

Biharmonic Products in the Normal Bundle

by

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Abstract. In this paper we characterize the product of submanifolds whose quadric representation φ satisfies $\Delta^\perp H_\varphi = \lambda H_\varphi$, for a real constant λ , where H_φ is the mean curvature vector field of φ . We also characterize the product of submanifolds whose quadric representation has mean curvature vector field proper for the Jacobi operator in terms of conditions on the factors of the product. As for curves and surfaces, a complete classification is given.

1. Introduction

In a recent paper, [8], we have introduced a representation of a product of indefinite Riemannian manifolds in a certain space of selfadjoint matrices. It is the so called quadric representation φ . This is a nice tool to study the geometry of submanifolds, into indefinite space forms, as determined by the geometry of their product isometrically immersed into the corresponding pseudo-Euclidean space. Actually, we found a characterization of indefinite Riemannian products whose quadric representation satisfies $\Delta H_\varphi = \lambda H_\varphi$, Δ being the Laplacian operator on the product and H_φ the mean curvature vector field of the quadric representation. Thus, in particular, for surfaces into De Sitter $S_1^3(1)$ and anti De Sitter $H_1^3(-1)$ spaces we have shown that such equation only holds either for products of minimal surfaces or products of a minimal surface and one of the following surfaces: $H^2(-1)$, $S^1(2/3) \times H^1(-2)$, $S_1^2(1)$, $S^1(2) \times H_1^1(-2/3)$, $S_1^1(2) \times H^1(-2/3)$ and a B -scroll over a null Frenet curve with torsion $\pm\sqrt{2}$.

The problem of finding non minimal submanifolds with harmonic mean curvature vector field was first stated and studied by B. Y. Chen (see [6]), who proposed to call them biharmonic submanifolds. Then, in [8], some contributions in that line are given. However, the condition $\Delta H_\varphi = \lambda H_\varphi$ always yields to hypersurfaces with both constant mean and scalar curvatures. Since the mean curvature vector field lies in the normal bundle, it seems natural to work with the normal Laplacian and study

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the new equation $\Delta^\perp H_\varphi = \lambda H_\varphi$. Now one might think that a characterization of biharmonic submanifolds in the normal bundle should improve that just obtained with regard to the Laplacian. In fact, $\Delta^\perp H_\varphi = \lambda H_\varphi$ implies that λ should vanish. This and the fact that the constancy neither the mean curvature nor scalar curvature can be asserted point out substantial differences between $\Delta^\perp H_\varphi = \lambda H_\varphi$ and $\Delta H_\varphi = \lambda H_\varphi$.

On the other hand, it is well known that the Jacobi operator J is defined on the normal bundle in terms of the normal Laplacian and the Simons operator (see [11]). Then it seems natural to ask for submanifolds whose quadric representation satisfies $JH_\varphi = \lambda H_\varphi$. We characterize the indefinite products hypersurfaces satisfying this equation finding that they have to be of constant mean and constant scalar curvatures with an appropriate relation between them. In particular, explicit formulas for the mean and scalar curvatures of the hypersurfaces can be given provided that their quadric representation have Jacobi mean curvature vector field (i.e., $JH_\varphi = 0$). A complete classification is achieved when we consider surfaces into De Sitter S_1^3 and anti De Sitter H_1^3 spaces. It is worth pointing out that, for surfaces into S_1^3 and H_1^3 , the characterizations obtained here from $JH_\varphi = \lambda H_\varphi$ have nothing to do with those got in [8] coming from $\Delta H_\varphi = \lambda H_\varphi$.

2. Preliminaries

Let $\bar{M}_\mu^{m+1}(k)$ and $\bar{N}_\nu^{n+1}(k)$ be two hyperquadrics of constant curvature k standardly embedded in \mathbb{R}_t^{m+2} and \mathbb{R}_s^{n+2} , respectively. We can define an immersion f from the pseudo-Riemannian product $\bar{M}_\mu^{m+1}(k) \times \bar{N}_\nu^{n+1}(k)$ into the space of real $(m+2) \times (n+2)$ matrices \mathfrak{M} by

$$\begin{aligned} f : \bar{M}_\mu^{m+1}(k) \times \bar{N}_\nu^{n+1}(k) &\longrightarrow \mathfrak{M} \\ (p, q) &\longrightarrow p \otimes q \end{aligned}$$

where $\otimes : \mathbb{R}_t^{m+2} \times \mathbb{R}_s^{n+2} \rightarrow \mathfrak{M}$ is given by $u \otimes v = G_1 u v^t G_2$, G_1 and G_2 standing for the matrices of the standard metrics on \mathbb{R}_t^{m+2} and \mathbb{R}_s^{n+2} , respectively.

