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The Behaviour of Chern Characters of Projective Manifolds under Certain Holomorphic Maps

by

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1. Introduction

The purpose of our paper is to study the behaviour of the (exponential) Chern characters of complex projective manifolds under certain holomorphic maps. Let X and Y be complex manifolds of dimension n and $S=\bigcup_{i=1}^{k} S_i$ be a hypersurface in Y with only simple normal crossings (i.e. S is a union of irreducible components $S_1, \cdots, S_k$ and the divisor $\sum_{i=1}^{k} S_i$ has only simple normal crossings). Let $\varphi: X \to Y$ be a surjective holomorphic map such that the restriction map of φ to $X-\varphi^{-1}(S)$ is locally biholomorphic and the map φ satisfies the following conditions:

(i) The inverse image $\varphi^{-1}(S)$ is a hypersurface in X which is a union of irreducible components $D_1, \cdots, D_t$ and the divisor $\sum_{i=1}^{t} D_i$ has only simple normal crossings.

(ii) The image $\varphi(D_i)$ of D_i is either $S_{j_1} \cap \cdots \cap S_{j_m}$ $(m<n,\ 1 \leq j_1 < \cdots < j_m \leq k)$ or a point of $S_{j_1} \cap \cdots \cap S_{j_n}$ $(1 \leq j_1 < \cdots < j_n \leq k)$.

(iii) For any point $p \in \varphi^{-1}(S)$, if $p \in D_1 \cap \cdots \cap D_r$ $(r \leq n)$ and $q=\varphi(p) \in S_1 \cap \cdots \cap S_m$, then we can choose local coordinate systems $(t_1, \cdots, t_n)$ and $(z_1, \cdots, z_n)$ around p and q respectively, such that (1) a local equation for D_i (resp. S_j) is $t_i=0$ (resp. $z_j=0$) and (2) φ is expressed as

$$z_j = t_1^{a_{j1}} \cdots t_r^{a_{jr}} t_{r+1}^{a_{jr+1}} \cdots t_n^{a_{jn}} \ (1 \leq j \leq m),$$

$$z_j = t_{r+1}^{a_{jr+1}} \cdots t_n^{a_{jn}} \ (m+1 \leq j \leq n),$$

where $a_{j1}, \cdots, a_{jr}$ $(1 \leq j \leq m)$ are positive integers and $a_{jr+1}, \cdots, a_{jn}$ $(1 \leq j \leq n)$ are nonnegative integers and moreover the determinant

$$\det \begin{pmatrix} a_{11} & \cdots & a_{1r} & a_{1r+1} & \cdots & a_{1n} \\ \vdots & & \vdots & \vdots & & \vdots \\ a_{m1} & \cdots & a_{mr} & a_{mr+1} & \cdots & a_{mn} \\ & & & a_{m+1r+1} & \cdots & a_{m+1n} \\ & & & \vdots & & \vdots \\ & & & a_{nr+1} & \cdots & a_{nn} \end{pmatrix} \neq 0 .$$

We simply say that the map φ satisfies the *monomial* (or *toroidal*) *condition over* S if it satisfies the above conditions (i), (ii) and (iii). In particular if $\varphi(D_i)=S_1\cap\cdots\cap S_m$, then for each point $p\in D_i$, we have local coordinate systems $(t_1,\cdots,t_n)$ and $(z_1,\cdots,z_n)$ around p and q respectively, such that a local equation for D_i (resp. S_i) is $t_1=0$ (resp. $z_i=0$) and φ is expressed as

$$z_j=t_1^{n_j}t_2^{a_{j2}}\cdots t_n^{a_{jn}}\ (1\leq j\leq m)\,,$$
$$z_j=t_2^{a_{j2}}\cdots t_n^{a_{jn}}\ (m+1\leq j\leq n)\,, \tag{1}$$

In this case, it is easy to check that the above positive integers $n_1,\cdots,n_m$ are uniquely determined by D_i. Therefore we call D_i either a *ramification locus of* φ *with type* $(n_1;S_1)$ (including the case $n_1=1$) or a *contraction locus of* φ *with type* $(n_1,\cdots,n_m;S_1,\cdots,S_m)$ according as $m=1$ or $m\geq 2$ respectively.

We denote by $\mathrm{ch}(X)$ the Chern character of the tangent bundle $\tau(X)$ of a complex manifold X. Our main result is the following theorem.

THEOREM 1.1. *Let* $\varphi: X\to Y$ *be a surjective holomorphic map between complex projective manifolds of dimension n and S be a hypersurface in Y which is a union of irreducible components* $S_1,\cdots,S_m$ *and the divisor* $\sum_{i=1}^m S_i$ *has only simple normal crossings. We assume that the restriction map of* φ *to* $X-\varphi^{-1}(S)$ *is locally biholomorphic and the map* φ *satisfies the monomial condition over S.*

(i) *If the map* φ *has only a contraction locus D with type* $(n_1,\cdots,n_m;S_1,\cdots,S_m)$ *and ramification loci* D_i *with type* $(\alpha_i;S_i)$ $(i=1,\cdots,m)$, *where* $m\leq n$, *then we have* $\varphi^*S_i=\alpha_iD_i+n_iD$ *and the formula*

$$\varphi^*\mathrm{ch}(Y)-\mathrm{ch}(X)=\sum_{i=1}^{m}(e^{\varphi^*S_i}-e^{D_i})-(e^D-1) \tag{2}$$

in $A(X)\otimes\boldsymbol{Q}$, *where* $A(X)$ *is the Chow ring of X and* $\boldsymbol{Q}$ *is the field of rational numbers.*

(ii) *If the map* φ *has only ramification loci* D_{ij} *with type* $(\alpha_{ij};S_i)$ $(i=1,\cdots,m,j=1,\cdots,r_i)$, *then we have* $\varphi^*S_i=\sum_{j=1}^{r_i}\alpha_{ij}D_{ij}$ *and*

$$\varphi^*\mathrm{ch}(Y)-\mathrm{ch}(X)=\sum_{i=1}^{m}\sum_{j=1}^{r_i}(e^{\varphi^*S_i}-e^{\varphi^*S_i-(\alpha_{ij}-1)D_{ij}})\,. \tag{3}$$

First of all, by extending the methods which can be found in Porteous [P] and Kawai [Ka, §3], we shall prove Theorem 4.2 in §4 which is a generalization of Theorem 2 in [Ka, §3]. The proof of Theorem 1.1 is given in §5 using Theorem 4.2 and the Grothendieck-Riemann-Roch theorem.

Next, applying Theorem 1.1, we shall examine the Chern classes of finite coverings of complex projective manifolds. Let $\varphi: X\to Y$ be a finite covering which is branched along S, where X and Y are complex projective manifolds and $S=\bigcup_{i=1}^m S_i$ is a hypersurface in Y with only simple normal crossings. Then we have the following theorem.

THEOREM 1.2. *If $\varphi: X \to Y$ has only ramification loci D_i with type $(\alpha_i; S_i)$ $(i=1, \cdots, m)$, then we have*

$$c_1(X)-\varphi^*c_1(Y)=-\sum_{i=1}^{m}(\alpha_i-1)D_i\,, \tag{4}$$

$$\begin{aligned} c_2(X)-\varphi^*c_2(Y)=&\sum_{i=1}^{m}(\alpha_i-1)(\alpha_iD_i-\varphi^*c_1(Y))D_i \\ &+\sum_{i<j}(\alpha_i-1)(\alpha_j-1)D_iD_j\,, \end{aligned} \tag{5}$$

where $c_i(X)$ (resp. $c_i(Y)$) is the i-th Chern class of X (resp. Y).

Putting $n_i=1$ and $\alpha_i=1$ $(i=1, \cdots, m)$ in (i) of Theorem 1.1, the map φ gives the blowing up of Y along $C=S_1\cap\cdots\cap S_m$ (which is the transversal intersection of $S_1, \cdots, S_m$) or at a point of $S_1\cap\cdots\cap S_n$ (in this case $m=n$). Using our formula (2) in Theorem 1.1, we can obtain the formulas expressing the behvaiour of the Chern classes under the above blowing up.

THEOREM 1.3. *Let Y be a complex projective manifold and S_i $(i=1, \cdots, m)$ be nonsingular hypersurfaces in Y such that the divisor $\sum_{i=1}^{m}S_i$ has only simple normal crossings, where $m\le n$. Let $\varphi: X\to Y$ be the blowing up of Y along $C=S_1\cap\cdots\cap S_m$ (in this case $m<n$) or at a point of $S_1\cap\cdots\cap S_n$ (in this case $m=n$) and let D be the exceptional divisor of the blowing up. Then we have*

$$c_1(X)-\varphi^*c_1(Y)=-(m-1)D\,, \tag{6}$$

$$c_2(X)-\varphi^*c_2(Y)=\varphi^*\left(\sum_{i=1}^{m}S_i-(m-1)c_1(Y)\right)D+\frac{m(m-3)}{2}D^2\,, \tag{7}$$

$$\begin{aligned} c_3(X)-\varphi^*c_3(Y)=&\varphi^*\left(\left(\sum_{i=1}^{m}S_i\right)c_1(Y)-(m-1)c_2(Y)-\left(\sum_{i=1}^{m}S_i^2\right)\right)D \\ &+\varphi^*\left(\frac{m(m-3)}{2}c_1(Y)-(m-2)\left(\sum_{i=1}^{m}S_i\right)\right)D^2 \\ &-\frac{m(m-1)(m-5)}{6}D^3\,. \end{aligned} \tag{8}$$

Finally, we shall apply Theorem 1.1 to investigate some properties of certain covering space (or its nonsingular model) of $\boldsymbol{P}^n$, where $\boldsymbol{P}^n$ is the n-dimensional complex projective space. Let X be a complex projective manifold of dimension n and K_X be the canonical divisor of X. It is known that if X is a minimal n-fold of general type (i.e. K_X is numerically effective and $0<K_X^n$), then the inequality $K_X^n\le\frac{2(n+1)}{n}c_2(X)K_X^{n-2}$ holds (see e.g. [M1], [M2]). First we have the following theorem.

