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# A Generalization of Shiokawa's Rational Approximations to the Rogers-Ramanujan Continued Fraction

by

S. BHARGAVA, Chandrashekar ADIGA and D. D. SOMASHEKARA

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**Abstract.** In this note we obtain rational approximations to the continued fraction

$$1+\frac{\alpha x}{1+}\,\frac{\beta x^2}{1+\cdots}\,\frac{\alpha x^{2n-1}}{1+}\,\frac{\beta x^{2n}}{1+\cdots}\ (\alpha,\beta \text{ and } x\text{: rational}).$$

The case $\alpha=\beta$ yields a recent result of Iekata Shiokawa.

## 1. Introduction

The following continued fraction expansion is found in the "Lost" notebook of Srinivasa Ramanujan [10]:

$$\frac{G(\alpha,\mu,\beta,x)}{G(\alpha x,\mu x,\beta,x)}=1+\frac{\alpha x+\mu x}{1+}\,\frac{\beta x+\mu x^2}{1+\cdots}\,\frac{\alpha x^{n+1}+\mu x^{2n+1}}{1+}\,\frac{\beta x^{n+1}+\mu x^{2n+2}}{1+\cdots}.$$

Here,

$$G(\alpha,\mu,\beta,x)=\sum_{n=0}^{\infty}\frac{x^{n(n+1)/2}}{(x)_n}\frac{(-\mu/\alpha)_n\alpha^n}{(-\beta x)_n},$$

$$(c;x)_\infty=(c)_\infty=\prod_{n=0}^{\infty}(1-cx^n)$$

and

$$(c;x)_n=(c)_n=\frac{(c)_\infty}{(cx^n)_\infty},\quad n\text{: integer}.$$

It is assumed here and throughout the paper that $|x|<1$.

Various proofs of the above and other expansions of $G(\alpha,\mu,\beta,x)/G(\alpha x,\mu x,\beta,x)$ can be found in the literature. For instance one may refer to [3], [4] and [5]. Many

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interesting special cases arise. See for instance [1], [2] and [3]. In particular we have the expansion

$$\frac{G(\alpha/x, o, \beta, x^2)}{G(\alpha x, o, \beta, x^2)}=1+\frac{\alpha x}{1+}\frac{\beta x^2}{1+\cdots}\frac{\alpha x^{2n-1}}{1+}\frac{\beta x^{2n}}{1+\cdots}.$$

The case $\alpha=1=\beta$ in this is the famous Rogers-Ramanujan continued fraction.

Setting

$$F(\alpha, \beta, x)=\frac{G(\alpha/x, o, \beta, x^2)}{G(\alpha x, o, \beta, x^2)} \tag{1}$$

and

$$f(\alpha, x)=F(\alpha, \alpha, x)$$

We have the following Theorems A and B of Iekata Shiokawa [11] with Theorem B establishing that Theorem A is best possible.

THOEREM A (Shiokawa). *Let a, b, c and d be non-zero integers with*

$$|d|>|c|^2.$$

*Then $f(a/b, c/d)$ is an irrational number and furthermore, there is a positive constant $C=C(a, b, c, d)$ such that*

$$\left|f\left(\frac{a}{b}, \frac{c}{d}\right)-\frac{p}{q}\right|>Cq^{-2-2A-B/\sqrt{\log q}}$$

*for all integers $p, q$ $(\geq 0)$, where*

$$A=\frac{\log|c|}{\log|d/c^2|} \quad \text{and} \quad B=\frac{\log|a^2d|-A\log|b/a^2|}{\sqrt{\log|d/c^2|}}.$$

THEOREM B (Shiokawa). *Let a, b and d be positive integers such that $(a, b)=1$, $d\geq 2$ and a divides d, and let*

$$C=\begin{cases}\sqrt{b/a} & \text{if } (a/b)^2>d,\\ \sqrt{a/bd} & \text{otherwise}.\end{cases}$$

*Then, for any $\varepsilon>0$,*

$$\left|f\left(\frac{a}{b}, \frac{1}{d}\right)-\frac{p}{q}\right|<(C+\varepsilon)q^{-2-\sqrt{\log d}/\sqrt{\log q}}$$

*for infinitely many integers $p, q$ $(\geq 0)$, while there is a positive constant $q_0=q_0$ $(a, b, d, \varepsilon)$ such that*

$$\left|f\left(\frac{a}{b}, \frac{1}{d}\right)-\frac{p}{q}\right|>(C-\varepsilon)q^{-2-\sqrt{\log d}/\sqrt{\log q}}$$

*for all integers* $p$, $q$ $(\geq q_0)$.

