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On the Complex Oscillation of Solutions of Non-homogeneous Linear Differential Equations with Polynomial Coefficients*

by

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Abstract. In this paper, we investigate the complex oscillation of solutions of non-homogeneous linear differential equations with polynomial coefficients, and obtain some results, and in some cases obtain estimates of the exponent of convergence of the zero-sequence of solutions. Theorem 2 and Theorem 4 are the main results among the Theorems in this paper.

1. Statement of the results

Various authors have produced works on the complex oscillation of solutions of second order homogeneous linear differential equations. In this paper, we investigate the complex oscillation of solutions of the following non-homogeneous linear differential equations:

$$f^{(k)}+a_{k-1}f^{(k-1)}+a_{k-2}f^{(k-2)}+\cdots+a_0f=F\,, \tag{1}$$

where $a_0, a_1, \cdots, a_{k-1}$ are polynomials, F is an entire function. To state our results, we need the following:

DEFINITION. If the entire function $g(z)$ has infinitely many zeros, we call $g(z)$ is oscillatory.

NOTATIONS. Denote the exponent of convergence of the zero-sequence of $g(z)$ by $\lambda(g)$, the exponent of convergence of the sequence of distinct zeros of $g(z)$ by $\bar{\lambda}(g)$, and the order of growth of $g(z)$ by $\sigma(g)$. In addition, we shall use the standard notations of the Nevanlinna theory (see [3]).

REMARK. All solutions of (1) are entire (see Lamma 2 in Section 2).

THEOREM 1. *Let F be a polynomial and $F\not\equiv 0$. Then:*
(a) *For every solution f of* (1) *with $k\geqslant 1$, $\lambda(f)=\bar{\lambda}(f)=\sigma(f)$.*
(b) *For every transcendental solution f of equation*

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$$f^{(k)}+a_0f=F\,, \tag{2}$$

where $k\geqslant 1$, a_0 *is a polynomial with* $\deg a_0=n$, $\lambda(f)=\bar{\lambda}(f)=\sigma(f)=(n+k)/k$. *Thus, f is oscillatory.*

If F is transcendental, the Theorem 1(b) does not hold. For example, $f=ze^z$ solves $f''+zf=(z^2+z+2)e^z$. But we have the following Theorem 2.

THEOREM 2. *Let F be a transcendental entire function with $\sigma(F)<+\infty$. Then for every solution f of* (1) *with* $k\geqslant 1$,

(a) *If F is oscillatory, then f is also oscillatory.*

(b) $\lambda(f)\geqslant\lambda(F)$.

(c) *If $\sigma(F)$ is not a positive integer, then* $\lambda(f)=\sigma(f)\geqslant\sigma(F)=\lambda(F)$.

(d) *If* $\sigma(f)>\sigma(F)$, *then* $\lambda(f)=\sigma(f)>\sigma(F)$.

The following Theorem 3 shows that the solution of (1) may be oscillatory although F with $\sigma(F)<+\infty$ is not oscillatory.

THEOREM 3. *For the equation*

$$f^{(k)}+a_{k-2}f^{(k-2)}+\cdots+a_0f=F\,, \tag{3}$$

where $k\geqslant 2$, $a_0,\cdots,a_{k-2}$ *are polynomials. If* $F=e^{P_0}$, P_0 *is a polynomial with*

$$\deg P_0>1+\max_{j=0,\cdots,k-2}\frac{\deg a_j}{k-j}\,,$$

Then every solution f of (3) *is oscillatory. In addition,* $\lambda(f)=\bar{\lambda}(f)$.

If

$$\deg P_0<1+\max_{j=0,\cdots,k-2}\frac{\deg a_j}{k-j}\,,$$

we have the following Theorem 4 which is quite interest.

THEOREM 4. *For the equation*

$$f''+a_0f=P_1e^{P_0}\,, \tag{4}$$

where a_0, P_0, P_1 *are polynomials,* $\deg a_0=n$, $\deg P_0<1+n/2$.

(a) *If* $n\geqslant 1$ *and* $\deg P_1<n$, *then every solution f of* (4) *satisfies* $\lambda(f)=\bar{\lambda}(f)=\sigma(f)=1+n/2>\deg P_0$.

