Hence from (6.18), we can write,

$$L_{\alpha_0/n}(y)=e^{\alpha_0 u}T(y), \tag{6.22}$$

where,

$$\text{all coefficients of} \quad T(y) \quad \text{are trivial in} \quad \operatorname{Log} F(-\pi,\pi). \tag{6.23}$$

We now set,

$$\delta^*=\max\{(\alpha_0/n)t(j)+\delta_0(z^{t(j)}B_{t(j)})\colon 0\leq j\leq\sigma\}, \tag{6.24}$$

so that by (6.4) we have,

$$[\alpha_0/n;G](y)=e^{-\delta^* u}G_{\alpha_0/n}(y). \tag{6.25}$$

Thus, in view of (6.13), (6.10), and (6.22), we have,

$$[\alpha_0/n;G](y)=e^{(\alpha_0-\delta^*)u}\left(\sum_{k\in J} A_k^*(1+V_k(u))y^k+T(y)\right). \tag{6.26}$$

Now since each $k\in J$ is some $t(j)$, clearly from (6.24) we have,

$$\delta^*\geq\max\{(\alpha_0/n)k+\delta_0(z^k B_k)\colon k\in J\}, \tag{6.27}$$

from which it follows using (6.8) and (6.6) that $\delta^*\geq\alpha_0$. If $\delta^*>\alpha_0$, it would follow from (6.26), (6.11), and Lemma E that all coefficients of $[\alpha_0/n;G](y)$ are trivial in Log $F(-\pi,\pi)$ which contradicts Lemma F. Thus, we must have $\alpha_0=\delta^*$ in (6.26), so that

$$[\alpha_0/n;G](y)=h(y)+T_1(y), \tag{6.28}$$

where h is given by (1.7), and $T_1(y)$ is a polynomial all of whose coefficients are trivial in Log $F(-\pi, \pi)$. Thus by [2; Lemma 12, Part (b)], the roots $b_1, \cdots b_n$ of $h(y)$ give rise to the critical monomials $b_1 z^{\alpha_0/n}, \cdots, b_n z^{\alpha_0/n}$ of $G(v)$, proving Lemma D.

Remark. We remark that (6.28) was derived without using the hypothesis that the roots of $h(y)$ be distinct, and thus in the terminology of [2; § 28], we can say that if $(b_1, \cdots, b_n)$ is the sequence (counting multiplicity) of roots of $h(y)$, then $(b_1 z^{\alpha_0/n}, \cdots, b_n z^{\alpha_0/n})$ is the sequence (counting multiplicity) of the critical monomials of $G(v)$.

Proof of Theorem 1. Assume the hypothesis and notation of Theorem 1, and let $f \not\equiv 0$ be any solution of (1.4), and let $\varepsilon > 0$. Let β be defined by (1.8) and set $g(\zeta) = f(e^{-i\beta}\zeta)$ for all complex ζ. Then, it is easy to check that $g(\zeta)$ solves the equation,

$$g^{(n)} + \sum_{j=0}^{n-1} d_j(\zeta) g^{(j)} = 0 , \tag{6.29}$$

where,

$$d_j(\zeta) = a_j(e^{-i\beta}\zeta) e^{(j-n)i\beta} . \tag{6.30}$$

Thus from (1.5), we have for $j=0, 1, \cdots, n-1$,

$$d_j(\zeta) = K_j \zeta^{\alpha_j} + O(\zeta^{\alpha_j - 1}) \quad \text{as} \quad \zeta \to \infty , \qquad \text{if} \quad a_j \not\equiv 0 , \tag{6.31}$$

where,

$$K_j = A_j e^{(j-n-\alpha_j)i\beta} . \tag{6.32}$$

Thus, the equation (6.29) satisfies (1.6), and if $\alpha_j = ((n-j)/n)\alpha_0$, we have,

$$K_j = A_j^* e^{-(n+\alpha_0)i\beta} [e^{i\beta(1+(\alpha_0/n))}]^j . \tag{6.33}$$

