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P-Lattices as Ideal Lattices and Submodule Lattices

by

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Throughout, all rings are assumed commutative with 1, all modules are assumed unitary. We denote the least and greatest elements of a lattice L by 0_L and I_L, respectively (or simply by 0 and I if no confusion will result). If $\mathscr{L}$ is a multiplicative lattice, we assume that $I_{\mathscr{L}}$ is a compact multiplicative identity and that multiplication is commutative. All lattices are assumed complete (though not necessarily modular).

Principal elements, as we use the term here, were introduced into the study of multiplicative lattices by R. P. Dilworth. Principally generated multiplicative lattices (r-lattices as they have come to be called) provide a rich framework in which to describe, for example, the lattice of homogeneous ideals of a graded ring, or the lattice of regular ideals of a Noetherian ring. However, there is a much older idea of a principal element in a multiplicative lattice, namely an element E satisfying M. Ward's Postulate D): $B \leq E$ implies $B = EC$ for some C [W1]. Multiplicative lattices satisfying D) for all E were called *P-lattices* or principal element lattices by Ward [W1]. We use the term *principal* in the newer sense of Dilworth [D] and call a lattice in which every element is principal a *PE-lattice*. We call a lattice satisfying the ascending chain condition *Noetherian*, and (following Dilworth) we call a principally generated, modular, Noetherian, multiplicative lattice a *Noether lattice*.

P-lattices have also been studied by McCarthy [Mc] and Janowitz [Ja]. McCarthy has shown that a Noether lattice is a P-lattice iff it is a PE-lattice. Janowitz has shown that a principally generated P-lattice is a PE-lattice iff it is Noether [Ja]. (McCarthy and Janowitz used the term *M-lattice* where we have used P-lattice.)

Elements satisfying property D) are also called weak-meet-principal. A *meet-principal element* is an element E satisfying the identity i) $AE \wedge B = (A \wedge (B : E))E$. *Weak-meet-principal elements* are defined by the weaker form of the identity obtained by setting $A = I$ in i). Elements satisfying the dual identity ii) $(A : E) \vee B = (A \vee BE) : E$ are said to be *join-principal*. Elements satisfying the weaker form of ii) obtained by setting $A = 0$ are said to be *weak-join-principal*. Elements which are both join-principal and meet-principal are said to be *principal*. (This is the definition of Dilworth [D] referred to above). Elements which are both weak-join-principal and weak-meet-principal are said to be *weak-principal*. Adopting the terminology of ideal theory, we say that a multiplicative lattice $\mathscr{L}$ is *h-local* if no non zero element is contained in an infinite number of maximal elements.

In this paper, we consider multiplicative lattices in which the primes are join-principally generated and weak-meet-principal. We show that such a lattice is a P-lattice, and Noether iff (for example) $\mathscr{L}$ is h-local (Theorem 1.5). We also consider similar problems in (fake) modules [A1]. We show that a (fake) module satisfying similar hypotheses over a Noether lattice is the lattice of submodules of a Noetherian module over a Noetherian ring (Theorem 2.8). The corollary that such (real) modules are necessarily Noetherian is new.

Section 1. Multiplicative lattices

PROPOSITION 1.1. *Let $\mathscr{L}$ be a P-lattice. Then $\mathscr{L}$ is completely (meet) distributive and any weak-join-principal element E is both compact and principal. If $\mathscr{L}$ is weak-join-principally generated, then $\mathscr{L}$ is Noether.*

Proof. $\mathscr{L}$ is completely distributive by [Ja, Theorem 2]. If $E \in \mathscr{L}$ is weak-join-principal, then E is weak principal. Since I is compact and $\mathscr{L}$ is completely distributive, weak-principal elements are join inaccessible. Hence it suffices to show that weak-join-principal elements are principal. Since $\mathscr{L}$ is modular, this follows from [Bo, Theorem 1], for example.