THEOREM 1.4. *Let $\varphi: X \to \boldsymbol{P}^n$ be a finite covering which is branched along S, where X is nonsingular and S is a hypersurface in $\boldsymbol{P}^n$ with only simple normal crossings. If the covering space X is a minimal n-fold of general type, then*

$$K_X^n < 2c_2(X)K_X^{n-2} . \tag{9}$$

If the covering space has singularities, then the explicit construction of its nonsingular model is generally difficult. We shall examine the Kawai cover X which is a nonsingular model of certain covering space of $\boldsymbol{P}^n$ constructed by Kawai [Ka]. Let $\varphi: X \to \boldsymbol{P}^n$ be the ν-th Kawai covering of $\boldsymbol{P}^n$ associated with S, where $S = \bigcup_{i=1}^m S_i$ is a hypersurface in $\boldsymbol{P}^n$ with only simple normal crossings (see e.g. Fujiwara [Fj]). Then we have the following theorem.

THEOREM 1.5. *For $2 \leq m \leq n-1$, if the Kawai cover X is a minimal n-fold of general type, then we have the inequality*

$$K_X^n < 2c_2(X)K_X^{n-2} . \tag{10}$$

2. Difference of tangent bundles and some lemmas

From now on, let $\varphi: X \to Y$ be the same as in Introduction and we denote by $\tau(X)$ and $\tau(Y)$ the holomorphic tangent bundles of X and Y respectively. Let $\varphi^*\tau(Y)$ be the induced bundle of $\tau(Y)$ by φ. We use the same notation for a holomorphic vector bundle and the coherent sheaf of its holomorphic sections. By the assumption, we may assume that for any point $p \in X$, there are local coordinate systems $(t_1, \cdots, t_n)$ and $(z_1, \cdots, z_n)$ around p and q respectively such that φ is expressed as $z_i = t_1^{a_{i1}} \cdots t_n^{a_{in}}$ $(i = 1, \cdots, n)$, where the determinant $|A| = |a_{ij}| \neq 0$. Then it is easily checked that

$$\det \frac{\partial(z_1, \cdots, z_n)}{\partial(t_1, \cdots, t_n)} = |A| t_1^{a_{11}+\cdots+a_{1n}-1} \cdots t_n^{a_{n1}+\cdots+a_{nn}-1} . \tag{11}$$

From (11), we see that the homorphism of locally free sheaves $\phi: \tau(X) \to \varphi^*\tau(Y)$ given at p by the equations

$$\phi\left(\frac{\partial}{\partial t_j}\right) = \sum_{i=1}^n \frac{\partial z_i}{\partial t_j} \frac{\partial}{\partial z_i} ,$$

is a monomorphism, where $\partial/\partial t_j$ and $\partial/\partial z_i$ are local sections which are local bases for $\tau(X)$ and $\varphi^*\tau(Y)$ respectively. Therefore we can regard $\tau(X)$ as a submodule of $\varphi^*\tau(Y)$. In this section, we study the quotient sheaf $\varphi^*\tau(Y)/\tau(X)$.

Let ρ be a holomorphic section of $\varphi^*\tau(Y)$ over X and is written at p in the form

$$\rho = \sum_{i=1}^n h_i(t) \frac{\partial}{\partial z_i} .$$

If its germ ρ_p at p belongs to the image of the induced map $\phi_p: \tau(X)_p \to \varphi^*(Y)_p$ on the stalks, then we shall express it as $\rho \equiv 0$ *mod* $\tau(X)$ *at* p.

Now, from (11), we can easily show that the equality

$$\phi\left(\sum_{j=1}^{n} g_j \frac{\partial}{\partial t_j}\right) = \sum_{i=1}^{n} h_i(t) \frac{\partial}{\partial z_i} \tag{12}$$

holds if and only if the following relations are satisfied

$$g_j = \frac{t_j}{|A|} \sum_{i=1}^{n} A_{ij} \frac{h_i(t)}{z_i} \qquad (j=1, \cdots, n), \tag{13}$$

where A_{ij} is the (i, j)-cofactor of $|A|$. The following lemma follows immediately from (13).

LEMMA 2.1. *If each $h_i(t)$ $(i=1, \cdots, n)$ satisfies the relations*

$$t_j A_{ij} h_i(t) = z_i \alpha_{ij}(t)$$

for $j=1, \cdots, n$, where the $\alpha_{ij}(t)$ is holomorphic at p, then $\rho = \sum_{i=1}^{n} h_i(t)\partial/\partial z_i \equiv 0$ mod $\tau(X)$ at p.

LEMMA 2.2. *If $\rho = \sum_{i=1}^{n} h_i(t)\partial/\partial z_i \equiv 0$ mod $\tau(X)$ at p and φ is expressed as*

$$z_i = t_{j_1}^{a_{ij_1}} \cdots t_{j_r}^{a_{ij_r}} p_i(\cdots, \hat{t}_{j_1}, \cdots, \hat{t}_{j_r}, \cdots) \ (1 \le j_1 < \cdots < j_r \le n),$$

where the $p_i(\cdots, \hat{t}_{j_1}, \cdots, \hat{t}_{j_r}, \cdots)$ is holomorphic at p and does not contain variables $t_{j_1}, \cdots, t_{j_r}$, then each h_i can be written in the form

$$h_i(t) = t_{j_1}^{a_{ij_1}-1} \cdots t_{j_r}^{a_{ij_r}-1} q_i(t),$$

where the $q_i(t)$ is holomorphic at p.

Proof. By (12), we have $h_i(t) = \sum_{j=1}^{n} g_j \partial z_i/\partial t_j$, from which Lemma 2.2 follows.

LEMMA 2.3. *Let D be a contraction (or ramification) locus of φ with type $(n_1, \cdots, n_m; S_1, \cdots, S_m)$ and p be a point of $D - \bigcup_k (D \cap D_k)$, where D_k is a contraction (or ramification) locus of φ which is different from D. Moreover we assume that there are local coordinate systems $(t_1, \cdots, t_n)$ and $(z_1, \cdots, z_n)$ around p and $q = \varphi(p)$ respectively, such that a local equation for D (resp. S_i $(i=1, \cdots, m)$) is $t_1 = 0$ (resp. $z_i = 0$) and φ is expressed as*

$$z_i = t_1^{n_i} p_i(t_2, \cdots, t_n) \ (1 \le i \le m),$$

$$z_i = p_i(t_2, \cdots, t_n) \ (m+1 \le i \le n).$$

Then $\rho = \sum_{i=1}^{n} h_i(t)\partial/\partial z_i \equiv 0$ mod $\tau(X)$ at p if and only if $h_1(t), \cdots, h_m(t)$ can be written in the forms

$$h_i(t) = t_1^{n_i - 1} q_i(t) \tag{14}$$

and $f_i = q_i/p_i$ is holomorphic at p and $f_1, \cdots, f_m$ satisfy the relations

$$\sum_{i=1}^{m} A_{ij} f_i = t_1 u_j(t) \qquad (j=2, \cdots, n), \tag{15}$$

where the $u_j(t)$ *is holomorphic at* p.

Proof. Let $h_i(t)$ be as in (14). Then, since $h_i/z_i=f_i/t_1$ ($i=1,\cdots,m$), the equation (13) can be rewritten in the form

$$|A|g_j=\frac{t_j}{t_1}\sum_{i=1}^{m}A_{ij}f_i+t_j\sum_{i=m+1}^{n}A_{ij}\frac{h_i}{z_i}. \tag{16}$$

Now we shall prove that $t_j\sum_{i=m+1}^{n}A_{ij}h_i/z_i$ is always holomorphic at p. Since D_k ($\neq D$) does not contain p, we may assume that the coordinates $z_{i_\alpha}(i_\alpha\geq m+1,\alpha=1,\cdots,l)$ can be written in the forms $z_{i_\alpha}=t_{j_\alpha}$ ($2\leq j_1<\cdots<j_l\leq n$) and if $i\neq i_\alpha$ and $i\geq m+1$, then z_i is a non vanishing holomorphic function. Then, since $A_{i_\alpha j}=0$ ($j\neq j_\alpha$) and $A_{ij_\alpha}=0$ ($i\neq i_\alpha$), we see that

$$t_j\sum_{i=m+1}^{n}A_{ij}\frac{h_i}{z_i}=\begin{cases}A_{i_\alpha j_\alpha}h_{i_\alpha} & (j=j_\alpha),\\ t_j\sum_{i\neq i_\alpha}A_{ij}\dfrac{h_i}{z_i} & (j\neq j_\alpha).\end{cases}$$

Note that p_i ($i=1,\cdots,m$) are non vanishing holomorphic functions and if f_i are holomorphic at p, then g_1 is holomorphic at p. Lemma 2.3 follows Lemma 2.2 and (16).

LEMMA 2.4. *Let D be a contraction locus of φ with type* $(n_1,\cdots,n_m;S_1,\cdots,S_m)$ *and* D_i ($i=1,\cdots,m$) *be a ramification locus of φ with type* $(\alpha_i;S_i)$. *Let p be an arbitrary point of either* (i) $D\cap D_1\cap\cdots\cap D_{r-1}$ ($1\leq r-1\leq m$) *or* (ii) $D_1\cap\cdots\cap D_r$ ($1\leq r\leq n$). *We assume that any other contraction* (*or ranification*) *locus which is different from the above D and D_i does not contain p and moreover there are local coordinate systems* $(t_1,\cdots,t_n)$ *and* $(z_1,\cdots,z_n)$ *around p and $q=\varphi(p)$ respectively, such that in case* (i) (*resp.* (ii)), *φ is expressed as*

$$\begin{aligned}&z_i=t_i^{\alpha_i}t_r^{n_i}p_i(t_{r+1},\cdots,t_n) && (i=1,\cdots,r-1),\\ &z_i=t_r^{n_i}p_i(t_{r+1},\cdots,t_n) && (i=r,\cdots,m),\\ &z_i=p_i(t_{r+1},\cdots,t_n) && (i=m+1,\cdots,n),\end{aligned} \tag{17}$$

$$\begin{aligned}(resp.\ &z_i=t_i^{\alpha_i}p_i(t_{r+1},\cdots,t_n) && (i=1,\cdots,r),\\ &z_i=p_i(t_{r+1},\cdots,t_n) && (i=r+1,\cdots,n),\quad)\end{aligned} \tag{18}$$

where a local equation for S_i is $z_i=0$ and local equations for D_i ($i=1,\cdots,r-1$) *and D are $t_i=0$ and $t_r=0$ respectively* (*resp. a local equation for D_i* ($i=1,\cdots,r$) *is $t_i=0$*).

