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Generalizations of Von Neumann Regular Rings and n-Like Rings

by

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As a generalization of von Neumann regular rings, D. D. Anderson [1] introduced the concept of n-von Neumann regular rings, in case rings are commutative. In this paper, we first treat a noncommutative generalization of n-von Neumann regular rings. Next we consider the structure of rings R satisfying the identity $(X_1-X_1^n)\cdots(X_m-X_m^n)=0$, where m and n are natural numbers with $n>1$. We show that the set of nilpotent elements of R forms a nilpotent ideal of index m. As a corollary we give a characterization of an n-like ring introduced by Yaqub [7]. As another corollary, we describe the structure of a generalized Boolean-like ring defined by Yakabe [6].

Throughout this paper, R denotes an associative ring with identity element 1, and $P(R)$ denotes the prime radical of R, that is the intersection of all prime ideals of R.

Let n be a natural number. A commutative ring R is called n-von Neumann regular if given $x_1, x_2, \cdots, x_n \in R$, there exist $a_1, a_2, \cdots, a_n \in R$ such that

$$(x_1-x_1a_1x_1)(x_2-x_2a_2x_2)\cdots(x_n-x_na_nx_n)=0 .$$

Anderson [1] proved that a commutative ring R is n-von Neumann regular if and only if $R/P(R)$ is von Neumann regular and $P(R)^n=(0)$. We attempt to generalize this result for noncommutative rings. The following example shows that it is impossible to generalize this result for noncommutative rings just as it is.

Example 1. Let F be a field, and set $S_i=M_2(F)$ for each $i\in N$. Then the ring $S=\prod_{i\in N} S_i$ consists of all sequences of 2 by 2 matrices. Let R denote the subring of S consisting of all sequences of matrices which are eventually upper triangular. We can easily see that R is semiprime and that given $x, y\in R$, there exist $a, b\in R$ such that $(x-xax)(y-yby)=0$. However, R $(=R/P(R))$ is not von Neumann regular. For example, consider the constant sequence $c=\left(\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}\right)$. It is easy to check that $c\notin cRc$.

As a generalization of [1, Theorem 1], we obtain the following theorem.

THEOREM 1. *Let R be a ring with* 1, *and n a natural number. Then the following statements are equivalent*:

(1) *Given* $x_1, \cdots, x_n \in R$, *there exist* $a_1, \cdots, a_n \in R$ *such that*

$$R(x_1 - x_1 a_1 x_1)R \cdots R(x_n - x_n a_n x_n)R = (0).$$

(2) $R/P(R)$ *is a von Neumann regular ring and* $P(R)^n = (0)$.

Proof. (1)$\Rightarrow$(2). Let x be an element of R. Then there exist $a_1, \cdots, a_n \in R$ such that $R(x - xa_1x)R \cdots R(x - xa_nx)R = (0)$. Let Spec$(R)$ denote the set of prime ideals of R. Then, for any $P \in \mathrm{Spec}(R)$, we have $x - xa_ix \in P$ for some i. Hence, if we set $S_i = \{P \in \mathrm{Spec}(R);\ x - xa_ix \in P\}$, we obtain that $\mathrm{Spec}(R) = S_1 \cup S_2 \cup \cdots \cup S_n$. Let I_i denote the intersection of all P in S_i. Then we have $x - xa_ix \in I_i$ for each i. Therefore we see that

$$\begin{aligned} x - x(a_1 + a_2 - a_1 x a_2)x &= (x - xa_1x)(1 - a_2x) \\ &= (1 - xa_1)(x - xa_2x) \in I_1 \cap I_2 . \end{aligned}$$

Continuing this process, we obtain an element $b \in R$ such that $x - xbx \in I_1 \cap I_2 \cap \cdots \cap I_n = P(R)$. This proves that $R/P(R)$ is von Neuman regular.

To prove $P(R)^n = (0)$, take $x_1, \cdots, x_n \in P(R)$. Then there exist $b_1, \cdots, b_n \in R$ such that $R(x_1b_1x_1 - x_1)R \cdots R(x_nb_nx_n - x_n)R = (0)$. Since $b_ix_i \in P(R)$, $b_ix_i - 1$ is invertible for each i. Hence we have

$$\begin{aligned} x_1x_2 \cdots x_n = (x_1b_1x_1 - x_1)(b_1x_1 - 1)^{-1}&(x_2b_2x_2 - x_2)(b_2x_2 - 1)^{-1} \\ &\cdots (x_nb_nx_n - x_n)(b_nx_n - 1)^{-1} \\ = 0 . \end{aligned}$$

This proves $P(R)^n = (0)$.