As noted by Shiokawa [11], case $c=1$ of Theorem A improves an earlier result of Osgood [8], [9], namely,

THEOREM C (Osgood). *If* $a$, $b$ *and* $d$ *are non-zero integers with* $|d|\geq 2$, *then for any* $\varepsilon>0$, *there is a positive constant* $q_0=q_0\,(a, b, d, \varepsilon)$ *such that*

$$\left|f\left(\frac{a}{b}, \frac{1}{d}\right)-\frac{p}{q}\right|>q^{-2-\varepsilon}$$

*for all integers* $p$, $q$ $(\geq q_0)$.

The purpose of the present note is to establish a generalization of Shiokawa's Theorem A. For convenience, we present the generalization in two parts by means of Theorem 1 and Theorem 2 below.

THEOREM 1. *Let* $a$, $b$, $c$, $d$, $e$ *and* $f$ *be non-zero integers with*

$$|b^2c^2e^2|<|a^2df^2|, \quad |a^2c^2f^2|<|b^2de^2|. \tag{2}$$

*Then* $F\,(a/b, e/f, c/d)$ *is an irrational number.*

THEOREM 2. *If* $a$, $b$, $c$, $d$, $e$ *and* $f$ *are non-zero integers satisfying* (2), *then there is positive constant* $C=C\,(a, b, c, d, e, f)$ *such that*

$$\left|F\left(\frac{a}{b}, \frac{e}{f}, \frac{c}{d}\right)-\frac{p}{q}\right|>Cq^{-2-2A-B/\sqrt{\log q}}$$

*for all integers* $p$, $q$ $(\geq 0)$ *where* $A=\log|c|/\log|d/c^2|$ *and*

$$B=\min\left[\frac{\log|b^2de^2|-2A\log|f/be^2|}{\sqrt{\log|d/c^2|}}, \frac{\log|a^2df^2|-2A\log|b/a^2f|}{\sqrt{\log|d/c^2|}}\right].$$

Again, Theorem 2 is best possible in the sense of Shiokawa [11]. In fact, since $f(a/b, c/d)=F(a/b, a/b, c/d)$ we can restate Theorem B in the following form.

THEOREM B′. *Let* $a$, $b$ *and* $d$ *be positive integers such that* $(a, b)=1$, $d\geq 2$ *and* $a$ *divides* $d$, *and let*

$$C=\begin{cases}\sqrt{b/a} & \textit{if}\quad (a/b)^2>d\\ \sqrt{a/bd} & \textit{otherwise}\,.\end{cases}$$

*Then for any* $\varepsilon>0$

$$\left|F\left(\frac{a}{b}, \frac{a}{b}, \frac{1}{d}\right)-\frac{p}{q}\right|<(C+\varepsilon)q^{-2-\sqrt{\log d}/\sqrt{\log q}}$$

*for infinitely many integers* $p$, $q$ $(\geq 0)$, *while there is a positive constant* $q_0=q_0(a, b, d, \varepsilon)$ *such that*

$$\left|F\left(\frac{a}{b}, \frac{a}{b}, \frac{1}{d}\right)-\frac{p}{q}\right|>(C-\varepsilon)q^{-2-\sqrt{\log d}/\sqrt{\log q}}$$

*for all integers* $p$, $q$ $(\geq q_0)$.

We prove Theorem 1 in Section 3 after obtaining some necessary Lemmas in Section 2. Theorem 2 and a necessary Lemma are proved in Sections 5 and 4 respectively.