Then $\rho=\sum_{i=1}^{n}h_i(t)\partial/\partial z_i\equiv 0$ *mod* $\tau(X)$ *at p if and only if* $h_1(t),\cdots,h_m(t)$ *can be written in the forms*

$$\begin{aligned}&h_i(t)=t_i^{\alpha_i-1}t_r^{n_i-1}q_i(t) && (i=1,\cdots,r-1),\\ &h_i(t)=t_r^{n_i-1}q_i(t) && (i=r,\cdots,m)\end{aligned} \tag{19}$$

and $f_i=q_i/p_i$ is holomorphic at p and $f_1,\cdots,f_m$ *satisfy the relations*

$$\begin{aligned} &A_{jj}f_j+\sum_{i=r}^{m}A_{ij}t_jf_i=t_ru_j(t) \qquad (j=1,\cdots,r-1)\,, \\ &\sum_{i=r}^{m}A_{ij}f_i=t_ru_j(t) \qquad (j=r+1,\cdots,n)\,, \end{aligned} \tag{20}$$

where the $u_j(t)$ is holomorphic at p (resp. $h_1(t),\cdots,h_r(t)$ can be written in the forms

$$h_i(t)=t_i^{\alpha_i-1}q_i(t) \tag{21}$$

where $q_i(t)$ is holomorphic at p).

Proof. In case (i), let $h_i(t)$ be the same as in (19). Since

$$A=\begin{pmatrix} \alpha_1 & & & n_1 & \cdots \\ & \ddots & 0 & \vdots & \\ & & \alpha_{r-1} & n_{r-1} & \cdots \\ & & & n_r & \cdots \\ & & & \vdots & \\ & & & n_m & \cdots \\ & 0 & & 0 & \cdots \\ & & & \vdots & \\ & & & 0 & \end{pmatrix}, \tag{22}$$

it is easy to check that $A_{ij}=0$ for $1\le i\le r-1$ and $i\neq j$. Therefore, in this case, the equation (13) can be written in the forms

$$\begin{aligned} g_j&=\frac{1}{|A|}\left\{\frac{1}{t_r}\left(A_{jj}f_j+\sum_{i=r}^{m}A_{ij}t_jf_i\right)+t_j\sum_{i=m+1}^{n}A_{ij}\frac{h_i}{z_i}\right\} \qquad (j=1,\cdots,r-1)\,, \\ g_r&=\frac{1}{|A|}\left\{\sum_{i=r}^{m}A_{ir}f_i+t_r\sum_{i=m+1}^{n}A_{ir}\frac{h_i}{z_i}\right\}, \\ g_j&=\frac{1}{|A|}\left\{\frac{t_j}{t_r}\left(\sum_{i=r}^{m}A_{ij}f_i\right)+t_j\sum_{i=m+1}^{n}A_{ij}\frac{h_i}{z_i}\right\} \qquad (j=r+1,\cdots,n)\,. \end{aligned}$$

By the same argument as in the proof of Lemma 2.3, first we see that $t_j\sum_{i=m+1}^{n}A_{ij}h_i/z_i$ is holomorphic at p. Since p_i $(i=1,\cdots,m)$ are non vanishing, again using Lemma 2.2 and these equations, we can prove that $g_1,\cdots,g_n$ are holomorphic at p if and only if $f_1,\cdots,f_m$ are holomorphic at p and satisfy (20).

In case (ii), let $h_i(t)$ be the same as in (21). In this case we have $A_{ij}=0$ for $1\le i\le r$ and $i\neq j$. Therefore we obtain

$$g_j=\frac{1}{|A|}\left\{A_{jj}\frac{t_jh_j}{z_j}+t_j\sum_{i=r+1}^{m}A_{ij}\frac{h_i}{z_i}\right\} \qquad (j=1,\cdots,r)\,,$$

$$g_j = \frac{1}{|A|} t_j \sum_{i=r+1}^{n} A_{ij} \frac{h_i}{z_i} \qquad (j=r+1, \cdots, n).$$

The assertion in the case (ii) follows from this.

3. Kawai's sheaves and several sheaf homomorphisms

DEFINITION 3.1. Let D_i $(i=1, \cdots, r)$ be contraction (or ramification) loci of φ and S_j be an irreducible hypersurface in Y with $\varphi(D_1 \cap \cdots \cap D_r) \cap S_j \neq \emptyset$. Then for integers $k_1, \cdots, k_r$ such that $k_i \geq -1$, we define $K_\varphi(k_1, \cdots, k_r; D_1, \cdots, D_r; S_j)$ to be

$$\begin{aligned} &K_\varphi(k_1, \cdots, k_r; D_1, \cdots, D_r; S_j) \\ &\quad = (\lambda_B)_* \{\varphi_B^*(\bar{\nu}(S_j)) \otimes (\bar{\nu}(D_1)^*)^{\otimes k_1} \otimes \cdots \otimes (\bar{\nu}(D_r)^*)^{\otimes k_r}\} \end{aligned}$$

if $k_i \geq 0$ $(i=1, \cdots, r)$ and

$$K_\varphi(k_1, \cdots, k_r; D_1, \cdots, D_r) = 0$$

if there is a k_i such that $k_i = -1$, where B is $D_1 \cap \cdots \cap D_r$ or a point of $D_1 \cap \cdots \cap D_n$, $\bar{\nu}(S_j)$ is the restriction of normal boundle $\nu(S_j)$ of S_j in Y to $\varphi(B)$, $\bar{\nu}(D_j)^*$ is the restriction of the dual bundle of normal boundle $\nu(D_j)$ of D_j in X to B, λ_B is the injection of B into X, φ_B is the restriction map of φ to B and $(\lambda_B)_*(\ )$ is the direct image induced by λ_B. We call it a *Kawai's sheaf associated to contraction* (or *ramification*) *loci* $(D_1, \cdots, D_r)$ *and a hypersurface* S_j *of orders* $(k_1, \cdots, k_r)$ (since such a sheaf was first introduced by Kawai in [Ka]).

Note that applying the projection formula in the Grothendieck group (see [P]) to the injection λ_B, if $k_i \geq 0$ $(i=1, \cdots, r)$, then the Kawai's sheaf $K_\varphi(k_1, \cdots, k_r; D_1, \cdots, D_r; S_j)$ can be identified with the coherent sheaf

$$(\lambda_B)_* \mathcal{O}_B \otimes \varphi^*[S_j] \otimes ([D_1]^*)^{\otimes k_1} \otimes \cdots \otimes ([D_r]^*)^{\otimes k_r},$$

where $[S_j]$ (resp. $[D_i]$) is the associated line boundle of the divisor S_j (resp. D_i) and $\mathcal{O}_B$ is the structure sheaf of B.

Now, to examine the difference of tangent bundles, we shall define two sheaf homomorphisms

$$\begin{aligned} \Phi(k_1, \cdots, k_r; D_1, \cdots, D_r; S_j) : K_\varphi(k_1, \cdots, k_r; D_1, \cdots, D_r; S_j) \\ \to \varphi^*\tau(Y)/\tau(X) \end{aligned} \tag{23}$$

and

$$\begin{aligned} &\Phi(k_1, \cdots, k_r; D_1, \cdots, D_r; D_{i_1}, \cdots, D_{i_p}; S_j) : \\ &K_\varphi(k_1, \cdots, k_r; D_1, \cdots, D_r; S_j) \to K_\varphi(k_{i_1}, \cdots, k_{i_p}; D_{i_1}, \cdots, D_{i_p}; S_j) \end{aligned} \tag{24}$$

for any nonempty subset $\{i_1, \cdots, i_p\}$ of $\{1, \cdots, r\}$.

Let p be any point of B such that $q=\varphi(p) \in S_j$. We take coordinate systems

$(t_1, \cdots, t_n)$ and $(z_1, \cdots, z_n)$ around p and q respectively, satisfying the condition (iii) in Introduction. We assume that $p \in B \cap D_{r+1} \cap \cdots \cap D_{r'}$ and $q \in S_j \cap S_{j_1} \cdots \cap S_{j_{m-1}}$. Then φ can be expressed as

$$z_j = t_1^{a_{j1}} \cdots t_{r'}^{a_{jr'}} t_{r'+1}^{a_{jr'+1}} \cdots a_n^{a_{jn}} \qquad (j=1, \cdots, m),$$

$$z_j = t_{r'+1}^{a_{jr'+1}} \cdots t_n^{a_{jn}} \qquad (j=m+1, \cdots, n),$$

where a local equation for D_i $(i=1, \cdots, r')$ is $t_i=0$ and a local equation for S_j (resp. S_{j_α}) is $z_1=0$ (resp. $z_{\alpha+1}=0$). With respect to these coordinate systems, we define $\Phi_1 = \Phi(k_1, \cdots, k_r; D_1, \cdots, D_r; S_j)$ to be

$$\Phi_1\left(g(t_{r+1}, \cdots, t_n) \frac{\partial}{\partial z_1} \otimes dt_1^{k_1} \otimes \cdots \otimes dt_r^{k_r}\right) = t_1^{k_1} \cdots t_r^{k_r} g(t_{r+1}, \cdots, t_n) \frac{\partial}{\partial z_1}$$

for $k_i \geq 0$ $(i=1, \cdots, r)$, where g is a holomorphic function and $\partial/\partial z_1$ is considered to be a local section of the line bundle $\varphi^*[S_j]$ on the left and a local section of $\varphi^*\tau(Y)$ on the right, dt_i is a local section of the line bundle $[D_i]^*$. To show that this is well defined, we take any other coordinate systems $(s_1, \cdots, s_n)$ and $(w_1, \cdots, w_n)$ such as above. Here we assume that a local equation for D_i (resp. S_j, S_{j_α}) is $s_i=0$ (resp. $w_1=0$, $w_{\alpha+1}=0$). Then we have $z_i = a_i w_i$ $(i=1, \cdots, m)$ and $t_i = b_i s_i$ $(i=1, \cdots, r')$, where a_i and b_i are non vanishing holomorphic functions. Therefore we have

$$\frac{\partial z_1}{\partial w_1} = \frac{\partial a_1}{\partial w_1} w_1 + a_1, \qquad \frac{\partial z_i}{\partial w_1} = \frac{\partial a_i}{\partial w_1} w_i \qquad (i=2, \cdots, m). \tag{25}$$