(b) *If* $\deg P_1\geqslant n\geqslant 0$, *then the solution f of* (4) *either satisfies* $\lambda(f)=\bar{\lambda}(f)=\sigma(f)=1+n/2>\deg P_0$, *or is of the form*

$$f=Qe^{P_0}\,,$$

where Q is a polynomial. And if (4) *has a solution of the form Qe^{P_0} with Q polynomial, then* (4) *must have solutions which satisfy* $\lambda(f)=\bar{\lambda}(f)=\sigma(f)=1+n/2>\deg P_0$.

From Theorem 3 and Theorem 4(a) we immediately obtain

COROLLARY. *If a_0, P_0 are polynomials, $\deg a_0 = n \geqslant 1$, $\deg P_0 \neq 1 + n/2$, P_1 is a non-zero constant, then every solution of* (4) *is oscillatory.*

In Section 7, we shall give examples to show that the conditions in Theorem 4 and the Corollary are sharp.

It is an open problem whether the higher order equations have the similar properties in Theorem 4.

If F is a transcendental entire function with $\sigma(F) = +\infty$, the Theorem 2(a) does not hold. For example, $f = \exp(e^z)$ solves $f'' + f = (e^{2z} + e^z + 1)\exp(e^z)$. But we have

THEOREM 5. *Let F be a transcendental entire function with $\sigma(F) = +\infty$.*

(a) *If $0 < n(r, 1/F) < K$, where $n(r, 1/F)$ is the number of zeros of $F(z)$ in the disc $|z| < r$, K = constant, then every solution f of* (3), *with $a_0, \cdots, a_{k-2}$ polynomials and $k \geqslant 2$, is oscillatory, and $\lambda(f) = \bar{\lambda}(f)$.*

(b) *If $F(z) \neq 0$ for any $z \in \mathbf{C}$ and $\sigma(F'/F)$ is not a positive integer, then every solution f of* (3), *with $a_0, \cdots, a_{k-2}$ polynomials and $k \geqslant 2$, is oscillatory, and $\lambda(f) = \bar{\lambda}(f)$.*

(c) *If F has finitely many zeros, then every solution f of equation*

$$f^{(k)} + a_{k-3} f^{(k-3)} + \cdots + a_0 f = F, \tag{5}$$

with $a_0, \cdots, a_{k-3}$ polynomials and $k \geqslant 3$, is oscillatory, and $\lambda(f) = \bar{\lambda}(f)$.

The above Theorem 2 and Theorem 4 are the main results in this paper. Theorem 1 is quite simple. Theorem 3 and Theorem 5 are, in fact, the corollaries of an important theorem which belongs to G. Frank and S. Hellerstein (see Theorem A in Section 2).

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2. Preliminary results needed for the proofs of theorems

The following Theorem A is the Theorem 1 in [1].

THEOREM A. *Let*

$$f^{(k)} + a_{k-2} f^{(k-2)} + \cdots + a_0 f = Q e^q,$$

with $a_0, \cdots, a_{k-2}$ polynomials and $k \geqslant 2$, have a solution

$$f = P e^p,$$

where P and Q are polynomials, p and q are entire. Then, either

(i) *p and q are polynomials with $p - q$ constant, or*

(ii) *P and Q are constants,*

$$p(z) = \omega \int e^{h(t)} dt - \frac{1}{2}(k-1)h(z), \qquad \omega^k = 1,$$

and $q-p=kh$, where h is a polynomial determined by a_{k-2} through the equation

$$k(k-1)(k-2)(h'^2-2h'')+24a_{k-2}=0\,.$$

If Q is a constant and p is a polynomial, then

$$\deg p \leqslant 1+\max_{j=0,\cdots,k-2}\frac{\deg a_j}{k-j}\,.$$

The following Lemma 1 belongs to H. Wittich (see [1, Lemma 4]).