Some properties of the immersion f are listed below (see [8]). Let (p, q) be a point in $\bar{M}_\mu^{m+1} \times \bar{N}_\nu^{n+1}$ and (X, Y) a vector field on $\bar{M}_\mu^{m+1} \times \bar{N}_\nu^{n+1}$, then

$$df_{(p,q)}(X_p, Y_q) = X_p \otimes q + p \otimes Y_q.$$

Let $\bar{\nabla}$ be the usual flat connection on \mathfrak{M} and (V, W) a vector field on $\bar{M}_\mu^{m+1} \times \bar{N}_\nu^{n+1}$. Then for the covariant derivative we have

$$\begin{aligned} \bar{\nabla}_{df(X,Y)} df(V, W) &= df(\nabla_X^1 V, \nabla_Y^2 W) + V \otimes Y + X \otimes W \\ &\quad - k(\langle X, V \rangle + \langle Y, W \rangle) f, \end{aligned} \tag{2.1}$$

where $\bar{\nabla}^1$ and $\bar{\nabla}^2$ are the usual flat connections on \mathbb{R}_t^{m+2} and \mathbb{R}_s^{n+2} , respectively, and ∇^1 and ∇^2 are the Levi-Civita connections on \bar{M}_μ^{m+1} and \bar{N}_ν^{n+1} , respectively.

Let \tilde{g} be the metric on \mathfrak{M} defined by $\tilde{g}(A, B) = k \operatorname{tr}(G_1 A G_2 B^t)$, for $A, B \in \mathfrak{M}$. Notice that \mathfrak{M} , endowed with \tilde{g} , is isometric to a pseudo-Euclidean space of index

$t(n+2)+s(m+2)-2st$ or $(m+2)(n+2)-t(n+2)-s(m+2)-2st$, provided that $k > 0$ or $k < 0$, respectively. Then

$$\tilde{g}(X \otimes V, Y \otimes W) = k \langle X, Y \rangle \langle V, W \rangle \tag{2.2}$$

and therefore f is an isometric immersion. Now, a straightforward computation from (2.1) allows us to obtain the second fundamental form $\tilde{\sigma}$ of f

$$\tilde{\sigma}((X, Y), (V, W)) = V \otimes Y + X \otimes W - k(\langle X, V \rangle + \langle Y, W \rangle)f. \tag{2.3}$$

Let $x: M_c^j \rightarrow \bar{M}_\mu^{m+1} \subset \mathbb{R}_t^{m+2}$ and $y: N_d^l \rightarrow \bar{N}_s^{n+1} \subset \mathbb{R}_s^{n+2}$ be two isometric immersions and let $\varphi: M_c^j \times N_d^l \rightarrow \mathfrak{M}$ be the isometric immersion defined by $\varphi(p, q) = f(x(p), y(q))$. From now on, φ will be called the *quadric representation of the pseudo-Riemannian product immersion* (x, y) .

By letting H_x and H_y the mean curvature vector fields of x and y , respectively, and \bar{H}_x and \bar{H}_y the mean curvature vector fields in the corresponding pseudo-Euclidean ambient spaces, we have

$$\begin{aligned} \bar{H}_x &= H_x - kx, & \bar{H}_y &= H_y - ky, \\ (j+l)H_\varphi &= j\bar{H}_x \otimes y + lx \otimes \bar{H}_y, \end{aligned} \tag{2.4}$$

where H_φ is the mean curvature vector field associated to φ .