Moreover we have the relations of the sections of $\varphi^*[S_j]$ and $[D_i]^*$

$$\frac{\partial}{\partial w_1} = a_1 \frac{\partial}{\partial z_1} \quad \text{and} \quad dt_i = b_i ds_i \tag{26}$$

respectively. Suppose that $g\partial/\partial z_1 \otimes dt_1^{\otimes k_1} \otimes \cdots \otimes dt_r^{\otimes k_r}$ and $h\partial/\partial w_1 \otimes ds_1^{\otimes k_1} \otimes \cdots \otimes ds_r^{\otimes k_r}$ are the same holomorphic sections. It follows from (26) that

$$g b_1^{k_1} \cdots b_r^{k_r} = h a_1. \tag{27}$$

Therefore, using (25) and (27), we obtain

$$s_1^{k_1} \cdots s_r^{k_r} h \frac{\partial}{\partial w_1} - t_1^{k_1} \cdots t_r^{k_r} g \frac{\partial}{\partial z_1} = \sum f_i \frac{\partial}{\partial z_i},$$

where

$$f_i = w_i s_1^{k_1} \cdots s_r^{k_r} h \frac{\partial a_i}{\partial w_1} \qquad (i=1, \cdots, m),$$

$$f_i = s_1^{k_1} \cdots s_r^{k_r} h \frac{\partial z_i}{\partial w_1} \qquad (i=m+1, \cdots, n).$$

Now, as in the proof of Lemma 2.3, assume that $z_{i_\alpha}=t_{j_\alpha}$ $(i_\alpha \geq m+1, \alpha=1, \cdots, l, r'+1 \leq j_1 < \cdots < j_l \leq n)$ and if $i \neq i_\alpha$ and $i_\alpha \geq m+1$, then z_i is a non vanishing holomorphic function. In this case, for $i=i_\alpha$ one gets

$$t_j A_{ij} f_i = \begin{cases} 0 & (j \neq j_\alpha)\,, \\ z_i A_{ij} f_i & (j=j_\alpha)\,. \end{cases}$$

From this, by Lemma 2.1 we see that Φ_1 is well defined. Hence we have (23).

Next we define $\Phi_2=\Phi\,(k_1, \cdots, k_r; D_1, \cdots, D_r; D_{j_1}, \cdots, D_{j_p})$ to be

$$\Phi_2\left(a(t_{r+1}, \cdots, t_n)\frac{\partial}{\partial z_1} \otimes dt_1^{\otimes k_1} \otimes \cdots \otimes dt_r^{\otimes k_r}\right)$$
$$= a(t_{r+1}, \cdots, t_n) t_{j_1}^{k_{j_1}} \cdots t_{j_{r-p}}^{k_{j_{r-p}}} \frac{\partial}{\partial z_1} \otimes dt_{i_1}^{k_{i_1}} \otimes \cdots \otimes dt_{i_p}^{k_{i_p}}$$

for $k_i \geq 0$ $(i=1, \cdots, r)$, where $(t_1, \cdots, t_n)$, $(z_1, \cdots, z_n)$, dt_i are the same as above, $\partial/\partial z_1$ is a local section of the line bundle $\varphi^*[S_j]$, a is a holomorphic function and $\{j_1, \cdots, j_{r-p}\}$ is the complement of $\{i_1, \cdots, i_p\}$ with respect to $\{1, \cdots, r\}$. Then, using (26), it is easy to check that the map Φ_2 is well defined. Therefore we have (24).

THEOREM 3.2. *For a contraction (resp. ramification) locus D of φ with type $(n_1, \cdots, n_m; S_1, \cdots, S_m)$ (resp. $(n_1; S_1)$) we have a sheaf homomorphism*

$$\Phi_D : \sum_{k=-1}^{n_1-2} K_\varphi(k; D; S_1) + \sum_{j=2}^{m} \sum_{k=0}^{n_j-1} K_\varphi(k; D; S_j) \to \varphi^*\tau(Y)/\tau(X)$$

$$(\textit{resp. } \Phi_D : \sum_{k=-1}^{n_1-2} K_\varphi(k; D; S_1) \to \varphi^*\tau(Y)/\tau(X))$$

such that Φ_D is an isomorphism over $D-\bigcup_i (D \cap D_i)$, where D_i is a contraction (or ramification) locus of φ which is distinct from D.

Proof. Using homomorphisms $\Phi(k; D; S_j)$, if $n_1 \geq 2$, then we define Φ_D to be

$$\Phi_D = \sum_{k=0}^{n_1-2} \Phi(k; D; S_1) + \sum_{j=2}^{m} \sum_{k=0}^{n_j-1} \Phi(k; D; S_j)\,.$$

Let p be an arbitrary point of $D-\bigcup_i (D \cap D_i)$. We may assume that there are local coordinate systems $(t_1, \cdots, t_n)$ and $(z_1, \cdots, z_n)$ around p and $q=\varphi(p)$ respectively, such that a local equation for D (resp. S_i) is $t_1=0$ (resp. $z_i=0$) and φ is expressed as (1) and we have

$$\Phi_D\left(\sum_{k=0}^{n_1-2} g_{k1}\frac{\partial}{\partial z_1} \otimes dt_1^{\otimes k} + \sum_{j=2}^{m} \sum_{k=0}^{n_j-1} g_{kj}\frac{\partial}{\partial z_j} \otimes dt_1^{\otimes k}\right) = \sum_{i=1}^{m} h_i \frac{\partial}{\partial z_i}\,,$$

where the g_{ki} are holomorphic functions of variables $t_2, \cdots, t_n$ and

$$h_1 = \sum_{k=0}^{n_1-2} g_{k1} t_1^k\,, \qquad h_i = \sum_{k=0}^{n_i-1} g_{ki} t_1^k \qquad (i=2, \cdots, m)\,.$$

Now suppose that $\sum_{i=1}^{m} h_i\partial/\partial z_i \equiv 0 \bmod \tau(X)$ at p. By Lemma 2.3, we can write

$$h_i = t_1^{n_i-1} q_i(t),$$

where each q_i is holomorphic at p. Therefore we see that $g_{k1}=0$ $(k=1, \cdots, n_1-2)$ and $g_{ki}=0$ $(k \le n_i-2$ and $i \ge 2)$. Consequently, we have $h_1=0$ and $h_i = t_1^{n_i-1} g_{n_i-1i}$ $(i \ge 2)$. Moreover, putting $f_i = q_i/p_i$ (in this case $f_1=0$ and $f_i = g_{n_i-1i}/p_i$), it follows from Lemma 2.3 that $\sum_{i=2}^{m} A_{ij} g_{n_i-1i}/p_i = 0$ $(j=2, \cdots, n)$. Now, using the relations $\sum_{i=1}^{r} n_i A_{ij} = |A|$ $(j=2, \cdots, n)$, where A_{ij} is the (i,j)-cofactor of $|A|$, we can easily verify that

$$\operatorname{rank}\begin{pmatrix} A_{22} & \cdots & A_{m2} \\ \vdots & \ddots & \vdots \\ A_{2n} & \cdots & A_{mn} \end{pmatrix} = m-1 .$$

From this we see that $g_{n_i i}=0$ $(i=2, \cdots, m)$. This implies that the induced map on the stalks $(\Phi_D)_p$ is injective.

Next we suppose that $\sum_{i=1}^{n} h_i\partial/\partial z_i$ is an arbitrary section of $\varphi^*\tau(Y)$ at p. We can write

$$h_i = \sum_{k=0}^{n_i-2} u_{ki}(t_2, \cdots, t_n) t_1^k + t_1^{n_i-1}\left(b_i(t_2, \cdots, t_n) + c_i(t_2, \cdots, t_n) t_1\right) \qquad (i=1, \cdots, m),$$

where u_{ki}, b_i and c_i are holomorphic at p. Now, since the equality

$$\operatorname{rank}\begin{pmatrix} A_{22} & \cdots & A_{m2} \\ \vdots & \ddots & \vdots \\ A_{2n} & \cdots & A_{mn} \end{pmatrix} = \operatorname{rank}\begin{pmatrix} A_{22} & \cdots & A_{m2} & \sum_{i=1}^{m} A_{i2}\frac{b_i}{p_i} \\ \vdots & \ddots & \vdots & \vdots \\ A_{2n} & \cdots & A_{mn} & \sum_{i=1}^{m} A_{in}\frac{b_i}{p_i} \end{pmatrix}$$

holds, we can choose the solutions $(x_2, \cdots, x_m) = (\beta_2(t_2, \cdots, t_n), \cdots, \beta_m(t_2, \cdots, t_n))$ of the system of equations

$$A_{1j}\frac{b_i}{p_i} + \sum_{i=2}^{m} A_{ij}\frac{b_i - x_i}{p_i} = 0 \qquad (j=2, \cdots, n).$$

Then, with the aid of Lemma 2.3, we have

$$\sum_{i=1}^{n} h_i \frac{\partial}{\partial z_i} \equiv \left(\sum_{k=0}^{n_i-2} u_{k1} t_1^k\right)\frac{\partial}{\partial z_1} + \sum_{i=2}^{m}\left(\sum_{k=0}^{n_i-2} u_{ki} t_1^k + \beta_i(t_2, \cdots, t_n) t_1^{n_i-1}\right)\frac{\partial}{\partial z_i}$$

(mod $\tau(X)$ at p). This proves that the induced map on the stalks $(\Phi_D)_p$ is surjective.