LEMMA 1. *Let f be a solution of the differential equation*

$$f^{(k)}+a_{k-1}f^{(k-1)}+\cdots+a_0f=0$$

with $a_0,\cdots,a_{k-1}$ polynomials and $k\geqslant 1$. Then f is entire of order

$$\sigma(f)\leqslant 1+\max_{j=0,\cdots,k-1}\frac{\deg a_j}{k-j}\,,$$

and of mean type.

LEMMA 2. *Let f be a solution of* (1) *with $a_0,\cdots,a_{k-1}$ polynomials, F entire function and $k\geqslant 1$. Then f is entire. And if $\sigma(F)<+\infty$, then $\sigma(f)<+\infty$.*

Proof. Assume $\{f_1,f_2,\cdots,f_k\}$ is a fundamental solution set of

$$f^{(k)}+a_{k-1}f^{(k-1)}+\cdots+a_0f=0\,.$$

By variation of parameters, we can write

$$f=A_1(z)f_1+A_2(z)f_2+\cdots+A_k(z)f_k\,,$$

where $A_1(z),\cdots,A_k(z)$ are determined by

$$\begin{aligned}&A_1'f_1+A_2'f_2+\cdots+A_k'f_k\equiv 0\\&A_1'f_1'+A_2'f_2'+\cdots+A_k'f_k'\equiv 0\\&\cdots\\&A_1'f_1^{(k-1)}+A_2'f_2^{(k-1)}+\cdots+A_k'f_k^{(k-1)}\equiv F\,.\end{aligned}\tag{6}$$

Noting that the Wronskian $W(f_1,f_2,\cdots,f_k)=\exp(-\int a_{k-1}dz)$, we have from (6)

$$A_j'=Fg_j\exp\left(\int a_{k-1}dz\right),\qquad j=1,\cdots,k\,,$$

where g_j are differential polynomials in $f_1,f_2,\cdots,f_k$ with constant coefficients. Thus, by Lemma 1, A_j' are entire, and so is f. And if $\sigma(F)<+\infty$, then, again by Lemma 1, $\sigma(A_j')<+\infty$, and so $\sigma(f)<+\infty$.

The following Lemma 3 is the Lemma 1 in [5].

LEMMA 3. *Suppose that $P=a_nz^n+\cdots+a_0$ is a non-constant polynomial with $\delta(P,\theta)=\mathrm{Re}(a_ne^{in\theta})\neq 0$ on $[a,b]$, and that $H(z)$ is analytic on the sectorial set S given by*

$$|z|\geqslant r_0\,, \qquad \arg z\in[a,\, b]$$

with

$$H'(z)=O(r^M)e^P \tag{7}$$

there. Then in a sectorial set

$$|z|\geqslant r_1\,, \qquad a+\varepsilon<\arg z<b-\varepsilon\,,$$

where ε *may be chosen arbitrarily small and positive, we have*

$$H(z)=O(r^M)e^P \tag{8}$$

if $\delta(P,\,\theta)>0$ *on* $[a,\,b]$, *while if* $\delta(P,\,\theta)<0$ *on* $[a,\,b]$ *we have*

$$H(z)=c+O(r^M)e^P \tag{9}$$

for some constant c. Also if $H'(z)$ *satisfies the stronger estimate*

$$H'(z)=\alpha z^\rho(1+o(1))e^P$$

on S, where ρ *is real and* α *is non-zero, then the term* $O(r^M)$ *in* (8) *and* (9) *may be replaced by*

$$(1+o(1))\alpha z^\rho/P'(z)\,.$$

3. Proof of Theorem 1

Part (a). We can write from (1)

$$\frac{1}{f}=\frac{1}{F}\left(\frac{f^{(k)}}{f}+a_{k-1}\frac{f^{(k-1)}}{f}+\cdots+a_0\right).$$

Thus, by Lemma 2 and (2.4) in Lemma 1 in [1],

$$\begin{aligned} m\left(r,\frac{1}{f}\right)&\leq m\left(r,\frac{1}{F}\right)+m\left(r,\frac{f^{(k)}}{f}\right)+\cdots \\ &=O(\log r)\,. \end{aligned}$$

Therefore

$$\begin{aligned} T(r,f)&=T\left(r,\frac{1}{f}\right)+O(1) \\ &=N\left(r,\frac{1}{f}\right)+m\left(r,\frac{1}{f}\right)+O(1) \\ &=N\left(r,\frac{1}{f}\right)+O(\log r)\,. \end{aligned} \tag{10}$$

In addition, if f has a zero z_0 with order $>k$, then z_0 is a zero of F. But F has only finitely many zeros, so

$$N\left(r,\frac{1}{f}\right)\leq k\bar{N}\left(r,\frac{1}{f}\right)+O(\log r). \tag{11}$$

(10) and (11) give $\sigma(f)=\lambda(f)=\bar{\lambda}(f)$.