If (X, Y) is a vector field along (x, y) then

$$(X \otimes Y)^T = k \langle X, x \rangle x \otimes Y^T + \langle Y, y \rangle X^T \otimes y, \tag{2.5}$$

where $()^T$ denotes the tangential component.

3. Products in \mathfrak{M} with mean curvature vector field proper for the normal Laplacian

In this section we are going to study the equation

$$\Delta^\perp H_\varphi = \lambda H_\varphi, \quad \lambda \in \mathbb{R}, \tag{3.1}$$

where Δ^\perp is the Laplacian operator in the normal bundle of $M_c^j \times N_d^l$ in \mathfrak{M} .

Let (p, q) be a point in $M_c^j \times N_d^l$ and choose local orthonormal frames $\{E_1, \dots, E_j\}$ and $\{F_1, \dots, F_l\}$ on M_c^j and N_d^l , respectively, such that $\nabla_{E_\alpha}^x E_\alpha(p) = 0$, for all $\alpha = 1, \dots, j$, and $\nabla_{F_\beta}^y F_\beta(q) = 0$, for all $\beta = 1, \dots, l$, where ∇^x and ∇^y are the Levi-Civita connections on M_c^j and N_d^l , respectively.

Taking covariant derivative in (2.4), we obtain at (p, q) that

$$\tilde{\nabla}_{df(E_i, 0)} H_\varphi = \frac{1}{(j+l)} (j \tilde{\nabla}_{E_i}^\perp \bar{H}_x \otimes y + l E_i \otimes \bar{H}_y),$$

by using (2.5), we get the normal component,

$$\tilde{\nabla}_{df(E_i, 0)}^\perp H_\varphi = \frac{1}{(j+l)} (j \tilde{\nabla}_{E_i}^{\perp 1} \bar{H}_x \otimes y + l E_i \otimes H_y).$$

By derivating here and taking the normal component we have

$$\begin{aligned}\tilde{\nabla}_{df(E_i,0)}^\perp \tilde{\nabla}_{df(E_i,0)}^\perp H_\varphi &= \frac{1}{(j+l)} (j\bar{\nabla}_{E_i}^{1\perp} \bar{\nabla}_{E_i}^{1\perp} \bar{H}_x \otimes y + l\bar{\nabla}_{E_i}^1 E_i \otimes H_y) \\ &= \frac{1}{(j+l)} (j\bar{\nabla}_{E_i}^{1\perp} \bar{\nabla}_{E_i}^{1\perp} \bar{H}_x \otimes y + l\bar{\sigma}_x(E_i, E_i) \otimes H_y),\end{aligned}$$

where $\bar{\sigma}_x$ is the second fundamental form of M_c^j in \mathbb{R}_t^{m+2} . Analogously

$$\tilde{\nabla}_{df(0,F_i)}^\perp \tilde{\nabla}_{df(0,F_i)}^\perp H_\varphi = \frac{1}{(j+l)} (jH_x \otimes \bar{\sigma}_y(F_i, F_i) + lx \otimes \bar{\nabla}_{F_i}^{2\perp} \bar{\nabla}_{F_i}^{2\perp} \bar{H}_y).$$

Therefore we have

$$\Delta^\perp H_\varphi = \frac{1}{(j+l)} (j\Delta^\perp \bar{H}_x \otimes y + lx \otimes \Delta^\perp \bar{H}_y - jl(\bar{H}_x \otimes H_y + H_x \otimes \bar{H}_y)). \quad (3.2)$$

By assuming that $\Delta^\perp H_\varphi = \lambda H_\varphi$, from (3.2) we deduce that

$$j\Delta^\perp \bar{H}_x \otimes y + lx \otimes \Delta^\perp \bar{H}_y - jl(\bar{H}_x \otimes H_y + H_x \otimes \bar{H}_y) = \lambda(j\bar{H}_x \otimes y + lx \otimes \bar{H}_y).$$

Since this can be viewed as an endomorphism on \mathbb{R}_s^{n+2} , we apply it to y to get

$$jk^{-1}G_1\Delta^\perp \bar{H}_x + l\langle \Delta^\perp \bar{H}_y, y \rangle G_1x - jl\langle \bar{H}_y, y \rangle G_1H_x = \lambda jk^{-1}G_1\bar{H}_x + \lambda l\langle \bar{H}_y, y \rangle G_1x.$$