## 2. Some preliminary results

LEMMA 1. *Let* $a_1, a_2, a_3, \cdots$, *be a sequence of real numbers such that*

$$|a_n a_{n+1}|>4 \quad (n\geq 1) \quad \text{and} \quad \sum_{n=1}^{\infty} |a_n a_{n+1}|^{-1}=\sigma<\infty .$$

*Define as usual*

$$p_n=a_n p_{n-1}+p_{n-2}, \qquad q_n=a_n q_{n-1}+q_{n-2} \quad (n\geq 1)$$

*with* $p_0=q_{-1}=0$, $p_{-1}=q_0=1$. *Then* $p_n/(a_2a_3\cdots a_n)$ *and* $q_n/(a_1a_2\cdots a_n)$ *converge to finite non-zero limits and they satisfy*

$$e^{-4\sigma}<|p_n/(a_2a_3\cdots a_n)|<e^{2\sigma}, \qquad e^{-4\sigma}<|q_n/(a_1a_2\cdots a_n)|<e^{2\sigma},$$

*so that the continued fraction*

$$\frac{1}{a_1+}\,\frac{1}{a_2+\cdots}\,\frac{1}{a_n+\cdots}=\lim_{n\to\infty}\frac{p_n}{q_n}$$

*is convergent.*

*Proof.* For a proof see [6, Section 4.4].

LEMMA 2. *If* $F(\alpha, \beta, x)$ *is an in* (1), *then*

(1′) $$F(\alpha, \beta, x)=1+\frac{1}{a_1+}\,\frac{1}{a_2+\cdots}\,\frac{1}{a_n+\cdots},$$

*where*

(3) $$a_{2n-1}=\frac{\beta^{n-1}}{\alpha^n x^n} \quad \text{and} \quad a_{2n}=\frac{\alpha^n}{\beta^n x^n}.$$

*Moreover,*

(4) $$\log|a_1a_2\cdots a_{2n-1}|=-\frac{(2n-1)^2}{4}\log|x|-\frac{(2n-1)}{2}\log|\alpha x|+O(1)$$

*and*

(5) $$\log|a_1a_2\cdots a_{2n}|=-n^2\log|x|-n\log|\beta x|+O(1).$$

*Proof.* (1′) follows easily on using the transformation [12, p. 20]

$$\frac{b_1}{1+}\,\frac{b_2}{1+\cdots}\,\frac{b_n}{1+\cdots}=\frac{1}{\dfrac{1}{b_1}+}\,\frac{1}{\dfrac{b_1}{b_2}+}\,\frac{1}{\dfrac{b_2}{b_1b_3}+}\,\frac{1}{\dfrac{b_1b_3}{b_2b_4}+\cdots}$$

$$\frac{1}{\dfrac{b_2b_4\cdots b_{2k}}{b_1b_3\cdots b_{2k+1}}+}\,\frac{1}{\dfrac{b_1b_3\cdots b_{2k+1}}{b_2b_4\cdots b_{2k+2}}+\cdots}.$$

To prove (4) and (5) note that

$$a_1a_2\cdots a_{2n-1}=a_{2n-1}\prod_{r=1}^{n-1}a_{2r-1}a_{2r}$$

$$=\frac{\beta^{n-1}}{\alpha^n x^n}\prod_{r=1}^{n-1}\frac{1}{\beta x^{2r}}=\frac{1}{\alpha^n x^{n^2}},$$

and

$$a_1a_2\cdots a_{2n}=\prod_{r=1}^{n}a_{2r-1}a_{2r}=\prod_{r=1}^{n}\frac{1}{\beta x^{2r}}=\frac{1}{\beta^n x^{n(n+1)}}.$$

Hence,

$$\log|a_1a_2\cdots a_{2n-1}|=-\frac{(2n-1)^2}{4}\log|x|-\frac{(2n-1)}{2}\log|\alpha x|+O(1)$$

and

$$\log|a_1a_2\cdots a_{2n}|=-n^2\log|x|-n\log|\beta x|+O(1).$$

LEMMA 3. *If $\alpha=a/b$, $\beta=e/f$ and $x=c/d$, where $a$, $b$, $c$, $d$, $e$ and $f$ are non-zero integers and if*

$$d_{2n-1}=|a^nc^{n^2}f^{n-1}|,\qquad d_{2n}=|b^nc^{n^2+n}e^n|,$$

*then $d_np_n$ and $d_nq_n$ are integers. Also*

(6) $$\log d_{2n-1}=\frac{(2n-1)^2}{4}\log|c|+\frac{(2n-1)}{2}\log|acf|+O(1)$$

*and*

(7) $$\log d_{2n}=n^2\log|c|+n\log|bce|+O(1).$$

*Proof.* Using the recurrence relations for $p_n$ and $q_n$ and employing induction on $n$ one can easily prove that $d_np_n$ and $d_nq_n$ are integers. (6) and (7) follow directly from the definition of $d_n$.