Part (b). It follows immediately from Wiman-Valiron theory and Part (a).

4. Proof of Theorem 2

Part (a). From Lemma 2, also $\sigma(f)<+\infty$. If f has only finitely many zeros, then we can write $f=Pe^p$, P and p are polynomials. Substituting it in (1), we find that F has also finitely many zeros. This is a contradiction.

Part (b). Since $\sigma(f)<+\infty$, $\sigma(F)<+\infty$, we can write

$$f=H_1e^{P_1}, \qquad F=H_2e^{P_2},$$

where H_1 and H_2 are canonical products formed respectively with the zeros of f and F, P_1 and P_2 are polynomials. It is easy to see that H_1 also solves the linear differential equation

$$H_1^{(k)}+b_{k-1}H_1^{(k-1)}+\cdots+b_0H_1=H_2e^{P_2-P_1}, \tag{12}$$

where $b_0, \cdots, b_{k-1}$ are also polynomials. From (12), we get

$$\lambda(F)=\sigma(H_2)\leqslant\sigma(H_2e^{P_2-P_1})\leqslant\sigma(H_1)=\lambda(f),$$

and then Part (b) holds.

Part (c). If $\sigma(F)$ is not a positive integer, then $\sigma(F)=\sigma(H_2)>\deg P_2$. Divide the discussion into two cases. Case I. Suppose that $\sigma(f)$ is not a positive integer. In this case, from (1) and the properties of canonical product, $\lambda(f)=\sigma(f)\geqslant\sigma(F)=\lambda(F)$, *and Part (c) holds. Case II. Suppose that* $\sigma(f)$ *is a positive integer.* In this case, if $\sigma(f)>\deg P_1$, then $\lambda(f)=\sigma(H_1)=\sigma(f)\geqslant\sigma(F)=\lambda(F)$, i.e. Part (c) holds. Otherwise, i.e. $\sigma(f)=\deg P_1$, $\deg P_1>\sigma(F)>\deg P_2$. But then from (12) we have

$$\deg P_1\leqslant\sigma(H_2e^{P_2-P_1})\leqslant\sigma(H_1).$$

Thus from $\lambda(f)=\sigma(H_1)\leqslant\sigma(f)=\deg P_1$ we get $\lambda(f)=\sigma(f)$, and Part (c) holds again.

Part (d). The proof is similar to the discussion of Case I and Case II in Part (c).

5. Proof of Theorem 3

Since $\sigma(F)<+\infty$, from Lemma 2 $\sigma(f)<+\infty$. If f has only finitely many zeros, then from Theorem A, f must have the form (i) or (ii) ($h\equiv$constant). But then in Theorem A, q and p are both polynomials, and

$$\deg p = \deg q = \deg P_0 \leq 1 + \max_{j=0,\cdots,k-2} \frac{\deg a_j}{k-j},$$

since $Q \equiv 1$ is a constant. A contradiction is obtained for P_0.

The proof of $\lambda(f) = \bar{\lambda}(f)$ is the same as Theorem 1(a).

6. Proof of Theorem 4

Part (a). We assert that $\sigma(f) \geqslant 1 + n/2$. Assume the contrary, i.e. $\sigma(f) < 1 + n/2$, and set

$$a_0 = d_n z^n + \cdots + d_0,$$

$d_0, \cdots, d_n$ are constants and $d_n \neq 0$. Let θ_j $(\theta_1 < \theta_2 < \cdots < \theta_{2(n+2)+1} = \theta_1 + 4\pi)$ be solutions of

$$\operatorname{Arg}(d_n) + (n+2)\theta = 0 \pmod{2\pi},$$

and take a small $\varepsilon > 0$ and a large r_1.

