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Generating Sets for Ideals

by

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Throughout, R will be a commutative ring with identity. For a finitely generated R-module A, $\mu(A)$ will denote the minimal number of generators for A. We say that A is *n-generated* if $\mu(A)\leq n$. If $\mu(A/Ra)\leq n-1$ for each $0\neq a\in A$, we say that A is *strongly n-generated.* Recently, Lantz and Martin [7] have studied strongly two-generated regular fractional ideals. Among other things, they showed that a strongly two-generated regular fractional ideal is invertible and that the set of such fractional ideals forms a group.

In this paper we study several related ideas. We say that a submodule C of A is *almost n-generated* (with respect to A) if for each $a\in A$, $\mu(C+Ra)\leq n+1$. (An ideal C of R is almost n-generated if C is almost n-generated with respect to R.) Suppose further that C is an ideal of R. Let $\{M_\lambda\}$ be the set of maximal ideals of R containing C and let $S(C)$ (or just S if no confusion can arise) be the multiplicatively closed subset $R-\bigcup M_\lambda$. It is easily seen that if I is an ideal of R with $I\subseteq\bigcup M_\lambda$, then I is contained in some M_λ ([5, Lemma 3, page 143]). Hence $\{M_{\lambda S}\}$ is the set of maximal ideals of R_S. Also, observe that for two ideals J and K of R, $J_S=K_S$ if and only if $J_{M_\lambda}=K_{M_\lambda}$ for each M_λ. We say that a subset $B\subseteq C$ *weakly generates* C if $C_S=BR_S$ and that C is *weakly n-generated* if C is weakly generated by a subset of n elements. Finally, if C is weakly one-generated, we will say that C is *weakly principal.*

It is well-known that for a finitely generated ideal I of R, $\mu(I/I^2)\leq\mu(I)\leq\mu(I/I^2)+1$. For example, see Nashier [8]. One interesting, but elementary, consequence of our investigation is that $\mu(I+(r))\leq\mu(I/I^2)+1$ for any finitely generated ideal I of R and any element $r\in R$. This follows from Theorem 2 which states that a weakly n-generated ideal is almost n-generated.

PROPOSITION 1. *Let $B\subseteq A$ be ideals of the commutative ring R. Let $S=R-\bigcup M_\lambda$ where $\{M_\lambda\}$ is the set of maximal ideals containing A. Then $B_S=A_S$ implies that $A=A^2+B$. If A_S is finitely generated, then $A=A^2+B$ implies that $B_S=A_S$.*

Proof. Since $B_S=A_S$, we have $B_{M_\lambda}=A_{M_\lambda}$ for each M_λ. Hence $A_{M_\lambda}=(A^2+B)_{M_\lambda}$. If M is a maximal ideal of R with $M\not\supseteq A$, then $A_M=R_M$. So again $A_M=(A^2+B)_M$.

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Hence $A=A^2+B$ locally and hence globally. Conversely, suppose that $A=A^2+B$ and that A_S is finitely generated. Then $A_S=A_S^2+B_S$. Observe that A_S is contained in the Jacobson radical of R_S. Since A_S is finitely generated, $A_S=B_S$ by Nakayama's Lemma.

THEOREM 2. *Suppose that $B\subseteq A$ are ideals of a commutative ring R with A finitely generated. Suppose that $B_M=A_M$ for each maximal ideal $M\supseteq A$. Then for every $r\in R$, $\mu(A+(r))\leq\mu(B)+1$. In particular, if A is finitely generated and weakly n-generated, then A is almost n-generated.*

Proof. By Proposition 1, $A=A^2+B$. Hence $\bar{A}$ is a finitely generated idempotent ideal of the ring $\bar{R}=R/B$. Thus $\bar{A}=\overline{Re}$ for some $e\in R$ where $\bar{e}\in\bar{R}$ is idempotent ([4, Corollary 6.3]). Then $\bar{A}+(\bar{r})=(\bar{e},\bar{r})=(\bar{e}+(\bar{1}-\bar{e})\bar{r})$ with the last equality following since $\bar{e}$ is idempotent. So $A+(r)=B+(e+(1-e)r)$ and hence $\mu(A+(r))\leq\mu(B)+1$.