In the case in which $n_1=1$, we can define Φ_D to be

$$\Phi_D = \sum_{j=2}^{m} \sum_{k=0}^{n_j-1} \Phi(k; D; S_j).$$

Then, by the same argument as above, we see that Φ_D induces an isomorphism over

$D-\bigcup_i(D\cap D_i)$. Moreover, if D is a ramification locus of φ with type $(n_1; S_1)$, then for $n_1\geq 2$, we define Φ_D to be

$$\Phi_D=\sum_{k=0}^{n_1-2}\Phi(k; D; S_1)\,.$$

Then the assertion follows immediately from Lemma 2.3.

4. Description of the difference of tangent bundles

DEFINITION 4.1. Let D be a contraction (resp. ramification) locus of φ with type $(n_1, \cdots, n_m; S_1, \cdots, S_m)$ (resp. $(n_1; S_1)$). We define $Ass_\varphi(D)$ to be

$$Ass_\varphi(D)=\sum_{k=-1}^{n_1-2}K_\varphi(k; D; S_1)+\sum_{j=2}^{m}\sum_{k=0}^{n_j-1}K_\varphi(k; D; S_j)$$

$$\left(\text{resp. } Ass_\varphi(D)=\sum_{k=-1}^{n_1-2}K_\varphi(k; D; S_1\right),$$

where $\sum$ denotes the direct sum of coherent sheaves, and we call it the *sheaf associated to the contraction (resp. ramification) locus D with type* $(n_1, \cdots, n_m; S_1, \cdots, S_m)$ *(resp.* $(n_1; S_1)$*).*

The following theorem is a generalization of Theorem 2 in [K, §3].

THEOREM 4.2. *Let $\varphi: X\to Y$ be a surjective holomorphic map of complex manifolds of dimension n and S be a hypersurface in Y which is a union of irredudible components $S_1, \cdots, S_m$, with only simple normal crossings. We assume that the restriction map of φ to $X-\varphi^{-1}(S)$ is locally biholomorphic and the map φ satisfies the monomial condition over S.*

(i) *If the map φ has only a contraction locus D with type $(n_1, \cdots, n_m; S_1, \cdots, S_m)$ and ramification loci D_i with type $(\alpha_i; S_i)$ $(i=1, \cdots, m)$, where $m\leq n$, then we have a sheaf homomorphism*

$$\Psi_\varphi: Ass_\varphi(D)+\sum_{i=1}^{m}Ass_\varphi(D_i)\to\varphi^*\tau(Y)/\tau(X)$$

such that

$$\operatorname{Ker}\Psi_\varphi\cong\sum_{j=-1}^{\alpha_1-2}\sum_{k=-1}^{n_1-2}K_\varphi(j, k; D_1, D; S_1)+\sum_{i=2}^{m}\sum_{j=-1}^{\alpha_i-2}\sum_{k=0}^{n_i-1}K_\varphi(j, k; D_i, D; S_i)\,, \tag{28}$$

$$\operatorname{Coker}\Psi_\varphi\cong K_\varphi(\alpha_1-1, n_1-1; D_1, D; S_1)\,. \tag{29}$$

(ii) *If the map φ has only ramification loci D_i $(i=1, \cdots, m)$, then we have a sheaf isomorphism*

$$\Psi_\varphi: \sum_{i=1}^{m}Ass_\varphi(D_i)\cong\varphi^*\tau(Y)/\tau(X)\,.$$

Proof. First we shall prove (i). Using Φ_D and Φ_{D_i} in Theorem 3.2, we define the map Ψ_φ to be

$$\Psi_\varphi=\Phi_D+\sum_{i=1}^{m}\Phi_{D_i}.$$

To begin with, we consider the case (I) in which $p\in D\cap D_1\cap\cdots\cap D_{r-1}$ $(1\leq r-1\leq m)$ and $p\notin D_r\cup\cdots\cup D_m$, and assume that $n_1\geq 2$, $\alpha_1\geq 2$ and $\alpha_i\geq 2$ $(2\leq i\leq l-1)$, $\alpha_i=1$ $(l\leq i\leq r-1)$. Then we have

$$\begin{aligned}\Psi_\varphi\Bigg(&\sum_{k=0}^{n_1-2} f_{k1}(t_1,\cdots,t_{r-1},t_{r+1},\cdots,t_n)\frac{\partial}{\partial z_1}\otimes dt_r^{\otimes k}\\&+\sum_{i=2}^{m}\sum_{k=0}^{n_i-1} f_{ki}(t_1,\cdots,t_{r-1},t_{r+1},\cdots,t_n)\frac{\partial}{\partial z_i}\otimes dt_r^{\otimes k}\\&+\sum_{i=1}^{l-1}\sum_{j=0}^{\alpha_i-2} g_{ji}(t_1,\cdots,t_{i-1},\cdots,t_{i+1},\cdots,t_n)\frac{\partial}{\partial z_i}\otimes dt_i^{\otimes j}\Bigg)\\=&\sum_{i=1}^{m} h_i\frac{\partial}{\partial z_i},\end{aligned}\tag{30}$$

where $(t_1,\cdots,t_n)$ and $(z_1,\cdots,z_n)$ are local coordinate systems as in case (i) of Lemma 2.4,

$$\begin{aligned}h_1=&\sum_{k=0}^{n_1-2} f_{k1}(t_1,\cdots,t_{r-1},t_{r+1},\cdots,t_n)t_r^k\\&+\sum_{j=0}^{\alpha_1-2} g_{j1}(t_2,\cdots,t_n)t_1^j,\end{aligned}$$

$$\begin{aligned}h_i=&\sum_{k=0}^{n_i-1} f_{ki}(t_1,\cdots,t_{r-1},t_{r+1},\cdots,t_n)t_r^k\\&+\sum_{j=0}^{\alpha_i-2} g_{ji}(t_1,\cdots,t_{i-1},t_{i+1},\cdots,t_n)t_i^j\qquad(2\leq i\leq l-1)\end{aligned}$$

and

$$h_i=\sum_{k=0}^{n_i-1} f_{ki}(t_1,\cdots,t_{r-1},t_{r+1},\cdots,t_n)t_r^k\qquad(l\leq l\leq m).\tag{31}$$

To prove (28), now suppose that $\sum h_i\partial/\partial z_i\equiv 0 \bmod \tau(X)$ at p. By Lemma 2.4, h_i can be written in the same form as in (19), where $q_1=0$ and each q_i $(i=2,\cdots,m)$ does not contain the variable t_r. Therefore, putting $f_i=q_i/p_i$ $(i=1,\cdots,m)$, the conditions (20) of Lemma 2.4 are equivalent to the following conditions

$$\sum_{i=r}^{m} A_{i1} f_i = 0 ,$$

$$A_{jj} f_j + \sum_{i=r}^{m} A_{ij} t_j f_i = 0 \qquad (2 \le j \le r-1) ,$$

$$\sum_{i=r}^{m} A_{ij} f_i = 0 \qquad (r+1 \le j \le n) .$$

We can easily verify that if the matrix A is the same as in (22), then

$$\operatorname{rank}\begin{pmatrix} A_{r1} & \cdots & A_{m1} \\ A_{rr+1} & \cdots & A_{mr+1} \\ \vdots & \ddots & \vdots \\ A_{rn} & \cdots & A_{mn} \end{pmatrix} = \operatorname{rank}\begin{pmatrix} A_{r1} & \cdots & A_{m1} & A_{11} \\ A_{rr+1} & \cdots & A_{mr+1} & 0 \\ \vdots & \ddots & \vdots & \vdots \\ A_{rn} & \cdots & A_{mn} & 0 \end{pmatrix} \tag{32}$$

$= m - r + 1$, where A_{ij} is the (i, j)-cofactor of the determinant $|A|$. Therefore we have $f_1 = \cdots = f_m = 0$. Hence $h_1 = \cdots = h_m = 0$. This implies that, putting

$$f_{ki} = \sum_j c_{jki} t_i^j ,$$

$$g_{ji} = \sum_k d_{kji} t_r^k ,$$

where c_{jki} and d_{kji} are holomorphic at p and do not contain variables t_i and t_r, the following relations

$$\begin{aligned} &c_{jk1} + d_{kj1} = 0 \quad (0 \le j \le \alpha_1 - 2, 0 \le k \le n_1 - 2) \quad \text{and} \\ &c_{jki} + d_{kji} = 0 \quad (2 \le i \le l-1, 0 \le j \le \alpha_i - 2, 0 \le k \le n_i - 1) \end{aligned} \tag{33}$$

are satisfied and moreover the functions c_{jki} and d_{kji} which do not appear in the relations (33) are zero functions. Since the maps $\Phi(j, k; D_i, D; D; S_i)$ and $\Phi(j, k; D_i, D; D_i; S_i)$ in Section 3 can be defined to be

$$\Phi(j, k; D_i, D; D; S_i)\left(c_{jki} \frac{\partial}{\partial z_i} \otimes dt_i^{\otimes j} \otimes dt_r^{\otimes k}\right) = c_{jki} t_i^j \frac{\partial}{\partial z_i} \otimes dt_r^{\otimes k} ,$$

$$\Phi(j, k; D_i, D; D_i; S_i)\left(c_{jki} \frac{\partial}{\partial z_i} \otimes dt_i^j \otimes dt_r^{\otimes k}\right) = c_{jki} t_r^k \frac{\partial}{\partial z_i} \otimes dt_i^{\otimes j}$$

respectively, it follows from (30) and (33) that the map

$$\sum_{j=0}^{\alpha_1 - 2} \sum_{k=0}^{n_1 - 2} (\Phi(j, k; D_1, D; D; S_1) - \Phi(j, k; D_1, D; D_1; S_1))$$

$$+ \sum_{i=2}^{l-1} \sum_{j=0}^{\alpha_i - 2} \sum_{k=0}^{n_i - 1} (\Phi(j, k; D_i, D; D; S_i) - \Phi(j, k; D_i, D; D_i; S_i))$$

induces an isomorphism on the stalks

$$(\operatorname{Ker}\Psi_\varphi)_p \cong \left(\sum_{j=0}^{\alpha_1-2}\sum_{k=0}^{n_1-2} K_\varphi(j,k;D_1,D;S_1)+\sum_{i=2}^{l-1}\sum_{j=0}^{\alpha_i-2}\sum_{k=0}^{n_i-1} K_\varphi(j,k;D_i,D;S_i)\right)_p .$$

