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# Essentially Purely Indecomposable Groups

by

John IRWIN and Frank OKOH

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**Abstract.** All groups in this paper are abelian. A group $G$ is said to be *essentially indecomposable* if whenever $G=A\dot{+}B$, then either $A$ or $B$ is free of finite rank. A group $G$ is said to be *essentially purely indecomposable* if every pure subgroup of $G$ is essentially indecomposable; $G$ is said to be *purely indecomposable* if every pure subgroup of $G$ is indecomposable. In this note results analogous to Griffith's results on purely indecomposable groups are obtained. It is shown, for instance, that an essentially purely indecomposable group that is torsion-free and reduced is isomorphic to a subgroup of $\prod_{p\in P} J_p \oplus F$, where $J_p$ is the group of $p$-adic numbers, $P$ the set of positive prime numbers, and $F$ is a free group of finite rank.

## § 1. Torsion-free essentially purely indecomposable groups

It is trivial to describe the torsion, mixed, and nonreduced essentially indecomposable abelian groups. So we shall assume in what follows that all groups $G$ are torsion-free and reduced. If $S$ is a subset of $G$, $\langle S\rangle$ and $\langle S\rangle_*$ will respectively denote the smallest subgroup and the smallest pure subgroup $G$ containing $S$. We conveniently regard $\langle 0\rangle$ as free of rank 0.

PROPOSITION 1. *If $G$ is essentially purely indecomposable then $G=G_1\dot{+}F$, where $F$ is free and $G_1$ is indecomposable.*

*Proof.* Suppose $G=A_0=A_1\dot{+}F_1$, where $F_1$ is free of finite rank. Suppose at the $k^{\text{th}}$ step we get $A_k=A_{k+1}\dot{+}F_{k+1}$, where $F_{k+1}$ is free of finite rank. If $A_{k+1}$ is indecomposable then let $F=F_{k+1}\dot{+}F_k\dot{+}\cdots\dot{+}F_1$ and let $G_1=A_{k+1}$. If for each $k=0, 1, 2, 3, \cdots, A_k$ is not indecomposable then $F=\bigcup_{k=1}^{\infty} F_k$ is a pure subgroup $G$ of infinite rank. By Pontryagin's criterion, [1, Theorem 19.1] $F$ is free. This contradicts the hypothesis that $G$ is essentially purely indecomposable. □

PROPOSITION 2. *If $G$ is purely indecomposable then $G\oplus F$ is essentially purely indecomposable for any free group $F$ of finite rank.*

*Proof.* Let $H$ be a pure subgroup of $G\oplus F$. We shall first show that $H=H_1\dot{+}F_1$, where $F_1$ is free of finite rank and $H_1$ has no free direct summand. If $H$ has no free direct summand then set $H_1=H$ and $F_1=\langle 0\rangle$. Suppose $H=H_1\oplus F_1$, where $F_1$ is free of rank $\geq 1$. We shall see below that $F_1$ must be of finite rank. Suppose that at the $k^{\text{th}}$ step we get that $H_k = H_{k+1} \oplus F_{k+1}$ where $F_{k+1}$ is free of finite rank $\geq 1$. If for each $k = 0, 1, 2, \cdots$, $H_k$ has a free direct summand, then we proceed as in the proof of Proposition 1 to get that $H$, hence $G \oplus F$, has a free pure subgroup, $A = \sum_{i=1}^{\infty} \cdot \langle a_i \rangle$, where $\langle a_i \rangle \cong Z$. Let $a_i = g_i + f_i$, $g_i \in G$, $f_i \in F_i$, $i = 1, 2, 3, \cdots$. Let rank $F = n < \infty$. Therefore for appropriate integers $k_1, k_2, \cdots, k_{n+1}$, we get that $b_1 = \sum_{i=1}^{n+1} k_i a_i = \sum_{i=1}^{n+1} k_i g_i$. Since $S = \{a_i\}_{i=1}^{\infty}$ is a basis for $A$, $b_1 \neq 0$. By partitioning $S$ into a denumerable set $\{S_i\}_{i=1}^{\infty}$ with card $S_i = n+1$ for all $i$, we get, in the same manner as we got $b_1$, a new set of $Z$-linearly independent elements $\{b_i\}_{i=1}^{\infty}$, $b_i \in G$. One readily sees that $\sum_{i=1}^{\infty} \cdot \langle b_i \rangle_*$ is a pure subgroup of $G$. This contradicts the hypothesis that $G$ is purely decomposable. Therefore every pure subgroup $H$ of $G \oplus F$ is of the form $H = H_1 \dot{+} F_1$ where $F_1$ is free of finite rank and $H_1$ has no free direct summand.

We now show that $H$ is essentially indecomposable. Suppose $H = A \dot{+} B$. Then $A = A_1 \dot{+} F_1$, $B = B_1 \dot{+} F_2$ where $F_1$ and $F_2$ are free of finite rank and neither $A_1$ nor $B_1$ has a free direct summand. Hence $A_1 \dot{+} B_1$ has no free direct summand. Hence $A_1 \oplus B_1$ has no nonzero component in $F$, i.e., $A_1 \dot{+} B_1 \subset G$. Since $G$ is purely indecomposable, either $A_1$ or $B_1$ is 0. Therefore either $A$ or $B$ is free of finite rank. This proves that $G \oplus F$ is essentially purely indecomposable. □

*Remark.* There are essentially purely indecomposable groups that are not of the form $G \oplus F$, where $G$ is purely indecomposable and $F$ is free, e.g., any indecomposable group of finite rank $\geq 3$ with the property that every pure proper subgroup is free. See [1, Section 88] for examples of such groups.