It follows from Proposition 1.1 that a weak-join-principally generated P-lattice is generated by a multiplicatively closed set of compact elements (the principal elements, as it turns out). In general, we call a multiplicative lattice $\mathscr{L}$ a $\mathscr{C}$-lattice if it is generated under joins by elements of a multiplicatively closed set $\mathscr{C}$ of compact elements. In any $\mathscr{C}$-lattice, one can localize at any multiplicatively closed subset S of $\mathscr{C}$ by taking $A_S = \bigvee \{B \in \mathscr{L} \mid BX \leq A \text{ for some } X \in S\}$ and $\mathscr{L}_S = \{A_S \mid A \in \mathscr{L}\}$. $\mathscr{L}_S$ is then a subposet of $\mathscr{L}$. The inf is the same. The product and sup are given by $A \cdot_S B = (AB)_S$ and $A \vee_S B = (A \vee B)_S$. It is assumed in the similar localization used in [J-S] that the elements of S are principal, but the properties listed there hold without this assumption. As usual, if P is a prime and is $S = \{E \in \mathscr{C} \mid E \nleq P\}$, we denote A_S by A_P. We observe in particular that $A_{\mathfrak{M}} = B_{\mathfrak{M}}$ for all maximal $\mathfrak{M}$ implies $A = B$.

We note that the product of compact elements in a $\mathscr{C}$ lattice is compact, so that a $\mathscr{C}$-lattice is a K-lattice in the sense of [N-A]. However we feel that the $\mathscr{C}$-lattice point of view is the natural one.

LEMMA 1.2. *Assume that the primes of $\mathscr{L}$ are weak-meet-principal. If Q is a weak-meet-principal element containing a power of a maximal prime $\mathfrak{M}$, then $\mathfrak{M}Q$ is weak-meet-principal. In particular, the powers of $\mathfrak{M}$ are weak-meet-principal.*

Proof. Assume $A \leq \mathfrak{M}Q$. Then $A \leq Q$, so $A = (A:Q)Q$. If $\mathfrak{M}Q = Q$ then $\mathfrak{M}Q$ is weak-meet-principal. Otherwise, $\mathfrak{M}Q$ is $\mathfrak{M}$-primary, so $(A:Q) \leq \mathfrak{M}$. Hence $A:Q = ((A:Q):\mathfrak{M})\mathfrak{M}$ and $A = (A:Q)Q = ((A:Q):\mathfrak{M})\mathfrak{M}Q = (A:\mathfrak{M}Q)\mathfrak{M}Q$.

LEMMA 1:3. *Let $\mathscr{L}$ be a $\mathscr{C}$-lattice with join-principally generated weak-meet-*

principal primes. Then $\mathscr{L}$ is completely distributive and locally a PE-lattice satisfying $\dim(\mathscr{L}) \leq 1$.

Proof. Fix $\mathfrak{M}$ maximal in $\mathscr{L}$ and set $P = \bigwedge_n \mathfrak{M}^n$. Assume $\mathfrak{M}^n = \mathfrak{M}^{n+1}$. Then P is weak-meet-principal (Lemma 1.2). If $E \leq P$ is join-principal, then $E = KP = K\mathfrak{M}P = \mathfrak{M}E$ (for some K), so $I = \mathfrak{M} \vee (0:E)$. It follows that $I = 0_{\mathfrak{M}} : E_{\mathfrak{M}}$, and hence that $E_{\mathfrak{M}} = 0_{\mathfrak{M}}$. Since the product of join-principal elements is join-principal, $P = \mathfrak{M}^n$ is join-principally generated. Since $(\mathfrak{M}^n)_{\mathfrak{M}} = \mathfrak{M}^n$, it follows that $P = P_{\mathfrak{M}} = 0_{\mathfrak{M}}$.

Assume $r \neq s$ implies $\mathfrak{M}^r \neq \mathfrak{M}^s$. If $AB \leq P$, $A \nleq P$ and $B \nleq P$, then we can write $A = C\mathfrak{M}^r$ and $B = D\mathfrak{M}^s$ with $C \nleq \mathfrak{M}$ and $D \nleq \mathfrak{M}$. Then $AB = CD\mathfrak{M}^{r+s} \leq \mathfrak{M}^{r+s+1}$, which is primary, and $\mathfrak{M}^{r+s} \nleq \mathfrak{M}^{r+s+1}$. But then $CD \leq \mathfrak{M}$. It follows that P is prime and therefore join-principally generated. As above, if $E \leq P$ is join-principal, then $I = 0_{\mathfrak{M}} : E_{\mathfrak{M}}$, so $P = P_{\mathfrak{M}} = 0_{\mathfrak{M}}$.