To prove (29), for an arbitrary section $\sum h_i\partial/\partial z_i$, we can write

$$h_i=\sum_{k=0}^{n_i-2} f_{ki}(t_1,\cdots,t_{r-1},t_{r+1},\cdots,t_n)t_r^k$$
$$+\sum_{j=0}^{\alpha_i-2} g_{ji}(t_1,\cdots,t_{i-1},t_{i+1},\cdots,t_n)t_i^j+t_i^{\alpha_i-1}t_r^{n_i-1}a_i \qquad (1\leq i\leq l-1),$$
$$h_i=\sum_{k=0}^{n_i-2} f_{ki}(t_1,\cdots,t_{r-1},t_{r+1},\cdots,t_n)t_r^k+t_r^{n_i-1}a_i \qquad (l\leq i\leq m),$$

where f_{ki}, g_{ji} are holomorphic at p and $f_{ki}=g_{ji}=0$ for $k=-1$ or $j=-1$. Furthermore we write

$$a_1=c_{01}(t_2,\cdots,t_{r-1},t_{r+1},\cdots,t_n)+c_{11}(t_1,\cdots,t_{r-1},t_{r+1},\cdots,t_n)t_1+c_{21}(t)t_r ,$$
$$a_i=c_{1i}(t_1,\cdots,t_{r-1},t_{r+1},\cdots,t_n)+c_{2i}(t)t_r \qquad (2\leq i\leq m),$$

where c_{ij} are holomorphic at p. In view of (32), we can find holomorphic functions $S_i(t_1,\cdots,t_{r-1},t_{r+1},\cdots,t_n)$ $(i=r,\cdots,m)$ which do not contain t_r and satisfy the system of equations

$$A_{11}\frac{c_{11}}{p_1}+\sum_{i=r}^{m} A_{i1}S_i=0 ,$$
$$\sum_{i=r}^{m} A_{ij}S_i=0 \qquad (r+1\leq j\leq n) .$$

Then, using Lemma 2.4, we can verify that, putting

$$f_{n_i-1i}=\left(c_{1i}+\frac{p_i}{A_{ii}}\sum_{j=r}^{m} A_{ji}t_iS_j\right)t_i^{\alpha_i-1} \qquad (2\leq i\leq r-1),$$
$$f_{n_i-1i}=d_{1i}-p_iS_i \qquad (r\leq i\leq m),$$

the relation

$$\sum h_i\frac{\partial}{\partial z_i}-(c_{01}t_1^{\alpha_1-1}t_r^{n_1-1})\frac{\partial}{\partial z_1}$$
$$=\Psi_\varphi\left(\sum_{k=0}^{n_1-2} f_{k1}\frac{\partial}{\partial z_1}\otimes dt_r^{\otimes k}+\sum_{i=2}^{m}\left(\sum_{k=0}^{n_i-1} f_{ki}\frac{\partial}{\partial z_i}\otimes dt_r^{\otimes k}\right)\right.$$
$$\left.+\sum_{i=1}^{l-1}\left(\sum_{j=0}^{\alpha_i-2} g_{ji}\frac{\partial}{\partial z_i}\otimes dt_i^{\otimes j}\right)\right)+\phi\left(\sum g_j(t)\frac{\partial}{\partial t_j}\right) \qquad (34)$$

holds, where the $g_j(t)$ are holomorphic functions. Since the map $\Phi(\alpha_1-1, n_1-1; D_1, D; S_1)$ in Section 3 is defined to be

$$\Phi(\alpha_1-1, n_1-1; D_1, D; S_1)\left(c_{01}\frac{\partial}{\partial z_1}\otimes dt_1^{\otimes\alpha_1-1}\otimes dt_r^{\otimes n_1-1}\right)=c_{01}t_1^{\alpha_1-1}t_r^{n_1-1}\frac{\partial}{\partial z_1},$$

(34) implies that the composition map $P\circ\Phi(\alpha_1-1, n_1-1; D_1, D; S_1)$ is surjective, where $P: \varphi^*\tau(Y)/\tau(X)\to\text{Coker}\,\Psi_\varphi$ is a canonical homomorphism. Moreover, to prove that this map is injective, we suppose that

$$c_{01}t_1^{\alpha_1-1}t_r^{n_1-1}\frac{\partial}{\partial z_1}\equiv\sum_{i=1}^{m}h_i\frac{\partial}{\partial z_i}$$

(mod $\tau(X)$ at p), where h_i is the same as in (31). By the relation (20) in Lemma 2.4, we have

$$-A_{11}\frac{c_{01}}{p_1}+t_1\sum_{i=r}^{m}A_{i1}\frac{f_{n_i-1}}{p_i}=u_1t_r,$$

where u_1 is holomorphic at p. Since c_{01} does not contain variables t_1 and t_r, this implies that $c_{01}=0$. Hence we infer that the map $P\circ\Phi(\alpha_1-1, n_1-1; D_1, D; S_1)$ induces an isomorphism on the stalks

$$(\text{Coker}\,\Psi_\varphi)_p\cong(K_\varphi(\alpha_1-1, n_1-1; D_1, D; S_1))_p. \tag{35}$$

If $n_1=1$ or $\alpha_1=1$, then, by the same argument as above, we can verify isomorphims

$$(\text{Ker}\,\Psi_\varphi)_p\cong\left(\sum_{i=2}^{l-1}\sum_{j=0}^{\alpha_i-2}\sum_{k=0}^{n_i-1}K_\varphi(j, k; D_i, D; S_i)\right)_p \tag{36}$$

and (35).

We consider the case (II) in which $p\in D\cap D_2\cap\cdots\cap D_r$ $(2\le r\le m)$ and $p\notin D_1\cup D_{r+1}\cup\cdots\cup D_m$. We assume that $\alpha_i\ge 2$ $(2\le i\le l-1)$ and $\alpha_i=1$ $(l\le i\le m)$. Using Lemma 2.4, we obtain, in a similar manner to the case (I), isomorphisms (36) and $(\text{Coker}\,\Psi_\varphi)_p\cong 0$.

Finally, we consider the case (III) in which $p\in D_1\cap\cdots\cap D_r$ $(1\le r\le m)$ and $p\notin D\cup D_{r+1}\cup\cdots\cup D_m$. If $\alpha_i\ge 2$ $(1\le i\le l-1)$ and $\alpha_i=1$ $(l\le i\le m)$, then, since the map Ψ_φ can be defined to be $\Psi_\varphi=\sum_{i=1}^{l-1}\Phi_{D_i}$, we have

$$\Psi_\varphi\left(\sum_{i=1}^{l-1}\sum_{j=1}^{\alpha_i-2}g_{ji}(t_1\cdots t_{i-1}t_{i+1}\cdots t_n)\right)\frac{\partial}{\partial z_i}\otimes t_i^{\otimes j}=\sum_{i=1}^{l-1}h_i\frac{\partial}{\partial z_i},$$

where $h_i=\sum_{j=1}^{\alpha_i-2}g_{ji}t_i^j$. Using the case (ii) of Lemma 2.4, we can easily prove isomorphisms $(\text{Ker}\,\Psi_\varphi)_p\cong 0$ and $(\text{Coker}\,\Psi_\varphi)_p\cong 0$. This completes the proof of (i) and the argument given in the case (III) proves (ii).

5. Proof of Theorem 1.1

In this section, using the Grothendieck-Riemann-Roch theorem, first we shall prove the formula (2). By Theorem 4.2, (i), there is an exact sequence

$$0 \to \sum_{j=0}^{\alpha_1-2} \sum_{k=0}^{n_1-2} K_\varphi(j, k; D_1, D; S_1) + \sum_{i=2}^{m} \sum_{j=0}^{\alpha_i-2} \sum_{k=0}^{n_i-1} K_\varphi(j, k; D_i, D; S_i)$$

$$\to Ass_\varphi(D) + \sum_{i=1}^{m} Ass_\varphi(D_i) \to \varphi^*\tau(Y)/\tau(X)$$

$$\to K_\varphi(\alpha_1 - 1, n_1 - 1; D_1, D; S_1) \to 0\,,$$

where $K_\varphi(j, k; D_i, D; S_i)$ is the Kawai's sheaf associated to (D_i, D) and S_i of order (j, k) and $Ass_\varphi(D)$ (resp. $Ass_\varphi(D_i)$) is the sheaf associated to D (resp. D_i) with type $(n_1, \cdots, n_m; S_1, \cdots, S_m)$ (resp. $(\alpha_i; S_i)$). From this, we have

$$\begin{aligned} &\varphi^*\mathrm{ch}(Y) - \mathrm{ch}(X) \\ &= \mathrm{ch}(Ass_\varphi(D)) + \sum_{i=1}^{m} \mathrm{ch}(Ass_\varphi(D_i)) \\ &- \sum_{j=0}^{\alpha_1-2} \sum_{k=0}^{n_1-2} \mathrm{ch}\, K_\varphi(i, k; D_1, D; S_1) - \sum_{i=2}^{m} \sum_{j=0}^{\alpha_i-2} \sum_{k=0}^{n_i-1} \mathrm{ch}\, K_\varphi(j, k; D_i, D; S_i) \\ &+ \mathrm{ch}\, K_\varphi(\alpha_i - 1, n_1 - 1; D_1, D; S_1)\,. \end{aligned}$$

LEMMA 5.1.

$$\mathrm{ch}\, K_\varphi(k_1, \cdots, k_r; D_1, \cdots, D_r; S_j) = Be^{\varphi^* S_j} \prod_{i=1}^{r} \left(\frac{1 - e^{-D_i}}{D_i} \right) e^{-k_i D_i}\,, \tag{38}$$

where the notation is the same as in Definition 3.1.