As $\langle \bar{H}_y, y \rangle = -1$ and G_1 is invertible, the above equation writes as

$$jk^{-1}\Delta^\perp \bar{H}_x + j(l - \lambda k^{-1})\bar{H}_x + l(\lambda + kj + \langle \Delta^\perp \bar{H}_y, y \rangle)x = 0. \quad (3.3)$$

A similar reasoning by applying to x^t leads to

$$lk^{-1}\Delta^\perp \bar{H}_y + l(j - \lambda k^{-1})\bar{H}_y + j(\lambda + kl + \langle \Delta^\perp \bar{H}_x, x \rangle)y = 0. \quad (3.4)$$

By multiplying now the above equation by y we find

$$j\langle \Delta^\perp \bar{H}_x, x \rangle + l\langle \Delta^\perp \bar{H}_y, y \rangle + \lambda(j+l) = 0. \quad (3.5)$$

Bringing (3.3) and (3.4) to $\Delta^\perp H_\varphi = \lambda H_\varphi$, bearing in mind (3.5), we obtain $H_x \otimes H_y = 0$. This equation yields $H_x = 0$ or $H_y = 0$ or both simultaneously. Now suppose that $H_x \neq 0$, then $\Delta^\perp \bar{H}_y = 0$ and so (3.3) can be rewritten as

$$jk^{-1}\Delta^\perp \bar{H}_x + j(l - \lambda k^{-1})\bar{H}_x + l(\lambda + kj)x = 0.$$

Since $\Delta^\perp x = 0$ and \bar{M}_μ^{m+1} is umbilic in \mathbb{R}_s^{n+2} , then $\Delta^\perp \bar{H}_x = \Delta^\perp H_x$. So we deduce from the last equation that $\lambda = 0$ and

$$\Delta^\perp H_x + kH_x = 0.$$

A straightforward computation shows that if one immersion is minimal and the other one verifies the above equation, then $\Delta^\perp H_\varphi = 0$. Therefore, we have just proved the following theorem.

THEOREM 1. *Let $\varphi: M_c^j \times N_d^l \rightarrow \mathfrak{M}$ be the quadric representation of a pseudo-Riemannian product (x, y) . Then $\Delta^\perp H_\varphi = \lambda H_\varphi$ if and only if $\lambda = 0$ and one of the following statements holds:*

- (1) M_c^j is minimal in \bar{M}_μ^{m+1} and y verifies the equation $\Delta^\perp H_y = -kjH_y$.
- (2) N_d^l is minimal in \bar{N}_ν^{n+1} and x verifies the equation $\Delta^\perp H_x = -klH_x$.

In particular, if M_c^m and N_d^n are hypersurfaces in \bar{M}_μ^{m+1} and \bar{N}_ν^{n+1} , respectively, we have the following consequences.

COROLLARY 2. *Let $\varphi: M_c^m \times N_d^n \rightarrow \mathfrak{M}$ be the quadric representation of a pseudo-Riemannian product of hypersurfaces. Then $\Delta^\perp H_\varphi = \lambda H_\varphi$ if and only if $\lambda = 0$ and one of the following statements holds:*

- (1) M_c^m is minimal in \bar{M}_μ^{m+1} and y verifies the equation $\Delta\alpha_y = -k\alpha_y$, where α_y stands for the mean curvature of N_d^n in \bar{N}_ν^{n+1} .
- (2) N_d^n is minimal in \bar{N}_ν^{n+1} and x verifies the equation $\Delta\alpha_x = -k\alpha_x$, where α_x stands for the mean curvature of M_c^m in \bar{M}_μ^{m+1} .

COROLLARY 3. *There is no compact Riemannian product of hypersurfaces in $\bar{M}_\mu^{m+1} \times \bar{N}_\nu^{n+1}$, $k > 0$, with proper mean curvature vector field in \mathfrak{M} .*

The above results can be sharpened in low dimensions. Indeed, we can apply Corollary 2 to products of curves to obtain the following result (see [7]).