PROPOSITION 3. *Every essentially purely indecomposable group $G$ is isomorphic to a subgroup of $\prod_{p \in P} J_p \oplus F$, where $F$ is free of finite rank.*

*Proof.* By Proposition 1 we may assume that $G$ is indecomposable. The proof that $G$ embeds in $\prod_{p \in P} J_p$ is essentially the same as the proof in [1, Theorem 88.5] for purely indecomposable groups.

Since $G$ is reduced, it follows from [1, Theorem 39.5] that $G$ is pure in its $Z$-adic completion, $\hat{G}$. Moreover, $\hat{G} = \prod_{p \in P} \hat{G}_p$ where $\hat{G}_p$ is a module over $Q_p^*$—the ring of $p$-adic integers, by [1, Theorem 40.1]. By Corollary 1 of Theorem 22 in [3] every reduced $Q_p^*$-module of rank one is isomorphic to $J_p$. Also by Lemma 17 of [3], $\hat{G}$ is a reduced $Q_p^*$-module.

Now let $0 \neq g \in G$. Then $g = \{g_p\}_{p \in P}$, $g_p \in \hat{G}_p$. If $g_p \neq 0$, then $g_p \in \langle g_p \rangle_* \cong J_p$. Since $J_p$ is algebraically compact, $\langle g_p \rangle_*$ is a direct summand of $\hat{G}_p$. Hence $g \in C = \prod_{p \in P} \langle g_p \rangle_*$, where $\langle g_p \rangle_* = 0$ if $g_p = 0$. Since $C$ is a direct summand of $\hat{G}$, $C \cap G$ and $D \cap G$ are pure in $\hat{G}$, hence in $G$, where $D$ is a direct complement of $C$ in $\hat{G}$. Hence $H = (C \cap G) \dot{+} (D \cap G)$ is pure in $G$. Therefore either $C \cap G$ or $D \cap G$ is free of finite rank. Suppose $C \cap G$ is free. Since $0 \neq g \in C \cap G$, the projection map

$$\rho_C : \hat{G} \to C$$

maps $G$ onto a nonzero free subgroup of $C \cap G$, thereby implying that $G$ has a free direct summand. Therefore $C \cap G$ is not free. So $D \cap G$ must be free. A similar argument with $\rho_D$ replacing $\rho_C$ implies that $D \cap G = 0$. Hence $\rho_D$ maps $G$ isomorphically into $C$. $\square$

From Theorem in Section 4 of [2] and Proposition 3 above we deduce.

PROPOSITION 4. *The collection of all nonisomorphic essentially purely indecomposable groups has cardinality $2^c$, c the cardinality of the continuum.*

The next result is analogous to the theorem in Section 3 of [2] and at a key point we borrow from [2].

PROPOSITION 5. *A pure subgroup $H$ of $\prod_{p \in P} J_p$ is essentially purely indecomposable if and only if every nonzero endomorphism of pure subgroups of $H$ has a kernel that is free of finite rank.*

*Proof.* Suppose $H$ is not essentially purely indecomposable. Then there is a pure subgroup $A = A_1 \dot{+} A_2$ of $H$, where neither $A_1$ nor $A_2$ is free. Then $pr_{A_1}: A \to A$ has a kernel that is not free.

Suppose $H$ is essentially purely indecomposable. Let $A$ be a pure subgroup of $H$. By Proposition 1, $A = A_1 \dot{+} F$, where $F$ is free of finite rank and $A_1$ is indecomposable. This implies that $A_1$ is fully invariant in $A$. Let $0 \neq \varphi \in End\,A$. Consider $\varphi$ as being in $End\,A_1$. Let $K_1 = Ker\,\varphi$ and let $K_2 = \langle Im\,\varphi \rangle_*$ where $Ker\,\varphi$ and $Im\,\varphi$ are the kernel of $\varphi$ and the image of $\varphi$ respectively. The argument in the proof of THEOREM in Section 3 of [2] gives that $K_1 \cap K_2 = 0$ and $K_1 \dot{+} K_2$ is a pure subgroup of $H$. Since $A_1$ has no free direct summand, $Im\,\varphi$, hence $K_2$, is not free. Therefore $K_1$ is free of finite rank. Now reconsider $\varphi$ in $End\,A$. Since $A_1$ is fully invariant, $\varphi = \varphi_1 + \varphi_2$ where $\varphi_1 \in End\,A_1$ and $\varphi_2 \in Hom(F, A)$. Moreover, $Ker\,\varphi = Ker\,\varphi_1 \dot{+} Ker\,\varphi_2$. Since $K_1 = Ker\,\varphi_1$ is free and $F$ is free and both $K_1$ and $F$ are of finite rank, $Ker\,\varphi$ is free of finite rank. $\square$

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Wayne State University
Detroit, Michigan 48202
U.S.A.