Proof. Applying the Grothendieck-Riemann-Roch theorem (see e.g. Hirzebruch [Hz], Fulton [F1]) to the injection λ_B, we see that

$$\mathrm{ch}(\lambda_B)_* \mathcal{O}_B = (\lambda_B)_*(\mathrm{ch}\, \mathcal{O}_B \cdot \mathrm{td}(\nu(B))^{-1})\,,$$

where $\mathrm{td}(\nu(B))$ is the Todd class of the normal bundle $\nu(B)$ of B in X. Note that in this case,

$$\nu(B) = \bar{\nu}(D_1) \oplus \cdots \oplus \bar{\nu}(D_r)\,,$$

where $\bar{\nu}(D_i)$ is the restriction of $\nu(D_i)$ to B. Using the projection formula, we can prove that

$$\mathrm{ch}(\lambda_B)_* \mathcal{O}_B = B \prod_{i=1}^{r} \left(\frac{1 - e^{D_i}}{D_i} \right).$$

It is easy to check that

$$\mathrm{ch}\,\varphi^*[S_j]=e^{\varphi^*S_j} \quad \text{and} \quad \mathrm{ch}([D_i]^*)^{\otimes k_i}=e^{-k_iD_i}\,.$$

From these relations, we have (38). This proves our lemma.

Since, by Lemma 5.1,

$$\mathrm{ch}\,K_\varphi(k;\,D;\,S_j)=De^{\varphi^*S_j}\,\frac{1-e^{-D}}{D}\,e^{-kD}\,,$$

we see that

$$\mathrm{ch}(Ass_\varphi D)=e^{\varphi^*S_1}(1-e^{-(n_1-1)D})+\sum_{i=2}^{m}e^{\varphi^*S_j}(1-e^{-n_iD})\,. \tag{39}$$

Similarly we see that

$$\mathrm{ch}(Ass_\varphi D_i)=e^{\varphi^*S_i}(1-e^{-(\alpha_i-1)D_i})\,. \tag{40}$$

Moreover, using Lemma 5.1 again, we have

$$\mathrm{ch}\,K_\varphi(j;\,k;\,D_i,\,D;\,S_i)=D_iDe^{\varphi^*S_i}\,\frac{1-e^{D_i}}{D_i}\,\frac{1-e^{-D}}{D}\,e^{-jD_i}e^{-kD}\,.$$

From this, by an easy computation, we can prove that

$$\sum_{j=0}^{\alpha_1-2}\sum_{k=0}^{n_1-2}\mathrm{ch}\,K_\varphi(j,\,k;\,D_1,\,D;\,S_1)=e^{\varphi^*S_1}(1-e^{-(\alpha_1-1)D_1})(1-e^{-(n_1-1)D})\,, \tag{41}$$

$$\sum_{j=0}^{\alpha_i-2}\sum_{k=0}^{n_i-1}\mathrm{ch}\,K_\varphi(j,\,k;\,D_i,\,D;\,S_i)=e^{\varphi^*S_i}(1-e^{-(\alpha_i-1)D_i})(1-e^{-n_iD}) \tag{42}$$

and

$$\mathrm{ch}\,K_\varphi(\alpha_1-1,\,n_i-1;\,D_1,\,D;\,S_1)=e^{\varphi^*S_1}[e^{-((\alpha_1-1)D_1+(n_1-1)D}-e^{-(\alpha_1D_1+(n_1-1)D)}$$
$$-e^{-((\alpha_1-1)D_1+n_1D)}+e^{-(\alpha_1D_1+n_1D)}]\,. \tag{43}$$

Note that in the case in which $n_1=1$ or $\alpha_i=1$ for some i, formulas (39), (40), (41) and (42) hold. Now, since the map φ is expressed as in (1), it is easy to see that $\varphi^*S_i=\alpha_iD_i+n_iD$. So, combining (39), (40), (41), (42) and (43) with (37), we can verify the formula (2). Using (ii) in Theorem 4.2, the same argument as above yields the formula (3).

6. Proof of Theorem 1.2

As an application of Theorem 1.1, first we shall prove Theorem 1.2. In this case we have $\varphi^*S_i=\alpha_iD_i$. Therefore, since $\varphi^*S_i-(\alpha_i-1)D_i=D_i$, it follows from the formula (3) in Theorem 1.1 that

$$\varphi^*\mathrm{ch}(Y)-\mathrm{ch}(X)=\sum_{i=1}^{m}(e^{\varphi^*S_i}-e^{D_i})\,.$$

Equating terms of degree 1, (4) is obvious. Next equating terms of degree 2 and using (4), the formula (5) is easily proved by a simple calculation. This completes the proof of Theorem 1.2.

7. Proof of Theorem 1.3 and Examples

Let D_i be the proper transform of S_i by φ. Then we have $\varphi^*S_i = D_i + D$. By the formula (2) in Theorem 1.1, we see that

$$\varphi^*\mathrm{ch}(Y) - \mathrm{ch}(X) = \sum_{r=1}^{n} \frac{1}{r!} E_r ,$$

where

$$E_r = \sum_{i=1}^{m} (\varphi^*S_i)^r - (\varphi^*S_i - D)^r - D^r .$$

In particular, for $r = 1, 2, 3$, we can write

$$E_1 = (m-1)D , \tag{44}$$

$$E_2 = 2\varphi^*\left(\sum_{i=1}^{m} S_i \right) D - (m+1)D^2 , \tag{45}$$

$$E_3 = 3\varphi^*\left(\sum_{i=1}^{m} S_i^2 \right) D - 3\varphi^*\left(\sum_{i=1}^{m} S_i \right) D^2 + (m-1)D^3 . \tag{46}$$

Therefore, equating terms of degree 1, (6) follows from (44). For terms of degree 2, using (6) and (45), we can verify (7). Finally, we equate terms of degree 3. Combining (6), (7) and (46), (8) is proved by a simple computation. This completes the proof of Theorem 1.3.

Now we shall give some examples.

Example 7.1. For $m = 2$, since it holds that $\varphi^*(S_1 + S_2)D \equiv \varphi^*(S_1S_2) + D^2$, one has

$$c_2(X) \equiv \varphi^*c_2(Y) - \varphi^*c_1(Y)D + \varphi^*(S_1S_2) ,$$

$$c_3(X) \equiv \varphi^*c_3(Y) + \varphi^*(c_1(Y)S_1S_2) - \varphi^*(c_2(Y) + S_1^2 + S_2^2) + D^3 ,$$

where $\equiv$ denotes the numerically equivalence.

Example 7.2. For $m = n$, note that if $r-1 \geq j \geq 1$, then $\varphi^*(\alpha)D^j \equiv 0$ for any $(r-j)$-cycle α on Y. From this, one has

$$c_2(X) \equiv \varphi^*c_2(Y) + \frac{n(n-3)}{2} D^2 ,$$

$$c_3(X) \equiv \varphi^*c_3(Y) - \frac{n(n-1)(n-5)}{6} D^3 .$$

In particular, in the case in which $n=3$, the formulas for $c_2(X)$ in Examples 7.1 and 7.2 are numerically equivalent to the formulas which can be found in Griffiths and Harris [GH, pp. 609–610].

8. Proof of Theorem 1.4

Let $\varphi: X\to \boldsymbol{P}^n$ be a finite covering of $\boldsymbol{P}^n$ branched along S and assume that X is nonsingular. Now note that $c_1(\boldsymbol{P}^n)=(n+1)H$ and $c_2(\boldsymbol{P}^n)=n(n+1)H^2/2$, where H is a hyperplane in $\boldsymbol{P}^n$. We apply the formula (3) in Theorem 1.1 to the case in which $Y=\boldsymbol{P}^n$. Since in this case we have $\alpha_{ij}=\alpha_i$ $(j=1, \cdots, r_i)$, we see that $\varphi^*(S_i)=\sum_j \alpha_i D_{ij}$. Equating terms of degree 1, we have

$$K_X=\varphi^*\left(\left(\sum_{i=1}^{m}\frac{\alpha_i-1}{\alpha_i}n_i-(n+1)\right)H\right), \tag{47}$$

where n_i is the degree of S_i. Moreover, using (47),

$$\varphi^*(H^2)(K_X)^{n-2}=\left(\sum_{i=1}^{m}\left(\frac{\alpha_i-1}{\alpha_i}\right)n_i-(n+1)\right)^{n-2}\varphi^*(H^n). \tag{48}$$

Next equating terms of degree 2, we have

$$2c_2(X)-c_1^2(X)=\varphi^*(2c_2(\boldsymbol{P}^n)-c_1^2(\boldsymbol{P}^n))+\sum_{i=1}^{m}(\alpha_i^2-1)D_i^2. \tag{49}$$

Using (49) together with (48), we obtain

$$\begin{aligned}&(2c_2(X)-c_1^2(X))K_X^{n-2}\\&=\deg\varphi\left(\sum_{i=1}^{m}\left(\frac{\alpha_i^2-1}{\alpha_i^2}\right)n_i^2-(n+1)\right)\left(\sum_{i=1}^{m}\left(\frac{\alpha_i-1}{\alpha_i}\right)n_i-(n+1)\right)^{n-2}.\end{aligned}$$

Now, using the projection formula, it follows from (47) that K_X is numerically effective and $K_X^n>0$ if and only if the relation $\sum_{i=1}^m \alpha_1\cdots(\alpha_i-1)\cdots\alpha_m n_i-l(n+1)>0$ holds. The inequality (9) follows from the above equality and this. This completes the proof of Theorem 1.4.