PROPOSITION 4. *Let $\gamma_1: I_1 \rightarrow \bar{M}_\mu^2$ and $\gamma_2: I_2 \rightarrow \bar{N}_\nu^2$ be two unit speed curves and $\varphi = \gamma_1 \otimes \gamma_2$ the quadric representation of the product. Then $\Delta^\perp H_\varphi = \lambda H_\varphi$ if and only if $\lambda = 0$, one curve is a geodesic and the other one has curvature ρ given by*

$$\begin{aligned} \rho(s) &= a \cos(\sqrt{-\varepsilon ks}) + b \sin(\sqrt{-\varepsilon ks}) && \text{if } \varepsilon k < 0, \\ \rho(s) &= a \cosh(\sqrt{\varepsilon ks}) + b \sinh(\sqrt{\varepsilon ks}) && \text{if } \varepsilon k > 0, \end{aligned}$$

where $a, b \in \mathbb{R}$ and ε is the character of the curve.

In [5] we have just constructed a new class of submanifolds in $\mathbb{H}_1^3(-1)$ defined by means of two semi-Riemannian submersions $\pi_s: \mathbb{H}_1^3(-1) \rightarrow \mathbb{H}_s^2(-1/4)$, $s = 0, 1$ (see details therein). By pulling back via π_s a non-null curve β in $\mathbb{H}_s^2(-1/4)$ we get the total horizontal lift of β , which is an immersed flat surface M_β in $\mathbb{H}_1^3(-1)$, that will be called the *semi-Riemannian Hopf cylinder associated to β* . Notice that if $s = 0$, M_β is a Lorentzian surface, whereas if $s = 1$, M_β is Riemannian or Lorentzian, according to β be spacelike or timelike, respectively.

Let $\beta: I \rightarrow \mathbb{H}_s^2(-1/4)$ be a unit speed curve with Frenet frame $\{T, \xi_2\}$ and curvature function ρ . Let $\bar{\beta}$ be a horizontal lift of β to $\mathbb{H}_1^3(-1)$ with Frenet frame $\{\bar{T}, \bar{\xi}_2, \xi_3^*\}$, curvature $\bar{\rho} = \rho \circ \pi_s$ and torsion $\tau = 1$. Recall that ξ_3^* is nothing but the unit tangent vector field to the fibers along $\bar{\beta}$. Then the Hopf Cylinder M_β can be orthogonally parametrized as

$$X(t, z) = \begin{cases} \cos(z)\bar{\beta}(t) + \sin(z)\xi_3^*(t) & \text{when } s = 0, \\ \cosh(z)\bar{\beta}(t) + \sinh(z)\xi_3^*(t) & \text{when } s = 1. \end{cases}$$

Notice that a unit normal vector field to M_β into $\mathbb{H}_1^3(-1)$ is obtained from the complete horizontal lift of ξ_2 and it is, of course, $\bar{\xi}_2$ along each horizontal lift of β . As a consequence we have that M_β is a flat surface and its mean curvature function α is given by $\alpha = \frac{1}{2}\bar{\rho}$.

Now from Theorem 1 and [7, Proposition 15] we have

PROPOSITION 5. *Let M_β be a Hopf cylinder into $\mathbb{H}_1^3(-1)$ and N_d^l a minimal submanifold in $\bar{N}_v^{n+1}(-1)$. Let φ be the quadric representation of $M_\beta \times N_d^l$. Then $\Delta^\perp H_\varphi = \lambda H_\varphi$ if and only if $\lambda = 0$ and one of the following statements holds:*

- 1) β is a timelike curve with $\rho(t) = a \cosh(\sqrt{l}t) + b \sinh(\sqrt{l}t)$.
- 2) β is a spacelike curve with $\rho(t) = a \cos(\sqrt{l}t) + b \sin(\sqrt{l}t)$.

Observe that this result produces infinitely many examples of products satisfying $\Delta^\perp H_\varphi = \lambda H_\varphi$ with non constant mean curvature in $\mathbb{H}_1^3(-1) \times \bar{N}_v^{n+1}(k)$.