## 3. Proof of Theorem 1

Since $a_{2n-1}a_{2n}=1/\beta x^{2n}$ and $a_{2n}a_{2n+1}=1/\alpha x^{2n+1}$, from (2) it follows that the series $\sum_{n=1}^{\infty}(a_n a_{n+1})^{-1}$ is absolutely convergent. Hence there exists an integer $N$ such that

$$|a_n a_{n+1}|>4, \qquad \text{for all} \quad n\geq N.$$

Now, put

$$\theta_n=\frac{1}{a_{n+1}}+\frac{1}{a_{n+2}}+\cdots. \tag{8}$$

Then by Lemma 1, $\theta_n$ converges for each $n\geq N$ and

$$\begin{aligned} &e^{-6\sigma}<|a_{n+k+1}\theta_{n+k}|<e^{6\sigma} \\ &e^{-6\sigma}<|a_{n+k+1}q_{n,k}/q_{n,k+1}|<e^{6\sigma} \end{aligned} \tag{9}$$

where $p_{n,k}/q_{n,k}$ is the $k$th convergent of the continued fraction (8) and $\sigma=\sum_{n=1}^{\infty}|a_n a_{n+1}|^{-1}$. For sufficiently large $k$

$$\left|\theta-\frac{p_{n,k}}{q_{n,k}}\right|=\frac{1}{|q_{n,k}(q_{n,k+1}+\theta_{n+k+1}q_{n,k})|}<\frac{2}{|q_{n,k}^2 a_{n+k+1}|}. \tag{10}$$

(10) follows on using

$$\theta_n=\frac{1}{a_{n+1}+}\cdots\frac{1}{a_{n+k+1}+\theta_{n+k+1}}=\frac{(a_{n+k+1}+\theta_{n+k+1})p_{n,k}+p_{n,k-1}}{(a_{n+k+1}+\theta_{n+k+1})q_{n,k}+q_{n,k-1}},$$

$p_{n,k+1}q_{n,k}-p_{n,k}q_{n,k+1}=\pm 1$ and, a consequence of (2) and (9) namely, $\lim_{k\to\infty}\theta_{n+k+1}=0=\lim_{k\to 0}(1/a_{n+k+1})$. Using Stolz's theorem [7, p. 75] that $\lim_{n\to\infty}(X_n/Y_n)=\lim_{n\to\infty}[(X_{n+1}-X_n)/(Y_{n+1}-Y_n)]$ if $\{Y_n\}$ is increasing and diverges to $+\infty$ and the fact $\{|q_{n,k}/(a_{n+1}a_{n+2}\cdots a_{n+k})|\}$ converges to a non-zero limit as $k$ tends to $\infty$, one can easily show that

$$\lim_{k\to\infty}\frac{\log|q_{n,k}^2 a_{n+k+1}|}{\log|d_{n+k+1}q_{n,k}|}=\begin{cases} 2-\dfrac{2\log|b^2ce^2/a^2f^2|}{\log|b^2de^2/a^2f^2|}, & \text{if} \quad n+k \quad \text{is even} \\ 2-\dfrac{2\log|a^2cf^2/b^2e^2|}{\log|a^2df^2/b^2e^2|}, & \text{if} \quad n+k \quad \text{is odd}. \end{cases}$$

Therefore, for a given $\varepsilon>0$ we have

$$(11)\qquad \frac{\log|q_{n,k}^2 a_{n+k+1}|}{\log|d_{n+k+1}q_{n,k}|}>\begin{cases} 2-\dfrac{2\log|b^2ce^2/a^2f^2|}{\log|b^2de^2/a^2f^2|}-\varepsilon, & \text{if } n+k \text{ is even} \\ 2-\dfrac{2\log|a^2cf^2/b^2e^2|}{\log|a^2df^2/b^2e^2|}-\varepsilon, & \text{if } n+k \text{ is odd} \end{cases}$$

for all sufficiently large $k$, using (11) in (10) we obtain

$$\left|\theta_n-\frac{d_{n+k}p_{n,k}}{d_{n+k}q_{n,k}}\right|<\begin{cases} 2|d_{n+k}q_{n,k}|^{-2+\frac{2\log|b^2ce^2/a^2f^2|}{\log|b^2de^2/a^2f^2|}+\varepsilon}, & \text{if } n+k \text{ is even} \\ 2|d_{n+k}q_{n,k}|^{-2+\frac{2\log|a^2cf^2/b^2e^2|}{\log|a^2df^2/b^2e^2|}+\varepsilon}, & \text{if } n+k \text{ is odd} \end{cases}$$

for all sufficiently large $k$. This proves that $\theta_n$ $(n\geq N)$ is irrational. Hence $F(a/b, e/f, c/d)$ is also irrational.