In either of the two cases considered, every non-zero element of $\mathscr{L}_{\mathfrak{M}}$ is a power of $\mathfrak{M}$: if $A \leq \mathfrak{M}^r$ and $A \nleq \mathfrak{M}^{r+1}$, then $A = (A : \mathfrak{M}^r)\mathfrak{M}^r \nleq \mathfrak{M}^{r+1}$, so $A : \mathfrak{M}^r \nleq \mathfrak{M}$, so $A = \mathfrak{M}^r$. Hence, $\mathscr{L}_{\mathfrak{M}}$ is a principal element lattice. It follows that $\mathscr{L}$ is locally distributive, and hence completely distributive, since $\mathscr{L}$ is a $\mathscr{C}$-lattice. $\operatorname{Dim}(\mathscr{L}) \leq 1$ is clear.

THEOREM 1.4. *Let $\mathscr{L}$ be a $\mathscr{C}$-lattice with join-principally generated weak-meet-principal primes. Then $\mathscr{L}$ is a P-lattice.*

Proof. In view of Lemma 1.3, the proof of Mott's theorem on multiplication rings [M, Theorem] carries over with relatively minor changes. The details are lengthy and we omit them. However, we note that for every element $A \in \mathscr{L}$, $A = \bigwedge A_P$, where P runs through the primes minimal over A. Also, for every prime P minimal over A, A_P is a power of P.

THEOREM 1.5. *Let $\mathscr{L}$ be a $\mathscr{C}$-lattice with join-principally generated weak-meet-principal primes. Then $\mathscr{L}$ is an r-lattice and the following are equivalent*:

1. *$\mathscr{L}$ is a Noether PE-lattice.*
2. *$\mathscr{L}$ is Noetherian.*
3. *The minimal primes of $\mathscr{L}$ are compact.*
4. *The set of minimal primes of $\mathscr{L}$ is finite.*
5. *$\mathscr{L}$ is h-local.*

Proof. By hypothesis, $\mathscr{L}$ is compactly generated. If E is compact, then E satisfies the principal identities i) and ii) locally and therefore globally. It follows that $\mathscr{L}$ is an r-lattice. 2) implies 1) by Theorem 1.4 and [Mc, Theorem 1].

Now assume the minimal primes of $\mathscr{L}$ are all compact. If $\mathfrak{M}$ is any non-minimal prime of $\mathscr{L}$, and if $\mathscr{J} = \mathscr{R}(\mathfrak{M})$ is the collection of all residuals $E : \mathfrak{M}$, with $E \leq \mathfrak{M}$ and E compact, and if J is the join of $\mathscr{R}$, then $J\mathfrak{M} = \mathfrak{M}$. But if $\mathfrak{M}^2 = \mathfrak{M}$, then $\mathfrak{M} = 0_{\mathfrak{M}}$, in contradiction to the non-minimality of $\mathfrak{M}$ as a prime. Hence $J = I$. But then since I is compact it follows that $\mathscr{J}$ has a finite subset $\mathscr{F}$ with $I = \bigvee_{E \in \mathscr{F}} E : \mathfrak{M}$, and hence that $\mathfrak{M} = \mathfrak{M}I = \bigvee_{E \in \mathscr{F}} \mathfrak{M}(E : \mathfrak{M}) = \bigvee_{E \in \mathscr{F}} E$ is the finite join of compact elements. Hence, all primes are compact. Hence 3) implies 2) by Cohen's Theorem for r-lattices [A1, Theorem 2.5].

Assume the set of minimal primes of $\mathscr{L}$ is finite. By passage to a direct factor if necessary, we may assume that $\mathscr{L}$ is indecomposable. From Lemma 1.3 it follows that $0 = \bigwedge 0_{\mathfrak{M}}$ ($\mathfrak{M}$ maximal) $= \bigwedge 0_P$ (P minimal) and that the components 0_P are pairwise comaximal. It follows that $\mathscr{L}$ has only one minimal prime. But if P is the only minimal prime of $\mathscr{L}$, then by Lemma 1.3, either P is maximal or $P_{\mathfrak{M}} = 0_{\mathfrak{M}}$, for every maximal $\mathfrak{M}$. In the first case, $\mathscr{L}$ is local so P is compact by Lemma 1.3. In the second case, $P = 0$ so is trivially compact. Hence, 4) implies 3).