It now easily follows that for the equation (6.29), the polynomial $h_1(t)$ defined in (1.7), namely

$$h_1(t) = t^n + \sum \{K_j t^j \colon \alpha_j = ((n-j)/n)\alpha_0, \, 0 \le j \le n-1\} , \tag{6.34}$$

is simply $e^{-i\beta(n+\alpha_0)} h(e^{i\beta(1+(\alpha_0/n))} t)$, where h is the polynomial in (1.7) for the original equation (1.4). Since h has the simple roots $b_1, \cdots, b_n$, clearly h_1 has the simple roots $b_1^*, \cdots, b_n^*$, where $b_j^* = b_j e^{-i\beta(1+(\alpha_0/n))}$. Thus, the equation (6.29) satisfies the complete hypothesis of Theorem 1, and hence by Lemma D, the factorization polynomial $G^*(v)$ for (6.29) has the n distinct critical monomials $N_j^* = b_j^* \zeta^{\alpha_0/n}$ for $j=1, \cdots, n$. Thus by the Corollary, if we let $\lambda_1 < \cdots < \lambda_p$ denote the union (over all (k, j) with $1 \le k < j \le n$) of the sets of zeros on $(-\pi, \pi)$ of the functions $IF(N_j^* - N_k^*, \varphi)$, and let $\lambda_{p+1} = \pi$, then all but finitely many zeros of $g(\zeta)$ lie in the union of the sectors,

$$W_j(\varepsilon) \colon |\arg \zeta - \lambda_j| < \varepsilon , \qquad \text{for} \quad j = 1, 2, \cdots, p+1 . \tag{6.35}$$

But clearly,

(6.36) $$N_j^* - N_k^* = (b_j - b_k)e^{-i\beta(1+(\alpha_0/n))}\zeta^{\alpha_0/n},$$

and so from the definition of indicial function in (2.2), and the definition of β in (1.8), it is easy to see that the function $IF(N_j^* - N_k^*, \varphi)$ is simply the negative of the function in (1.9). Hence, for the set D defined in Theorem 1, we have $D = \{\lambda_1, \cdots, \lambda_p\}$, and so $p=q$, and the sector $W_j(\varepsilon)$ in (6.35) is $|\arg \zeta - \sigma_j| < \varepsilon$, for $j = 1, \cdots, q+1$. Since $f(z) = g(\zeta)$, where $\zeta = e^{i\beta}z$ (and so $\arg \zeta = \beta + \arg z$), clearly all but finitely many zeros of f lie in the union of the sectors (1.10) for $j=1, \cdots, q+1$, which proves Theorem 1.

7. Examples

(1) The example (1.11) affords an opportunity to check the sharpness of Theorem 1. In the notation of Theorem 1, we have for (1.11) that $\alpha_0 = 3$, and $h(t) = t^3 - t^2 - 4t + 4$, so that h has the simple roots $b_1 = 1$, $b_2 = 2$, and $b_3 = -2$. For $j=2$ and $k=1$, the function in (1.9) is $\mathrm{Sin}(2(\varphi - \pi))$, while for $(j, k) = (3, 1)$ and $(j, k) = (3, 2)$, the functions in (1.9) are just $-\mathrm{Sin}(2(\varphi - \pi))$. Thus in Theorem 1, we have $q=3$ and $\sigma_1 = -\pi/2$, $\sigma_2 = 0$, $\sigma_3 = \pi/2$, and $\sigma_4 = \pi$. The constant β in (1.8) is $3\pi/4$, and so the ε-sectors (1.10) are sectors centered on the rays $\arg z = -5\pi/4$, $-3\pi/4$, $-\pi/4$, and $\pi/4$. In fact, all of these ε-sectors contain infinitely many zeros of any solution of (1.11) of the form $c_1 e^{z^2} + c_2 e^{-z^2}$ where c_1 and c_2 are nonzero constants. (In the special case $c_1 = 1$, $c_2 = -1$, all of the zeros are on the center rays.)