Note first that the sectors

$$S_j(\varepsilon, r_1) = \{z : |z| > r_1,\ \theta_j + \varepsilon < \arg z < \theta_{j+2} - \varepsilon\}$$

are such that for any samll $\varepsilon > 0$,

$$\{z : |z| > r_1\} \subseteq \bigcup S_j(\varepsilon, r_1).$$

Take such a sector $S_j(\varepsilon, r_1)$, and set

$$\zeta(z) = \int_0^z a_0(t)^{1/2} dt = (1 + o(1)) D_n z^N \tag{13}$$

with $N = 1 + n/2$ and $D_n = \sqrt{d_n} \neq 0$. Now

$$y'' + a_0 y = 0 \tag{14}$$

has solutions (see [4])

$$y_1 = a_0^{-1/4}(1 + o(1)) e^{i\zeta}, \qquad y_2 = a_0^{-1/4}(1 + o(1)) e^{-i\zeta} \tag{15}$$

in $S_j(2\varepsilon, r_1)$. Also the Wronskian $W(y_1, y_2) = c \neq 0$, say. We can now write

$$f = Ay_1 + By_2$$

where, using (4) and variation of parameters,

$$A'y_1 + B'y_2 \equiv 0, \qquad A'y_1' + B'y_2' \equiv P_1 e^{P_0}.$$

This gives

$$A' = -\frac{1}{c} y_2 P_1 e^{P_0}, \qquad B' = \frac{1}{c} y_1 P_1 e^{P_0}.$$

For a suitable point a,

$$f=\frac{1}{c}\left(y_2\int_a^z y_1P_1e^{P_0}dt-y_1\int_a^z y_2P_1e^{P_0}dt\right)+c_1y_1+c_2y_2\,. \tag{16}$$

We estimate (16) in the sectors

$$S_j'(2\varepsilon, r_2)=\{z:\,|z|>r_2,\ \theta_j+2\varepsilon<\arg z<\theta_{j+1}-2\varepsilon\}\,,$$

$$S_j''(2\varepsilon, r_2)=\{z:\,|z|>r_2,\ \theta_{j+1}+2\varepsilon<\arg z<\theta_{j+2}-2\varepsilon\}$$

for some $r_2\geqslant r_1$. Now because of the choice of the θ_j, in $S_j'(2\varepsilon, r_2)$ one of y_1, y_2 is large, say y_1, and the other is small, while in $S_j''(2\varepsilon, r_2)$ y_2 is large and y_1 is small. We can integrate asymptotically using Lemma 3 to get (noting $\deg P_0'<n/2$)

$$\begin{aligned}\int_a^z y_1P_1e^{P_0}dt&=\int_a^z P_1(t)a_0(t)^{-1/4}(1+o(1))e^{i\zeta(t)}e^{P_0(t)}dt\\&=\frac{1+o(1)}{i\zeta'+P_0'}y_1P_1e^{P_0}=\frac{1+o(1)}{i\zeta'(1+o(1))}y_1P_1e^{P_0}=\frac{1+o(1)}{i\zeta'}y_1P_1e^{P_0}\end{aligned} \tag{17}$$

and similarly

$$\int_a^z y_2P_1e^{P_0}dt=c_3+\frac{1+o(1)}{-i\zeta'+P_0'}y_2P_1e^{P_0}=c_3+\frac{1+o(1)}{-i\zeta'}y_2P_1e^{P_0} \tag{18}$$