The following corollary improves the well-known result mentioned in the introduction.

COROLLARY 3. *Let I be a finitely generated ideal and let $r_1,\cdots,r_s\in R$. Then $\mu(I+(r_1,\cdots,r_s))\leq\mu(I/I^2)+s$.*

Proof. It suffices to do the case $s=1$. Let $\mu(I/I^2)=n$ and take $B=(i_1,\cdots,i_n)$ where $i_1,\cdots,i_n\in I$ and $\bar{i}_1,\cdots,\bar{i}_n$ generate I/I^2. Then $I=I^2+B$. Hence by the proof of Theorem 2, $\mu(I+(r_1))\leq\mu(B)+1\leq n+1=\mu(I/I^2)+1$.

PROPOSITION 4. *Let A be a finitely generated ideal of R that is weakly generated by B. Then $AR[\{X_\alpha\}]$ is also weakly generated by B. In particular, if A is finitely generated and weakly n-generated, then so is $AR[\{X_\alpha\}]$.*

Proof. By Proposition 1 we have $A=A^2+BR$. Hence $AR[\{X_\alpha\}]=(AR[\{X_\alpha\}])^2+BR[\{X_\alpha\}]$. Since A is finitely generated, so is $AR[\{X_\alpha\}]$. By Proposition 1, $AR[\{X_\alpha\}]$ is weakly generated by B.

A well-known result of Davis and Geramita [3] states that if R is a regular Noetherian Hilbert ring, then every maximal ideal of $R[X_1,\cdots,X_n]$ can be generated by an R-sequence. Hence if $\dim R=s$, every maximal ideal of $R[X_1,\cdots,X_n]$ can be generated by $n+s$ elements. We give an alternative proof of this result using Theorem 2.

THEOREM 5. *Let R be a regular Noetherian Hilbert ring. Then every maximal ideal of $R[X_1,\cdots,X_n]$ can be generated by an R-sequence. In particular, if $\dim R=s$, every maximal ideal of $R[X_1,\cdots,X_n]$ can be generated by $n+s$ elements.*

Proof. We first consider the case $n=1$. Let M be a maximal ideal of $R[X_1]$. Let $N=M\cap R$ and $m=htN$. Since R is a Hilbert ring, N is a maximal ideal of R. (For this fact about Hilbert rings and for other unreferenced results used in this proof, see [6].) Since N is a maximal ideal, $M=NR[X_1]+(f)$ for some $f\in R[X_1]$. Now since R is a regular ring, R_N is a regular local ring with $\dim R_N=m$. So N_N can be generated by m

elements. Hence N is weakly m-generated. By Proposition 4, $NR[X_1]$ is also weakly m-generated. By Theorem 2, $M=NR[X_1]+(f)$ can be generated by $m+1$ elements. Since $htM=m+1$, M can be generated by an R-sequence.

Suppose that $n>1$. Let M be a maximal ideal of $R[X_1, \cdots, X_n]$ and let $N=M\cap R[X_1, \cdots, X_{n-1}]$. Then N is a maximal ideal of $R[X_1, \cdots, X_{n-1}]$. Hence by induction N can be generated by an R-sequence, say $f_1, \cdots, f_t$. Now $M=NR[X_1, \cdots, X_n]+(f)$ for some $f\in R[X_1, \cdots, X_n]$ and it is easily seen that $f_1, \cdots, f_t, f$ is an R-sequence.