Since $\mathscr{L}$ is a $\mathscr{C}$-lattice, it is immediate that h-local and locally Noether imply Noether. That the number of primes minimal over 0 is finite in a Noether lattice was shown in [D]. Hence 5) implies 4).

The implication 1) implies 5) is clear: such lattices are ideal lattices of principal ideal rings [J-L1].

Section 2. (Fake) modules

Let $\mathscr{L}$ be a multiplicative lattice and let L be a (complete) lattice on which $\mathscr{L}$ acts. If the action is reasonable, then it is natural to think of L as a module of sorts over $\mathscr{L}$. In particular, we assume of a (fake) module that $I_{\mathscr{L}}N = N$, $0_{\mathscr{L}}N = 0_L$ and $(\bigvee_\alpha A_\alpha)(\bigvee_\beta N_\beta) = \bigvee_{\alpha,\beta} A_\alpha N_\beta$ are identities.

If $\mathscr{L}$ is $\mathscr{C}$-lattice and L is generated under joins by a set $\mathscr{C}'$ of compact elements closed under multiplication by elements of $\mathscr{C}$, we will say that L is a $(\mathscr{C}, \mathscr{C}')$-module over $\mathscr{L}$. $(\mathscr{C}, \mathscr{C}')$-modules have a localization procedure that works as one would hope: If S is a multiplicatively closed subset of $\mathscr{C}$ and $N \in L$, then $N_S = \bigvee \{T \in L \mid XT \leq N$, for some $X \in S\}$ and $L_S = \{N_S \mid N \in L\}$. L_S is a subposet of L with inf and sup and product satisfying properties similar to those satisfied in $\mathscr{L}_S$.

We will call a (fake) module L a *P-module* if $N \leq N'$ implies $N = AN'$ for some $A \in \mathscr{L}$ (i.e., if every element of L is weak-meet-principal). We call an element $P \in L$ *prime* if $AN \leq P$ implies $N \leq P$ or $AI_L \leq P$. In particular, I_L is prime.

We assume from now on that $\mathscr{L}$ is an r-lattice and that L is a $(\mathscr{C}, \mathscr{C}')$-module over $\mathscr{L}$. We call a module L a Noether module if it is principally generated, modular and Noetherian.

LEMMA 2.1. *If $(\mathscr{L}, \mathfrak{M})$ is quasi-local and $F \in L$ is weak-meet-principal and not completely join irreducible, then $\mathfrak{M}F = F$.*

Proof. Assume F is the join of elements $F_\alpha < F$. Since F is multiplication, it is immediate that $F_\alpha \leq \mathfrak{M}F$ for all α, and hence that $F \leq \mathfrak{M}F$.

LEMMA 2.2. *Let F be a weak-meet-principal element of L which is generated by compact, join-principal elements. Then F is locally weak-principal. If F is compact, then F is weak-principal.*

Proof. An element $F \in L$ is join-principal over $\mathscr{L}$ iff it is join-principal over $\mathscr{L}/(0_L : F)$. Similarly, F is join-principal as an element of L iff it is join-principal as an element of the submodule $[0, F]$. Also, if F is compact, F is join-principal iff F is locally join-principal. Hence it suffices to consider the case $F=I_L$ with $0_L:I_L=0_{\mathscr{L}}$, $\mathscr{L}$ quasi-local with unique maximal element $\mathfrak{M}$ and $I_L=\bigvee_\alpha E_\alpha$ where each E_α is compact and join-principal.

If I_L is completely join-irreducible, then $I_L=E_\alpha$, for some α, so I_L is weak-principal.

On the other hand, if I_L is not completely join-irreducible, then $\mathfrak{M}I_L=I_L$ (Lemma 2.1). Since each E_α is a multiple of I_L, also $\mathfrak{M}E_\alpha=E_\alpha$, for all α. Then $\mathfrak{M}\vee(0_L:E_\alpha)=I_{\mathscr{L}}$, so $E_\alpha=0_L$, for all α. It follows that $I_L=0_L$, so I_L is principal.