## 4. A lemma

We now prove a Lemma which will be used in proving Theorem 2.

LEMMA 4. *If $a$, $b$, $c$, $d$, $e$ and $f$ are non-zero integers satisfying* (2), *then there exists a positive integer $n=n(q)$ such that*

$$(12)\qquad \left|F\left(\frac{a}{b},\frac{e}{f},\frac{c}{d}\right)-\frac{p}{q}\right|>\frac{1}{2}q^{-1-[\log|d_nq_n|/\log q]}$$

*for all integers $p$, $q$ $(\geq 0)$.*

*Proof.* On using (4) and (5) we have

$$(13)\qquad \log|q_{2m-1}|=\frac{(2m-1)^2}{4}\log\left|\frac{d}{c}\right|+\frac{(2m-1)}{2}\log\left|\frac{bd}{ac}\right|+O(1)$$

and

$$(14)\qquad \log|q_{2m}|=m^2\log\left|\frac{d}{c}\right|+m\log\left|\frac{df}{ce}\right|+O(1).$$

Further from (6), (7), (13) and (14) we have

$$(15)\qquad \log\left|\frac{q_{2m+1}}{d_{2m+1}}\right|-\log\left|\frac{q_{2m}}{d_{2m}}\right|=m\log\left|\frac{b^2e^2d}{a^2c^2f^2}\right|+O(1)$$

and

(16)
$$\log\left|\frac{q_{2m}}{d_{2m}}\right|-\log\left|\frac{q_{2m-1}}{d_{2m-1}}\right|=m\log\left|\frac{a^2df^2}{b^2c^2e^2}\right|+O(1).$$

Hence, from (2), (9), (15) and (16) it follows that there exists an integer $N_o$ $(\geq N)$ such that

(17)
$$|\theta_m|<\frac{1}{2},\quad |q_{m-1}|<|q_m|,\quad |q_{m-1}/d_{m-1}|<|q_m/d_m|,$$

for all $m\geq N_o$. Now, let $p$ and $q$ $(\geq o)$ be given integers. Then we may assume that $|q_{N_o}/d_{N_o}|<4q$. Therefore by (15), (16) and (17) there exists a positive integer $n=n(q)\geq N_o$ such that

(18)
$$|q_{n-1}/d_{n-1}|\leq 4q<|q_n/d_n|.$$

Since $p_nq_{n-1}-p_{n-1}q_n=\pm 1$ at least one of $p_{n-1}q-pq_{n-1}$, $p_nq-q_np$ is different from zero. So we first assume that $p_nq-q_np\neq 0$ and consider

(19)
$$d_nq_n\left[F\left(\frac{a}{b},\frac{e}{f},\frac{c}{d}\right)-\frac{p}{q}\right]=\frac{d_n(p_nq-q_np)}{q}+d_n\left[q_nF\left(\frac{a}{b},\frac{e}{f},\frac{c}{d}\right)-p_n\right],$$

where $|d_n(p_nq-q_np)|\geq 1$. But

(20)
$$\left|d_n\left[q_nF\left(\frac{a}{b},\frac{e}{f},\frac{c}{d}\right)-p_n\right]\right|=\frac{d_n}{|q_{n+1}+\theta_{n+1}q_n|}\leq\frac{2d_n}{|q_n|}\leq\frac{1}{2q}$$

by (17) and (18). Substituting (20) in (19) we obtain after simplification

$$\left|F\left(\frac{a}{b},\frac{e}{f},\frac{c}{d}\right)-\frac{p}{q}\right|>\frac{1}{2}q^{-1}\frac{1}{|d_nq_n|}=\frac{1}{2}q^{-1-[\log|d_nq_n|/\log q]}.$$

The same inequality is obtained in the other case namely $p_{n-1}q-q_{n-1}p\neq 0$. This completes the proof of the Lemma 4.