4. Products in \mathfrak{M} with mean curvature vector field proper for the Jacobi operator

Let $f: M \rightarrow \tilde{M}$ be a pseudo-Riemannian immersion, with \tilde{M} of constant curvature c . Let J be the Jacobi operator associated to the immersion f defined by

$$J = \Delta^\perp - \tilde{A} - \dim(M)cI,$$

where \tilde{A} is the Simons operator defined by $\langle \tilde{A}\xi, \zeta \rangle = \text{tr}(A_\xi \circ A_\zeta)$, A_ξ being the Weingarten endomorphism associated to ξ , and I is the identity map on the normal bundle. In this section we want to characterize the products in \mathfrak{M} whose mean curvature vector field verifies the equation $JH_\varphi = \lambda H_\varphi$, $\lambda \in \mathbb{R}$, in terms of conditions on the immersions x and y .

By using a formula proved in [4] we have

$$JH_\varphi = 2\Delta^\perp H_\varphi - (\Delta H_\varphi)^\perp. \quad (4.1)$$

In [8] we found that

$$\Delta H_\varphi = \frac{1}{(j+l)} (j\Delta\bar{H}_x \otimes y + lx \otimes \Delta\bar{H}_y - 2jl\bar{H}_x \otimes \bar{H}_y),$$

and so, we get

$$(\Delta H_\varphi)^\perp = \frac{1}{(j+l)} (j(\Delta\bar{H}_x)^\perp \otimes y + lx \otimes (\Delta\bar{H}_y)^\perp - 2jl\bar{H}_x \otimes \bar{H}_y).$$

Bringing (3.2) and the above equation to (4.1) we obtain

$$JH_\varphi = \frac{1}{(j+l)} (j\bar{J}_x\bar{H}_x \otimes y + lx \otimes \bar{J}_y\bar{H}_y - 2jl(H_x \otimes H_y - k^2x \otimes y)), \quad (4.2)$$

where \bar{J}_x and \bar{J}_y are the Simons operators associated to the immersions $x: M_c^j \rightarrow \mathbb{R}_s^{n+2}$

and $y: N_d^l \rightarrow \mathbb{R}_t^{m+2}$, respectively.

By assuming that $JH_\phi = \lambda H_\phi$, from (4.2) we have

$$j\bar{J}_x\bar{H}_x \otimes y + lx \otimes \bar{J}_y\bar{H}_y - 2jl(H_x \otimes H_y - k^2x \otimes y) = \lambda(j\bar{H}_x \otimes y + lx \otimes \bar{H}_y),$$

which we apply to y to get

$$k^{-1}jG_1\bar{J}_x\bar{H}_x + l\langle \bar{J}_y\bar{H}_y, y \rangle G_1x + 2kjlG_1x = \lambda k^{-1}jG_1\bar{H}_x - \lambda lG_1x.$$

As G_1 is invertible, the above equation writes as

$$j\bar{J}_x\bar{H}_x = \lambda j\bar{H}_x - (2k^2jl + \lambda kl + kl\langle \bar{J}_y\bar{H}_y, y \rangle)x. \quad (4.3)$$

A similar reasoning by applying to x^t leads to

$$l\bar{J}_y\bar{H}_y = \lambda l\bar{H}_y - (2k^2jl + \lambda kj + kj\langle \bar{J}_x\bar{H}_x, x \rangle)y. \quad (4.4)$$

By multiplying now the above equation by y we find

$$j\langle \bar{J}_x\bar{H}_x, x \rangle + l\langle \bar{J}_y\bar{H}_y, y \rangle = -\lambda(j+l) - 2kjl.$$

Bringing (4.3) and (4.4) to $JH_\phi = \lambda H_\phi$, using the above relation, we obtain $H_x \otimes H_y = 0$. This equation yields $H_x = 0$ or $H_y = 0$. Now if we suppose $H_x \neq 0$, then $\bar{J}_y\bar{H}_y = ly$ and so (4.3) can be rewritten as

$$j\bar{J}_x\bar{H}_x = \lambda j\bar{H}_x - (2k^2jl + \lambda kl + k^2l^2)x.$$

Now, let J_x be the Jacobi operator associated to the immersion $x: M_c^j \rightarrow \bar{M}_\mu^{m+1}$. By a straightforward computation we get

$$\bar{J}_x\bar{H}_x = J_xH_x + kj\langle \bar{H}_x, \bar{H}_x \rangle x.$$

Therefore, from the last two equations, we deduce that

$$J_xH_x = \lambda H_x, \quad \text{with } \lambda = -\frac{j^2}{j+l}\langle H_x, H_x \rangle - k(j+l). \quad (4.5)$$