LEMMA 2.3. *Assume I_L is the join of compact, join-principal elements and that the primes of L are weak-meet-principal. Then L is a completely distributive P-module and locally a PE-module.*

Proof. We may assume $0_L:I_L=0_{\mathscr{L}}$. Fix $\mathfrak{M}$ maximal in $\mathscr{L}$. By Lemma 2.2, $I_{L_\mathfrak{M}}$ is weak principal over $\mathscr{L}_\mathfrak{M}$. It is easy to see that the primes of $L_\mathfrak{M}$ are primes of L and therefore weak-meet-principal. Set $\bar{\mathscr{L}}=\mathscr{L}_\mathfrak{M}/(0_{L_\mathfrak{M}}:I_{L_\mathfrak{M}})$. Then the map $A\to AI_{L_\mathfrak{M}}$ is an $\bar{\mathscr{L}}$-isomorphism of $\bar{\mathscr{L}}$ onto $L_\mathfrak{M}$. The conclusion now follows from Lemma 1.3 and Theorem 1.4.

LEMMA 2.4. *Assume $P\in\mathscr{L}$ is prime and $E\in\mathscr{L}$ is principal. If I_L is weak-meet-principal and generated by compact join-principal elements and if $0_L:I_L=0_{\mathscr{L}}$, then $EI_L\le PI_L$ implies $E\le P$ or $I_L=PI_L$.*

Proof. Let $\mathscr{J}$ be the collection of join-principal elements of L. We first consider the case $E\le\bigvee_{F\in\mathscr{J}}F:I_L$. Since E is compact, it follows that $E\le\bigvee_{F\in\mathscr{F}}F:I_L$ for some finite subset $\mathscr{F}$ of $\mathscr{J}$, say $\mathscr{F}=\{F_1,\cdots,F_n\}$. Then $E\le F:I_L$, where $F=F_1\vee\cdots\vee F_n$. From $EI_L\le PI_L$ we get $E(F_i:I_L)I_L\le P(F_i:I_L)I_L$, and hence $EF_i\le PF_i$, for $i=1,\cdots,n$. Since each F_i is join-principal, $E\le P\vee(0:F_i)$, $i=1,\cdots,n$, and hence $E\le\bigwedge_i(P\vee(0:F_i))=P\vee\bigwedge_i(0:F_i)=P\vee(0:\bigvee_iF_i)=P\vee(0_L:F)$.

From $EI_L\le F$ we get $0_L:F\le 0_L:EI_L\le(0_L:I_L):E=0_{\mathscr{L}}:E$ (since $0_L:I_L=0_{\mathscr{L}}$). Now, $E(0:E)\le P$ implies $E\le P$ or $0:E\le P$. In the latter case, $E\le P\vee(0:F)\le P\vee(0:E)\le P$.

On the other hand, if $E\not\le\bigvee_{F\in\mathscr{J}}F:I_L$ and $E'\le\bigvee_{F\in\mathscr{J}}F:I_L$ is principal, then $EE'I_L\le PI_L$, so by the above, $EE'\le P$. If $E\not\le P$, then it follows that $E'\le P$ for all $E'\le\Theta(I_L)$, and hence that $\bigvee_{F\in\mathscr{J}}F:I_L\le P$. Since $I_L=(\bigvee_{F\in\mathscr{J}}F:I_L)I_L$, it follows that $I_L\le PI_L$.

LEMMA 2.5. *Assume I_L is weak-meet-principal and generated by compact join-principal elements. If $P\in\mathscr{L}$ is prime and $0_L:I_L=0_{\mathscr{L}}$, then PI_L is prime.*

Proof. Assume $AN\le PI_L$ and $AI_L\not\le PI_L$. Then $A\not\le P$. Fix E principal, $E\le A$, $E\not\le P$. Then $EN\le PI_L$. Set $N=BI_L$ and let $E'\le B$ be principal. Then $E(E'I_L)=(EE')I_L\le PI_L$. Assume $PI_L\ne I_L$. Then by Lemma 2.4, $EE'\le P$, whence $E'\le P$, by the choice of E. Since $E'\le B$ is arbitrary, it follows that $B\le P$, so $N=BI_L\le PI_L$.