## 5. Proof of Theorem 2

In what follows $C_i=C_i\,(a, b, c, d, e, f)$, $i=1, 2, \cdots, 10$ are independent of $q$ and $n$. If $n$ (of Lemma 4) is odd, say $n=2k-1$, by (3), (6), (7), (13), (14) and (18) we have

(21)
$$\log|d_{2k-1}q_{2k-1}|=\log|d_{2k-1}d_{2k-2}|+\log\left|\frac{q_{2k-1}}{q_{2k-2}}\right|+\log\left|\frac{q_{2k-2}}{d_{2k-2}}\right|$$
$$<\log q+\frac{(2k-1)^2}{2}\log|c|+\frac{(2k-1)}{2}\log|b^2de^2|+C_1.$$

Again if $n$ (of Lemma 4) is even, say $n=2k$, by (3), (6), (7), (13), (14) and (18) we have as before,

$$\log|d_{2k}q_{2k}|<\log q+2k^2\log|c|+k\log|a^2df^2|+C_2\,. \tag{22}$$

Further from (6), (7), (13), (14), (18), (21) and (22) we obtain

$$\frac{(2k-1)^2}{4}\log\left|\frac{d}{c^2}\right|+\frac{(2k-1)}{2}\log\left|\frac{f}{be^2}\right|-C_3<\log q \tag{23}$$
$$<\frac{(2k-1)^2}{4}\log\left|\frac{d}{c^2}\right|+\frac{(2k-1)}{2}\log\left|\frac{bd}{a^2c^2f}\right|+C_4$$

and

$$k^2\log\left|\frac{d}{c^2}\right|+k\log\left|\frac{b}{a^2f}\right|-C_5<\log q<k^2\log\left|\frac{d}{c^2}\right|+k\log\left|\frac{df}{bc^2e^2}\right|+C_6\,. \tag{24}$$

Thus if $n=2k-1$ or if $n=2k$ by (23) and (24) we have

$$n=[2\sqrt{\log q}/\sqrt{\log|d/c^2|}]+O(1)\,. \tag{25}$$

From (23), (24) and (25) we obtain

$$n^2\leq\begin{cases}\dfrac{4\log q}{\log|d/c^2|}-\dfrac{4\sqrt{\log q}\log|f/be^2|}{\sqrt{\log|d/c^2|}\log|d/c^2|}+C_7\,, & \text{if}\quad n=2k-1\,,\\[2ex] \dfrac{4\log q}{\log|d/c^2|}-\dfrac{4\sqrt{\log q}\log|b/a^2f|}{\sqrt{\log|d/c^2|}\log|d/c^2|}+C_8\,, & \text{if}\quad n=2k\,.\end{cases} \tag{26}$$

On using (25) and (26) in (21) and (22) respectively we obtain

$$\frac{\log|d_nq_n|}{\log q}<\begin{cases}1+2A+[B_1/\sqrt{\log q}]+C_9\,, & \text{if}\quad n=2k-1\,,\\ 1+2A+[B_2/\sqrt{\log q}]+C_{10}\,, & \text{if}\quad n=2k\,,\end{cases} \tag{27}$$

where,

$$A=\frac{\log|c|}{\log|d/c^2|}\,,\qquad B_1=\frac{\log|b^2de^2|-2A\log|f/be^2|}{\sqrt{\log|d/c^2|}}\,,$$

and

$$B_2=\frac{\log|a^2df^2|-2A\log|b/a^2f|}{\sqrt{\log|d/c^2|}}\,.$$

Substituting (27) in (12) and putting

$C=\max\left\{\frac{1}{2}q^{-C_9},\frac{1}{2}q^{-C_{10}}\right\}$ and $B=\min\{B_1, B_2\}$ we obtain

$$\left|F\left(\frac{a}{b},\frac{e}{f},\frac{c}{d}\right)-\frac{p}{q}\right|>Cq^{-2-2A-B/\sqrt{\log q}}\,.$$

This completes the proof of Theorem 2.

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Bhargava and Adiga:
Department of Mathematics
University of Mysore
Manasagangotri
MYSORE–570 006
India

Somashekara:
Department of Mathematics
Yuvaraja's College
University of Mysore
MYSORE–570 005
India