(2) To illustrate the results in § 3, we consider the equation,

(7.1) $$w''' + zw' - zw = 0,$$

which is not covered by Theorem 1 since $\alpha_1 = \alpha_0$. We will first apply the Corollary directly to this equation. We rewrite the equation in θ which yields (by (2.3) and (2.4)),

(7.2) $$z^{-3}\theta^3 w - 3z^{-3}\theta^2 w + (1 + 2z^{-3})\theta w - zw = 0.$$

Clearly (3.3) is satisfied, and we construct the sequence $t(j)$ defined in Theorem 2. Clearly $t(0) = 0$, $t(1) = 1$, and $t(2) = 3$ so that in (3.4), we have

(7.3) $$G(v) = v^3 + (z + 2z^{-2})v - z.$$

Applying the algorithm in [2; § 26], we find the critical monomials $N_1 = 1$ (from degree 1 and degree 0), and $N_2 = -iz^{1/2}$, $N_3 = iz^{1/2}$ (from degree 3 and degree 1). The three functions $IF(N_j - N_k, \varphi)$ for $1 \le k < j \le 3$, all reduce to $\pm\mathrm{Sin}((3/2)\varphi)$, which on $(-\pi, \pi)$ has zeros $\lambda_1 = -2\pi/3$, $\lambda_2 = 0$, $\lambda_3 = 2\pi/3$. According to the Corollary, we must also add $\lambda_4 = \pi$, and the union of the sectors (3.7) for $j = 1, 2, 3, 4$, contains all but finitely many zeros of any solution $f \not\equiv 0$ of (7.1). However, by using a rotation, as mentioned in § 1, we will now show that the sector around $\arg z = \lambda_4$ is extraneous.

For any solution $w(z)$ of (7.1), and any real constant α, let $u(\zeta) = w(e^{-i\alpha}\zeta)$. We compute the differential equation for $u(\zeta)$ and write it in terms of $\theta u = \zeta u'$. This yields, (7.4) $$e^{3i\alpha}\zeta^{-3}\theta^3u-3e^{3i\alpha}\zeta^{-3}\theta^2u+(1+2e^{3i\alpha}\zeta^{-3})\theta u-e^{-i\alpha}\zeta u=0\,.$$

Forming the factorization polynomial in (3.4), we find the critical monomials $N_1=e^{-i\alpha}$, and

(7.5) $$N_2=e^{i(\pi-3\alpha)/2}\zeta^{1/2}\,,\quad \text{and}\quad N_3=-N_2\,.$$

The indicial functions $IF(N_j-N_k,\varphi)$ for $k<j$ all reduce to,

(7.6) $$\pm\,\mathrm{Cos}((3/2)\varphi+(\pi-3\alpha)/2)\,.$$

We now choose a value of α for which $\varphi=\pi$ is automatically a zero of (7.6). Clearly $\alpha=\pi$ will suffice, and for this value of α, the zeros on $(-\pi,\pi)$ of the functions in (7.6) are $\lambda_1=-\pi/3$ and $\lambda_2=\pi/3$. To apply the Corollary, we must also adjoin $\lambda_3=\pi$, and thus all but finitely many zeros of $u(\zeta)$ must lie in the union of the sectors $|\arg\zeta-\lambda_j|<\varepsilon$ for $j=1, 2, 3$. Since $w(z)=u(\zeta)$ where $\zeta=e^{i\alpha}z$, we find that all but finitely many zeros of a solution $w(z)$ of (7.1) must lie in the union (over $j=1, 2, 3$,) of the sectors $|\arg z-\beta_j|<\varepsilon$, where $\beta_1=-4\pi/3$, $\beta_2=-2\pi/3$, and $\beta_3=0$. (Of course β_1 gives the same sector as $|\arg z-2\pi/3|<\varepsilon$, and we have thus shown that the sector around $\arg z=\pi$ is extraneous.)

8. Remark

In this section, we show that for any $n>2$, there exist linear differential equations of the form (1.4), having polynomial coefficients, for which no analogue of Theorem 1 exists. Consider the equation,

(8.1) $$w^{(n)}+z^2w''+zw'+w=0\,,\qquad \text{where}\quad n\geq 3\,.$$

We will show that this equation has the following property: For any $\varepsilon>0$ and any value of θ in $(-\pi,\pi]$, there exists a solution $f\not\equiv 0$ of (8.1), which has infinitely many zeros in $|\arg z-\theta|<\varepsilon$.