LEMMA 2.6. *Assume I_L is generated by compact join-principal elements and that the primes of L are weak-meet-principal. If L has only a finite number of minimal primes, then I_L is compact.*

Proof. Let K be the inf of the minimal primes of L. We first show that $I_{L/K}$ is compact. For notational simplicity, we assume $K=0$. Since L is distributive, it suffices to show that $I_{(L/P)}$ is compact for every prime P of L. Hence, we may assume 0_L is prime, $0_L \neq I_L$. Also, by passing to $\mathscr{L}/(0_L : I_L)$ if necessary, we may assume $0_L : I_L = 0_{\mathscr{L}}$, and hence that $0_{\mathscr{L}}$ is prime.

Fix $\mathfrak{M}$ maximal in $\mathscr{L}$. Then $I_{L_{\mathfrak{M}}}$ is principal. Hence $(NI_L : I_L)_{\mathfrak{M}} \leq N_{\mathfrak{M}} I_{L_{\mathfrak{M}}} : I_{L_{\mathfrak{M}}} \leq N_{\mathfrak{M}} \vee (0_{L_{\mathfrak{M}}} : I_{L_{\mathfrak{M}}})$. Since $0_{L_{\mathfrak{M}}} = 0_L$ is prime, it follows that $(NI_L : I_L)_{\mathfrak{M}} \leq N_{\mathfrak{M}}$. Since this is so for every maximal element $\mathfrak{M}$, $NI_L : I_L \leq N$. It follows that I_L is weak-join-principal. Since I_L is weak-meet-principal, it follows that the map $A \to AI_L$ of $\mathscr{L}$ to L is an isomorphism, and hence that I_L is compact. Hence $I_{L/K}$ is compact.

By the preceding paragraph, $I_{L/K}$ is compact, so $I_L = C \vee K$ for some compact $C \in L$. If $P \in L$ is prime, then $P = (P : I_L)I_L$, and $P : I_L$ is prime in $\mathscr{L}$. Hence $\mathfrak{M}I_L \geq K$ for every maximal element $\mathfrak{M} \in \mathscr{L}$. It follows that $I_L = C \vee \mathfrak{M}I_L$ and hence that $I_{L_{\mathfrak{M}}} = C_{\mathfrak{M}} \vee \mathfrak{M}I_{L_{\mathfrak{M}}}$ in $L_{\mathfrak{M}}$. But $I_{L_{\mathfrak{M}}} = I_L$ is principal in $L_{\mathfrak{M}}$, so $\mathfrak{M} \vee (C_{\mathfrak{M}} : I_{L_{\mathfrak{M}}}) = I_{L_{\mathfrak{M}}}$ and therefore $C_{\mathfrak{M}} = I_{L_{\mathfrak{M}}} = I_L$, for every maximal element $\mathfrak{M}$. It follows that $I_L = C$, and hence that I_L is compact.

THEOREM 2.7. *Let L be a $(\mathscr{C}, \mathscr{C}')$-module over an r-lattice $\mathscr{L}$. Assume that I_L is the join of compact, join-principal-elements. If L has only a finite number of minimal primes, and the primes of L are weak-meet-principal then L is a PE-module. In particular, L is Noether.*

Proof. By passage to $\mathscr{L}/(0_L : I_L)$, we may assume that $0_L : I_L = 0_{\mathscr{L}}$. By Lemma 2.3 and Lemma 2.6, I_L is principal. Hence, the map $A \to AI_L$ of $\mathscr{L}$ to L is an isomorphism. It follows from Theorem 1.5 that L is a (Noether) *PE*-module.

THEOREM 2.8. *Let L be a $(\mathscr{C}, \mathscr{C}')$-module over a Noether lattice $\mathscr{L}$. Assume that I_L is generated by join-principal-elements and that the primes of L are weak-meet-principal. If $0_L : I_L = 0_{\mathscr{L}}$, then there exist a Noetherian ring R and a Noetherian module M over R with isomorphisms $\rho : \mathscr{L} \to \mathscr{L}(R)$ and $\tau : L \to \mathscr{L}_R(M)$ with $\tau(AN) = \rho(A)\tau(N)$, for all $A \in \mathscr{L}$, $N \in L$.*

Proof. As in the proof of Theorem 2.7, $L \cong \mathscr{L}$, so $\mathscr{L}$ is a Noether principal element lattice. It follows [J-L1, Theorem 5] that $\mathscr{L}$ is the lattice of ideals of a Noetherian ring. Since $L \cong \mathscr{L}$, the remainder of the result follows easily.

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