Actually, the last part of Theorem 5 can be extended to intersections of maximal ideals. Let R be an s-dimensional regular Noetherian Hilbert ring. If $M_1, \cdots, M_r$ are maximal ideals of $R[X_1, \cdots, X_n]$, then $M_1\cap\cdots\cap M_r$ can also be generated by $n+s$ elements. The proof of this result is similar to the proof of Theorem 5. We sketch the proof for the heart of the argument which is the case $n=1$. Let $I=M_1\cap\cdots\cap M_r$ and $J=I\cap R$. Then $J=N_1\cap\cdots\cap N_{r'}$ where $1\le r'\le r$ and the N_i are distinct maximal ideals of R. It suffices to show that J is weakly s-generated. R/J is a finite direct product of fields and hence $(R/J)[X_1]$ is a finite direct product of polynomial rings over fields and hence is a principal ideal ring. Hence $I=JR[X_1]+(f)$ for some $f\in R[X_1]$. Now J and hence $JR[X_1]$ is weakly s-generated. Hence by Theorem 2, I can be generated by $s+1$ elements. To show that J is weakly s-generated, it suffices to show that J/J^2 can be generated by s elements. Now $R/J^2\cong R/N_1^2\times\cdots\times R/N_{r'}^2$. Since R is an s-dimensional regular ring, N_i/N_i^2 can be generated by s elements, say N_i/N_i^2 is generated by $x_{1i}, \cdots, x_{si}$. Then $J/J^2\cong N_1/N_1^2\times\cdots\times N_{r'}/N_{r'}^2$ is generated by the s elements $(x_{11}, \cdots, x_{1r'}), \cdots, (x_{s1}, \cdots, x_{sr'})$.

In the case where A is weakly 1-generated, more can be said. The following theorem extends and gives the converse to [5, Theorem 1]. Recall that an ideal A is called a multiplication ideal if for each ideal $B\subseteq A$, $B=AC$ for some ideal C. Principal ideals and invertible ideals are multiplication ideals. A finitely generated ideal is a multiplication ideal if and only if it is locally principal. For results on multiplication ideals, the reader is referred to [1] and [2].

THEOREM 6. *For an ideal A of R consider the following conditions.*

(1) *$A=A^2+(x)$ for some $x\in A$.*

(2) *A is a multiplication ideal and $A\supsetneq AM_\lambda$ where $\{M_\lambda\}$ is the set of maximal ideals containing A.*

(3) *A_S is principal where $S=R-\bigcup M_\lambda$ and $\{M_\lambda\}$ is as in* (2).

Then (2) $\Rightarrow$ (3) $\Rightarrow$ (1). *If A is finitely generated and* $(0:A)$ *is contained in each M_λ, then* (1) $\Rightarrow$ (2).

Proof. (2) $\Rightarrow$ (3). Let $x\in A-\bigcup AM_\lambda$. Since A is a multiplication ideal, $(x)=AB$ for some ideal B. Also $B\not\subseteq M_\lambda$ for each M_λ. Thus $B_S=R_S$, so that $(x)_S=(AB)_S=A_S$.

(3) $\Rightarrow$ (1). Let $A_S=(x)_S$ where $x\in A$. By Proposition 1, $A=A^2+(x)$.

Suppose that A is finitely generated and that $(0:A)$ is contained in each M_λ. Under these conditions we prove that (1) $\Rightarrow$ (2). By Proposition 1, $A=A^2+(x)$ implies that $A_S=(x)_S$ and hence that $A_{M_\lambda}=(x)_{M_\lambda}$ for each maximal ideal $M_\lambda \supseteq A$. If M is a maximal ideal of R with $M \not\supseteq A$, then $A_M=R_M$ is again principal. So A is finitely generated and locally principal. Hence A is a multiplication ideal ([1, Theorem 3]). Suppose that $A=\bigcup AM_\lambda$. Then $x \in AM_\lambda$ for some M_λ. Now $(x)=AB$ for some ideal B of R. Hence $A=A^2+(x)=A^2+AB=A(A+B)$. Since A is a finitely generated multiplication ideal, we have $R=A+B+(0:A)$ ([1, Theorem 3]). Now $AB=(x)\subseteq AM_\lambda$ gives that $B\subseteq M_\lambda+(0:A)$. Hence $R=A+B+(0:A)\subseteq A+M_\lambda+(0:A)\subseteq M_\lambda$, a contradiction.

COROLLARY 7. *For a finitely generated regular ideal A, the following conditions are equivalent.*

(1) $A=A^2+(x)$ *for some* $x\in A$.