As a consequence $\langle H_x, H_x \rangle$ is constant. It is easy to see that if one immersion is minimal and the other one verifies the above two equations then $JH_\phi = \lambda H_\phi$. If both submanifolds are minimal then a direct computation shows that H_ϕ satisfies the required equation with $\lambda = -k(j+l)$. So we have proved the following result.

THEOREM 6. *Let $\phi: M_c^j \times N_d^l \rightarrow \mathfrak{M}$ be the quadric representation of a pseudo-Riemannian product (x, y) . Then $JH_\phi = \lambda H_\phi$ if and only if one of the following statements holds:*

- (1) M_c^j is minimal in \bar{M}_μ^{m+1} and y verifies the equation $J_yH_y = \lambda H_y$ with $\lambda = (-l^2/(j+l))\langle H_y, H_y \rangle - k(j+l)$ and $\langle H_y, H_y \rangle$ constant.
- (2) N_d^l is minimal in \bar{N}_ν^{n+1} and x verifies the equation $J_xH_x = \lambda H_x$ with $\lambda = (-j^2/(j+l))\langle H_x, H_x \rangle - k(j+l)$ and $\langle H_x, H_x \rangle$ constant.

Now, let M_c^m and N_d^n be hypersurfaces in \bar{M}_μ^{m+1} and \bar{N}_ν^{n+1} , respectively. Suppose that $H_x \neq 0$, then from (4.5) we obtain

$$\varepsilon_x m^2 \alpha_x^2 = -(m+n)(\lambda + k(m+n)),$$

where ε_x and α_x are the sign and mean curvature of M_c^m in \bar{M}_μ^{m+1} . Therefore α_x is constant. From the equation (4.5), as $J_x H_x = -(\varepsilon_x \operatorname{tr}(S_x^2) + km)H_x$, we get

$$\varepsilon_x \operatorname{tr}(S_x^2) = -(\lambda + km),$$

where S_x stands for the shape operator of M_c^m in \bar{M}_μ^{m+1} . By the Gauss equation we have

$$\begin{aligned} \tau_x &= m^2 \langle \bar{H}_x, \bar{H}_x \rangle - km - \varepsilon_x \operatorname{tr}(S_x^2) \\ &= m^2(k + \varepsilon_x \alpha_x^2) + \lambda, \end{aligned}$$

where τ_x is the scalar curvature of M_c^m in \bar{M}_μ^{m+1} . This equation leads to τ_x is constant, and by eliminating λ we deduce

$$(m+n)\tau_x - \varepsilon_x m^2(m+n-1)\alpha_x^2 = k(m+n)(m^2 - m - n).$$

Thus we have proved the following results.

THEOREM 7. *The quadric representation φ of a pseudo-Riemannian product of hypersurfaces satisfies the equation $JH_\varphi = \lambda H_\varphi$ if and only if one of the following statements holds:*

- (1) $M_c^m \times N_d^n$ is minimal in $\bar{M}_\mu^{m+1} \times \bar{N}_\nu^{n+1}$ and $\lambda = -k(m+n)$.
- (2) M_c^m is minimal in \bar{M}_μ^{m+1} and N_d^n has nonzero constant mean curvature α_y and constant scalar curvature τ_y such that

$$\begin{aligned} \tau_y &= \frac{1}{m+n} \{ \varepsilon_y n^2(m+n-1)\alpha_y^2 + k(m+n)(n^2 - m - n) \} \\ \lambda &= -\frac{n^2}{m+n} \varepsilon_y \alpha_y^2 - k(m+n). \end{aligned}$$