To prove this fact, we note first that by [3; §4], there exist finitely many points $\theta_1<\theta_2<\cdots<\theta_q$ in $(-\pi,\pi)$ such that in each of $F(-\pi,\theta_1)$, $F(\theta_1,\theta_2)$, $\cdots$, $F(\theta_q,\pi)$ separately, the equation (8.1) possesses solutions g_1 and g_2 with $g_1\sim z^i$ and $g_2\sim z^{-i}$. Let θ be any number in $(-\pi,\pi)$ which is distinct from $\theta_1,\cdots,\theta_q$, and let $\varepsilon>0$ be arbitrary. We can assume ε is so small that g_1 and g_2 exist and are admissible in $F(\theta-\varepsilon,\theta+\varepsilon)$, and satisfy $g_1\sim z^i$ and $g_2\sim z^{-i}$. (Of course, g_1 and g_2 can be extended to be entire functions in view of the form of (8.1).) Set $u_j(\zeta)=g_j(\zeta e^{i\theta})$ for $j=1, 2$, so that u_1 and u_2 are admissible in $F(-\varepsilon,\varepsilon)$, and satisfy $u_1(\zeta)\sim e^{-\theta}\zeta^i$ and $u_2(\zeta)\sim e^{\theta}\zeta^{-i}$ in $F(-\varepsilon,\varepsilon)$. Set $v_1=e^{\theta}u_1$ and $v_2=e^{-\theta}u_2$, so that we can write $v_1(\zeta)=\zeta^ih_1(\zeta)$ and $v_2(\zeta)=\zeta^{-i}h_2(\zeta)$, where $h_j(\zeta)\to 1$ in $F(-\varepsilon,\varepsilon)$ for $j=1, 2$. Now set $h=h_2/h_1$, $F(\zeta)=\zeta^{2i}-1$, and $r_m=e^{\pi m}$ for $m=1, 2,\cdots$, so that $F(r_m)=0$ for all m. By [4; Lemma D, p. 127], there exists δ in $(0, 1)$ such that if D_m denotes the closed disk $|\zeta-r_m|\leq\delta r_m$, then each D_m lies in $|\arg\zeta|<\varepsilon/2$, and, (8.2) $$|F(\zeta)|\geq\delta \quad \text{on} \quad |\zeta-r_m|=\delta r_m\,.$$

Since $h\to 1$ in $F(-\varepsilon, \varepsilon)$, there is an element V in $F(-\varepsilon, \varepsilon)$ on which h is analytic and,

(8.3) $$|1-h(\zeta)|<\delta \quad \text{for all} \quad \zeta \quad \text{in} \quad V\,.$$

By Lemma A in § 2(a), clearly V contains the disks D_m for all sufficiently large m, say for $m\geq m_0$, and in view of (8.2) and (8.3), it follows from Rouche's theorem that the function $F(\zeta)+(1-h(\zeta))$ possesses a zero ζ_m in D_m for each $m\geq m_0$. (Since $\delta<1$, clearly $|\zeta_m|\to+\infty$ as $m\to\infty$.) From the definitions of F, h, u_1 and u_2, it follows that $u_1(\zeta_m)=e^{-2\theta}u_2(\zeta_m)$ for each $m\geq m_0$. Thus, if we set $z_m=\zeta_m e^{i\theta}$ for $m\geq m_0$, then clearly $|\arg z_m-\theta|<\varepsilon$, and each z_m is a zero of the solution $g_1-e^{-2\theta}g_2$ of (8.1). Since $|z_m|=|\zeta_m|\to+\infty$ as $m\to\infty$, we have established the desired property of (8.1) if θ is distinct from $\theta_1, \cdots, \theta_q$, and clearly it then also holds for $\theta_1, \cdots, \theta_q$, and π.

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