(2)$\Rightarrow$(1). Let $x_1, \cdots, x_n$ be elements of R. Since $R/P(R)$ is von Neumann regular, there exist $a_1, \cdots, a_n \in R$ such that $x_1a_1x_1 - x_1, \cdots, x_na_nx_n - x_n \in P(R)$. Then we have $R(x_1a_1x_1 - x_1)R \cdots R(x_na_nx_n - x_n)R = (0)$, because $P(R)^n = (0)$.

A ring R is called a duo ring if every one-sided ideal of R is a two-sided ideal. Clearly commutative rings are duo rings. A von Neumann regular ring R is called strongly regular if R has no non-zero nilpotent elements. It is easily checked that a von Neumann regular ring R is strongly regular if and only if R is a duo ring.

COROLLARY 1 (cf. [1, Theorem 1]). *Let* R *be a duo ring. Then the following statements are equivalent*:

(1) *Given* $x_1, \cdots, x_n \in R$, *there exist* $a_1, \cdots, a_n \in R$ *such that*

$$(x_1a_1x_1 - x_1) \cdots (x_na_nx_n - x_n) = 0 .$$

(2) $R/P(R)$ *is a strongly regular ring and* $P(R)^n = (0)$.

Proof. For a duo ring R, the statement (1) is equivalent to the statement (1) of Theorem 1. Thus the equivalence of (1) and (2) follows from Theorem 1.

Example 2. Let D be a strongly regular ring, and $D[X]$ denote the polynomial ring. Then $R=D[X]/X^nD[X]$ is a duo ring satisfying the equivalent statements in Corollary 1.

Let m be a natural number. In [1, Theorem 4], Anderson studied the structure of a commutative ring R satisfying the identity $(X_1-X_1^2)\cdots(X_m-X_m^2)=0$. He proved that $R/P(R)$ is a Boolean ring and $P(R)^m=(0)$. We shall generalize this result as follows:

THEOREM 2. *Let R be a ring with* 1, *and m and n be natural numbers with $n>1$. Then the following statements are equivalent*:

(1) *R satisfies the identity $(X_1-X_1^n)\cdots(X_m-X_m^n)=0$.*

(2) *$R/P(R)$ is a commutative ring satisfying the identity $X=X^n$ and $P(R)^m=(0)$.*

Proof. It suffices to prove that (1) implies (2). So assume that (1) holds. Let N denote the set of nilpotent elements in R. Let x be an element of the Jacobson radical J of R. Then we see that $x^m=(x-x^n)^m(1-x^{n-1})^{-m}=0$. This shows that $J\subseteq N$. We claim that N coincides with J. To see this, we need to show that R/J has no non-zero nilpotent elements. By [2, Theorem I.8.1, p. 14], R/J is a subdirect sum of primitive rings. Hence it suffices to show that a primitive ring S satisfying the identity in (1) is a division ring. Suppose, to the contrary, that S is not a division ring. Then, by [2, Theorem II.4.3, p. 33], either S is isomorphic to a matrix ring $M_k(D)$ $(k>1)$ over a division ring D, or for each natural number t, there exists a homomorphism of a subring of S on $M_t(D)$, D a division ring. In either case, we obtain an integer $k>1$ and a division ring D such that $M_k(D)$ satisfies the identity in (1). Let $\{e_{ij}\}$ denote the matrix units of $M_k(D)$. Set $x_i=e_{12}$ if i is odd and $x_i=e_{21}$ if i is even. Then we see that $(x_1-x_1^n)\cdots(x_m-x_m^n)=x_1\cdots x_m\neq 0$, which is a contradiction. Thus we proved $N=J$. Hence R/N satisfies the identity $X=X^n$. By [2, Theorem X.1.1, p. 217], R/N is a commutative ring.

Next we claim that $N^m=(0)$. To see this, let $x_1,\cdots,x_m\in N$. We construct elements $y_1,\cdots,y_m$ of N inductively as follows:

$$y_1=x_1 \quad\text{and}\quad y_{i+1}=(1-y_i^{n-1})^{-1}x_{i+1} \qquad \text{for}\quad i=1,\cdots,m-1\,.$$

Then we have

$$\begin{aligned}0&=(y_1-y_1^n)(y_2-y_2^n)\cdots(y_m-y_m^n)\\&=x_1x_2\cdots x_m(1-y_m^{n-1})\,.\end{aligned}$$

Thus we obtain $x_1x_2\cdots x_m=0$. This implies $N^m=(0)$. Therefore we conclude $N=P(R)$.