9. Proof of Theorem 1.5

We start by recalling the definition of the ν-th Kawai covering of $\boldsymbol{P}^n$ associated with S, which was introduced in [Ka]. Let H be a subgroup of the fundamental group $\pi_1(\boldsymbol{P}^n-S)$ of $\boldsymbol{P}^n-S$ generated by $\nu n_1\gamma_1, \cdots, \nu n_m\gamma_m$ with $2\le m\le n-1$, where ν is an integer and γ_i is the class of a positively oriented little loop around S_i. Let $\psi: Z\to \boldsymbol{P}^n$ be the covering associated to the subgroup H and X be the nonsingular model of Z constructed by Kawai in [Ka]. We denote by $\varphi: X\to\boldsymbol{P}^n$ the composition map of the desingularization of Z and the covering ψ, which is called the *ν-th Kawai covering associated with S*. By virtue of Theorem 1 in [Ka], it is easily checked that

$\varphi^{-1}(S)$ can be written as a union $\varphi^{-1}(S)=D\cup D_1\cup\cdots\cup D_m$ of irreducible components, where D is isomorphic to $C\times \boldsymbol{P}^{m-1}$ $(C=S_1\cap\cdots\cap S_m)$ and D_i is the proper transform of S_i by φ. Moreover we infer readily that the restriction map of φ to $X-\varphi^{-1}(S)$ is locally biholomorphic and the map φ satisfies the monomial condition over S and that D is a contraction locus of φ with type $(n_1,\cdots,n_m; S_1,\cdots,S_m)$ and each D_i is a ramification locus of φ with type $(\nu n_i; S_i)$. Moreover we have $D_1\cap\cdots\cap D_m=\varnothing$.

Now, by the formula (2) in Theorem 1.1, we have

$$\varphi^*S_i=\nu n_iD_i+n_iD\,, \tag{50}$$

$$\varphi^*\mathrm{ch}(\boldsymbol{P}^n)-\mathrm{ch}(X)=\sum_{r=1}^{n}E_r\,,$$

where

$$E_r=\frac{1}{r!}\left[\sum_{i=1}^{m}(\varphi^*S_i)^r-\left(\sum_{i=1}^{m}D_i^r+D^r\right)\right].$$

From this, equating terms of degree 1 and noting that

$$D_1\equiv\cdots\equiv D_m\,,$$

where $\equiv$ denotes numerically equivalence, we have

$$K_X\equiv(\nu(d-n-2)+(\nu-m))D_1+(d-n-2)D\,, \tag{51}$$

where $d=\sum n_i$ is the degree of S. Moreover, since $c_1^2(\boldsymbol{P}^n)-2c_2(\boldsymbol{P}^n)=(n+1)H^2$, equating terms of degree 2, we have

$$c_1^2(X)-2c_2(X)=\sum_{i=1}^{m}(n_i^2(\nu D_i+D)^2-D_i^2)-D^2-(n+1)\varphi^*(H^2)\,.$$

From this, we get

$$c_1^2(X)-2c_2(X)\equiv\left(\sum_{i=1}^{m}n_i^2-n-2\right)(\nu D_1+D)^2+(\nu^2-m)D_1^2+2\nu D_1D\,. \tag{52}$$

LEMMA 9.1. *We have the following intersection formula*

$$D_1^{m-i}D^{n-m+i}=(-\nu)^{i-1}\binom{n-m+i-1}{n-m}n_1\cdots n_m \qquad (i=1,\cdots,m)\,,$$

where $\binom{n-m+i-1}{n-m}$ *is the binomial coefficient.*

Proof. From (50), we have $\varphi^*S_1=\nu n_1D_1+n_1D$. First, note that $\deg\varphi=\nu^{m-1}n_1\cdots n_m$. By the relation

$$\varphi^*(S_1)^n=n_1^n\varphi^*(H^n)=n_1^n\deg\varphi\,,$$

we obtain

$$(vD_1+D)^n=v^{m-1}n_1\cdots n_m\,. \tag{53}$$

Furthermore, for $i=1,\cdots,m-1$, since $\dim\varphi_*(D_1^{i-1}D)=\dim C=n-m$, by the projection formula, one has

$$\varphi^*(S_1^{n-i})D_1^{i-1}D=0\,.$$

So we obtain

$$(vD_1+D)^{n-i}D_1^{i-1}D=0 \qquad (i=1,\cdots,m-1)\,. \tag{54}$$

Now, noting that $D_1\cap\cdots\cap D_m=\varnothing$, we have $D_1^mD^{n-m}=D_1^{m+1}D^{n-m-1}=\cdots=D_1^n=0$. Therefore, putting $X_i=(vD_1)^{m-i}D^{n-m+i}$ $(i=1,\cdots,m)$, we see from (53) and (54) that X_i $(i=1,\cdots,m)$ satisfy the system of equations

$$\binom{n}{n-m+1}X_1+\cdots+\binom{n}{n}X_m=v^{m-1}n_1\cdots n_m \tag{55}$$

and

$$\binom{n-i}{n-m}X_1+\cdots+\binom{n-i}{n-i}X_{m-i+1}=0 \qquad (i=1,\cdots,m-1)\,. \tag{56}$$

It is easy to check that (55) and (56) are equivalent to the system of equations

$$X_1=v^{m-1}n_1\cdots n_m \quad\text{and} \tag{57}$$

$$\binom{k}{n-m}X_1+\binom{k}{n-m+1}X_2+\cdots+\binom{k}{k}X_{k-n+m+1}=0$$

$$(k=n-m+1,\cdots,n-1)\,. \tag{58}$$

Then, by induction on k, we can show that $X_i=(-1)^{i-1}\binom{n-m+i-1}{n-m}v^{m-1}n_1\cdots n_m$ $(i=1,\cdots,m)$ are solutions of (57) and (58). Hence we can prove Lemma 9.1.

Now we put $k=v(d-n-2)+(v-m)$, $l=d-n-2$ and $L=\sum n_i^2-n-2$. Then, by (51) and (52), we see that

$$K_X\equiv kD_1+lD\,, \tag{59}$$

$$c_1^2(X)-2c_2(X)\equiv L(vD_1+D)^2+(v^2-m)D^2+2vD_1D\,. \tag{60}$$

Moreover, note that $D_1^iD^{n-i}=0$ for $n\le i$. From (59), we obtain

$$K_X^{n-2}\equiv\sum_{i=0}^{m-1}\binom{n-2}{n-m-1+i}k^{m-1-i}l^{n-m-1+i}D_1^{m-1-i}D^{n-m-1+i}\,.$$

Using Lemma 9.1 and this, we have

$$(vD_1+D)^2K_X^{n-2}=n_1\cdots n_m\sum_{i=0}^{m-1}\binom{n-2}{n-m-1+i}(-v)^i k^{m-1-i}$$
$$\times l^{n-m-1+i}\left(\binom{n-m+i-2}{n-m}-2\binom{n-m+i-1}{n-m}+\binom{n-m+i}{n-m}\right),$$

where $\binom{r}{n-m}=0$ $(r\leq n-m-1)$. Since it holds that

$$\binom{n-m+i-2}{n-m}-2\binom{n-m+i-1}{n-m}+\binom{n-m+i}{n-m}$$
$$=\begin{cases}\binom{n-m+i-2}{n-m-2} & (m\leq n-2,\ i\geq 0),\\ 1 & (m=n-1,\ i=0),\\ 0 & (m=n-1,\ i\geq 1),\end{cases}$$

we obtain

$$(vD_1+D)^2K_X^{n-2}$$
$$=\begin{cases}n_1\cdots n_m\sum_{i=0}^{m-1}\binom{n-2}{n-m+i-1}\binom{n-m+i-2}{n-m-2}(-v)^i k^{m-i-1}l^{n-m+i-1} & (m\leq n-2),\\ k^{n-2}n_1\cdots n_{n-1} & (m=n-1).\end{cases} \tag{61}$$

Furthermore, for $m\leq n-2$, using the relation

$$\binom{n-2}{n-m+i-1}=\sum_{j=n-m+i-2}^{n-3}\binom{j}{n-m+i-2},$$

we see that

$$(vD_1+D)^2K_X^{n-2}=n_1\cdots n_m l^{n-m-1}\sum_{j=n-m-2}^{n-3}\ \sum_{i=0}^{j-(n-m-2)}\binom{j}{n-m+i-2}$$
$$\times\binom{n-m+i-2}{n-m-2}k^{m-i-1}(-vl)^i.$$

From this, using the following relation

$$\binom{j}{n-m+i-2}\times\binom{n-m+i-2}{n-m-2}=\binom{j}{n-m-2}\times\binom{j-(n-m-2)}{i},$$

we obtain

$$(vD_1+D)^2K_X^{n-2}=n_1\cdots n_m l^{n-m-1}$$
$$\times\sum_{j=n-m-2}^{n-3}\binom{j}{n-m-2}k^{n-j-3}(k-vl)^{j-(n-m-2)}. \tag{62}$$

By the same argument as above, we can verify that

$$D_1^2 K_X^{n-2} = \begin{cases} n_1 \cdots n_m l^{n-m+1} \sum_{j=n-m}^{n-3} k^{n-j-3} \binom{j}{n-m} (k-\nu l)^{j-(n-m)} & (m \geq 3), \\ 0 & (m=2) \end{cases} \tag{63}$$

and

$$D_1 D K_X^{n-2} = n_1 \cdots n_m l^{n-m} \binom{n-2}{m-2} (k-\nu l)^{m-2}. \tag{64}$$

Note that $k-\nu l=\nu-m$. Using (60), we have

$$\begin{aligned}&(c_1^2(X)-2c_2(X))K_X^{n-2}\\ &=L(\nu D_1+D)^2 K_X^{n-2}+(\nu^2-m)D_1^2 K_X^{n-2}+2\nu D_1 D K_X^{n-2}\end{aligned} \tag{65}$$

Now, according to a result of [Fj], we see that K_X is numerically effective and $K_X^n>0$ if and only if $\nu \geq m$ and $d>n+2$. Therefore, in this case, combining (61), (62), (63), (64) with (65), we can prove (10). This completes the proof of Theorem 1.5.

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