(It is true that ζ might not be a polynomial but the proof is the same in Lemma 3). The formulas (17) and (18) hold in $S_j'(3\varepsilon, r_2)$. Now (16), (17), (18) give

$$f=c_4y_1+c_5y_2+\frac{2+o(1)}{i\zeta'}\,\frac{1}{c}\,y_1y_2P_1e^{P_0}=c_4y_1+c_5y_2+\frac{2+o(1)}{ica_0}P_1e^{P_0}$$

in $S_j'(3\varepsilon, r_2)$. Because $\sigma(f)<1+n/2$, we must have $c_4=0$, and so

$$\frac{ica_0}{P_1}e^{-P_0}f-2\to 0\,.$$

Similarly in $S_j''(3\varepsilon, r_2)$ we get

$$f=c_6y_1+c_7y_2+\frac{2+o(1)}{ica_0}P_1e^{P_0}\,,$$

$c_7=0$ and

$$\frac{ica_0}{P_1}e^{-P_0}f-2\to 0\,.$$

Since $\sigma(f)<1+n/2$ and ε can be chosen initially as small as we like, by the Phragme'n-Lindelöf Theorem (see e.g. [2, p. 104]) we get

$$\frac{ica_0}{P_1}e^{-P_0}f-2\to 0$$

(in fact, uniformly) in $S_j(3\varepsilon, r_2)$, or

$$f=\frac{2+o(1)}{ica_0}P_1e^{P_0}.$$

Noting $\{z: |z|>r_2\}\subseteq\bigcup S_j(3\varepsilon, r_2)$, this implies that f has only finitely many zeros in the whole plane. So by Theorem A we can write $f=Qe^{P_0}$, where Q is a polynomial. Substituting it in (4) we get

$$Q''+2Q'P_0'+QP_0''+QP_0'^2+a_0Q\equiv P_1 .$$

This is impossible, since $\deg(Q''+2Q'P_0'+QP_0'^2+QP_0''+a_0Q)=\deg(a_0Q)\geqslant n>\deg P_1$.

So we now know that $\sigma(f)\geqslant 1+n/2$. On the other hand, using the Wiman-Valiron theory in (14) we get $\sigma(y_1)=1+n/2$, $\sigma(y_2)=1+n/2$. Thus from (16) we get $\sigma(f)\leqslant 1+n/2$. So we must have $\sigma(f)=1+n/2$. Since

$$\sigma(f)=1+\frac{n}{2}>\deg P_0=\sigma(P_1e^{P_0}),$$

by Theorem 2(d) we get $\lambda(f)=\sigma(f)=1+n/2>\deg P_0$.

The proof of $\lambda(f)=\bar{\lambda}(f)$ is the same as Theorem 1(a).

Part (b). By checking the proof of Part (a), it is easy to see that the first part of Part (b) holds. Noting that the solution of (4) are of the form

$$f=c_1y_1+c_2y_2+Qe^{P_0}$$

then, where y_1, y_2 are linearly independent solutions of (14) satisfying $\sigma(y_1)=\sigma(y_2)=1+n/2$, the second part of Part (b) is clear.

7. Examples for Theorem 4 and Corollary

Example 1. $f=e^z$ solves $f''+zf=(z+1)e^z$, and $f=ze^z$ solves $f''+z^2f=(z^3+z+2)e^z$. These show that Part (a) of Theorem 4 does not hold if $\deg P_1\geqslant n$ and $\deg P_0<1+n/2$, and also show that there is a solution of form Qe^{P_0} in Part (b) of Theorem 4.

Example 2. $f=\exp(z^3)$ solves $f''-9z^4f=6z\exp(z^3)$, and $f=z\exp(z^2)$ solves $f''-4z^2f=6z\exp(z^2)$. These show that Part (a) of Theorem 4 does not hold if $\deg P_0=1+n/2$ and $\deg P_1<n$.

Example 3. $f=\exp(z^2)$ solves $f''-4z^2f=2\exp(z^2)$. This shows that the Corollary does not hold if $\deg P_0=1+n/2$.

8. Proof of Theorem 5

Part (a) and Part (b). Assume f has only finitely many zeros. Then, from Theorem A, F can only have the form in (ii) of Theorem A. This contradicts the assumptions of F. The proof of $\lambda(f)=\bar{\lambda}(f)$ is the same as Theorem 1(a).

Part (c). Since $a_{k-2}\equiv 0$, h is a constant in Theorem A. If f has only finitely many zeros, then, by Theorem A, $\sigma(F)<+\infty$. This is a contradiction. The proof of $\lambda(f)=\bar{\lambda}(f)$ is the same as Theorem 1(a).

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