An ideal I of a ring R is said to be right T-nilpotent if given any sequence of elements $\{x_i\}\subseteq I$, there exists a natural number k such that $x_1x_2\cdots x_k=0$. It is well known that any right T-nilpotent ideal of a ring R is contained in $P(R)$. Taking in consideration this fact, we can prove the following theorem similarly as in the proof of Theorem 2.

THEOREM 3. *Let R be a ring with 1, and $n>1$ be an integer. Then the following statements are equivalent*:

(1) *Given any sequence of elements $\{x_i\}\subseteq R$, there exists a natural number k such that $(x_1-x_1^n)\cdots(x_k-x_k^n)=0$.*

(2) *$R/P(R)$ is a commutative ring satisfying the identity $X=X^n$ and $P(R)$ is right T-nilpotent.*

A ring R is called a generalized n-like ring if R satisfies the identity $(XY)^n-XY^n-X^nY+XY=0$. This ring was first investigated by H. G. Moore [4]. A generalized n-like ring R with $\text{char}(R)=n$ is called an n-like ring ([7]).

COROLLARY 2. 1) *If R is a generalized n-like ring, then $R/P(R)$ is a commutative ring satisfying the identity $X=X^n$, and $P(R)^2=(0)$.*

2) *Let R be a ring with 1, and n a natural number. Then R is an n-like ring if and only if R is commutative of characteristic n, $R/P(R)$ satisfies $X=X^n$ and $P(R)^2=(0)$.*

3) *Let R be a p-like ring, where p is a prime. Then $B=\{a\in R;\ a^p=a\}$ forms a subring of R and R is the trivial extension of B by $P(R)$.*

Proof. 1) By [5, Lemma 3(1)], R satisfies the identity $(X-X^n)(Y-Y^n)=0$. Hence the assertion follows from Theorem 2.

2) This follows from [5, Lemma 3(1)] and [7, Theorem 2].

3) Since R is commutative of characteristic p, we can easily check that B forms a subring of R. For any $x\in R$, we see that $0=(x-x^p)^p=x^p-x^{p^2}$, that is $x^p\in B$. Thus, $x=x^p+(x-x^p)\in B+P(R)$. Therefore we have that $R=B+P(R)$ as a B-bimodule. Since $P(R)^2=(0)$, R is the trivial extension of B by $P(R)$.

Following I. Yakabe [6], a ring R is called a generalized Boolean-like ring if $\text{char}(R)=2$ and $(x-x^2)(y-y^2)=0$ for all $x, y\in R$.

COROLLARY 3. *Let R be a ring of characteristic 2. Then the following statements are equivalent*:

(1) *R is a generalized Boolean-like ring.*

(2) *$R/P(R)$ is a Boolean ring and $P(R)^2=(0)$.*

(3) *There exists a Boolean ring B and a B-bimodule M such that R is embedded in $\begin{pmatrix} B & M \\ 0 & B \end{pmatrix}$ as a subring.*

Proof. The equivalence of (1) and (2) follows from Theorem 2, and (3) $\Rightarrow$ (2) is clear. The implication (2) $\Rightarrow$ (3) follows from [3, Theorem 2].

A ring R is called a generalized n-Boolean ring if $\text{char}(R)=2$ and $(x_1-x_1^2)\cdots(x_n-x_n^2)=0$ for all $x_1,\cdots,x_n\in R$ (cf. [1, p. 72]). Thus a generalized 2-Boolean ring is nothing but a generalized Boolean-like ring. We conclude this paper with the following corollary.

COROLLARY 4. *Let R be a generalized n-Boolean ring, and B denote the set of idempotents of R. Then B forms a subring of R if and only if B is contained in the center* *of R. In this case,* $R=B\oplus P(R)$ *as a B-bimodule.*

Proof. The first assertion is easily checked. Suppose now that B forms a subring of R. Let r be an element of R. Take an integer k such that $n\leq 2^k$. Since $\mathrm{char}(R)=2$, we see that $0=(r-r^2)^{2k}=r^{2k}-r^{2k+1}$, that is $r^{2k}\in B$. Since $R/P(R)$ is Boolean by Theorem 2, we see that $r-r^{2k}\in P(R)$. Hence we see $r\in B+P(R)$. Thus we conclude that $R=B+P(R)=B\oplus P(R)$.

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