(2) *A is invertible and* $A \supsetneq \bigcup AM_\lambda$ *where* $\{M_\lambda\}$ *is the collection of maximal ideals containing A.*

(3) A_S *is principal where* $S=R-\bigcup M_\lambda$ *with* $\{M_\lambda\}$ *as in* (2).

The following theorem sums up the relationships among the types of generation that we have defined.

THEOREM 8. *Let A be a finitely generated ideal of a commutative ring R. Then* (1) *A is n-generated* $\Rightarrow$ (2) *A is strongly $n+1$-generated* $\Rightarrow$ (3) *A is weakly n-generated (provided $A^2\neq 0$)* $\Rightarrow$ (4) *A is almost n-generated* $\Rightarrow$ (5) *A is $n+1$-generated. However, in general none of these implications can be reversed.*

Proof. The implications (1) $\Rightarrow$ (2) and (4) $\Rightarrow$ (5) are obvious. (2) $\Rightarrow$ (3). Suppose that A is strongly $n+1$-generated and $A^2\neq 0$. Let $0\neq a\in A^2$. Then $A/(a)$ and hence A/A^2 can be generated by n elements. Hence $A=A^2+B$ where $\mu(B)\leq n$. So A is weakly n-generated. The implication (3) $\Rightarrow$ (4) is Theorem 2.

The following examples show that none of these implications can be reversed. (2) $\not\Rightarrow$ (1). Let A be a nonprincipal ideal in a Dedekind domain. Then A is strongly 2-generated, but not 1-generated. (3) $\not\Rightarrow$ (2). Let A be a nonprincipal ideal in a Dedekind domain R. Then A is weakly 1-generated. Hence $AR[X]$ is also weakly 1-generated by Proposition 4. By [7, Corollary 7] the only strongly 2-generated ideals of $R[X]$ that are extended are the principal ones. Hence $AR[X]$ is weakly 1-generated but not strongly 2-generated. (4) $\not\Rightarrow$ (3). Let $A=(X, Y)$ in the power series ring $K[[X, Y]]$, K a field. Since both A and $K[[X, Y]]$ can be generated by 2 elements, A is almost 1-generated. But since A is not invertible, A is not weakly 1-generated. (5) $\not\Rightarrow$ (4). The ideal $A=(X, Y)$ is a 2-generated ideal of the ring $K[X, Y, Z]$, K a field. Now $(X, Y)+(Z)$ can not generated by two elements, so (X, Y) is not almost 1-generated.

The product of two 1-generated ideals is 1-generated and the product of two strongly two-generated ideals is strongly two-generated ([7, Lemma 4]). This raises the natural question of whether the product of two weakly principal ideals is weakly principal. This is *not* the case. For let A be a weakly principal ideal that is not principal. Now $AR[X]$ is again weakly principal. Suppose that $AR[X](X)$ is weakly principal. Then $XAR[X]=X^2A^2R[X]+hR[X]$ for some $h\in R[X]$. One easily sees that $h=Xg$ for some $g\in R[X]$ and that then $A=g(0)R$ is principal. This contradiction shows that $AR[X](X)$ is not weakly principal.

We end with a remark on minimal bases. A minimal basis for a module A is defined as an irredundant generating set for A. Hence $\mu(A)$ is the length of the shortest minimal basis for A. If R is quasi-local with maximal ideal M, then any two minimal bases for A have the same length, namely $\mu(A)=\dim_{R/M} A/MA$. However, if R is not quasi-local, then A can have minimal bases of different lengths. An interesting result of Ratliff and Robson [9, (2) Theorem] states that if A has a minimal basis of length s then for all integers t with $s\leq t\leq \lambda(A)$, there is a minimal basis for A of length t. ($\lambda(A)$ is the length of a composition series for $A/J(A)$ or ∞ if no composition series exists; here $J(A)$ is the intersection of the maximal submodules of A). Actually, part of this result is a special case of a beautiful result due to Tarski [10]. It follows from Tarski's Irredundant Basis Theorem that if A is an algebra all of whose fundamental operations have arity not exceeding two, then the set of lengths of irredundant bases for A forms a convex subset of the natural members.

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