- (3) N_d^n is minimal in \bar{N}_ν^{n+1} and M_c^m has nonzero constant mean curvature α_x and constant scalar curvature τ_x such that

$$\begin{aligned} \tau_x &= \frac{1}{m+n} \{ \varepsilon_x m^2(m+n-1)\alpha_x^2 + k(m+n)(m^2 - m - n) \} \\ \lambda &= -\frac{m^2}{m+n} \varepsilon_x \alpha_x^2 - k(m+n). \end{aligned}$$

COROLLARY 8. *The quadric representation φ of a pseudo-Riemannian product of hypersurfaces has Jacobi mean curvature vector field, that is, $JH_\varphi = 0$, if and only if one of the following statements holds:*

- (1) M_c^m is minimal in \bar{M}_μ^{m+1} and N_d^n has constant mean and scalar curvatures given by

$$\alpha_y^2 = -\varepsilon_y k \left(\frac{m+n}{n} \right)^2, \quad \tau_y = -km(2n+m), \quad \text{with } k\varepsilon_y < 0;$$

(2) N_d^n is minimal in \bar{N}_v^{n+1} and M_c^m has constant mean and scalar curvatures given by

$$\alpha_x^2 = -\varepsilon_x k \left(\frac{m+n}{m} \right)^2, \quad \tau_x = -kn(2m+n), \quad \text{with } k\varepsilon_x < 0.$$

Since $k\varepsilon < 0$ in the above proposition, we get

COROLLARY 9. (1) *There is no Riemannian product of hypersurfaces in $\bar{M}_0^{m+1} \times \bar{N}_0^{n+1}$, $k > 0$, with Jacobi mean curvature vector field in \mathfrak{M} .*

(2) *There is no Riemannian product of hypersurfaces in $\bar{M}_1^{m+1} \times \bar{N}_1^{n+1}$, $k < 0$, with Jacobi mean curvature vector field in \mathfrak{M} .*

The above results can be sharpened for surfaces. Actually we have the following.

PROPOSITION 10. *Let M_c^2 and N_d^2 be two surfaces in $\mathbb{S}_u^3(1)$, $u=0, 1$. Then the quadric representation φ of the product satisfies $JH_\varphi = \lambda H_\varphi$ if and only if one of the following statements holds:*

(1) $M_c^2 \times N_d^2$ is minimal in $\mathbb{S}_1^3 \times \mathbb{S}_1^3$, where $\lambda = -4$.

(2) $u=1$, a surface is minimal in \mathbb{S}_1^3 and the other one is a B -scroll over a null-curve with torsion equal to $\pm\sqrt{2}$, where $\lambda = -6$.

PROPOSITION 11. *Let M_c^2 and N_d^2 be two surfaces in $\mathbb{H}_u^3(-1)$, $u=0, 1$. Then the quadric representation φ of the product satisfies $JH_\varphi = \lambda H_\varphi$ if and only if $M_c^2 \times N_d^2$ is minimal in $\mathbb{H}_u^3 \times \mathbb{H}_u^3$, where $\lambda = 4$.*

Proof of Propositions. In view of Theorem 7, we can assume that $M_c^2 \times N_d^2$ is not minimal. Then either M_c^2 or N_d^2 has to be minimal, so we can suppose N_d^2 is minimal. Therefore Theorem 7 yields M_c^2 is an isoparametric surface. Hence M_c^2 is totally umbilical, a B -scroll, a pseudo-Riemannian product or a complex circle (see [1], [2] and [9]). A straightforward computation shows that the surfaces in Propositions are the only isoparametric surfaces satisfying the relation in Theorem 7.

Since $\lambda \neq 0$ for the above products of surfaces, we can state the following.

COROLLARY 12. *There is no pseudo-Riemannian product of surfaces with Jacobi mean curvature vector field in \mathfrak{M} .*

We can apply Theorem 6 to products of curves. Let \tilde{M} be a semi-Riemannian manifold and consider an immersed curve $\gamma: I \rightarrow \tilde{M}$. As usual, the metric will be denoted by \langle , \rangle and the Riemannian connection by ∇ . Let $V(t)$ be the tangent vector to γ at $\gamma(t)$ and $T(t)$ the unit tangent vector, so we have $\gamma'(t) = v(t)T(t)$, where $v(t) = (\varepsilon_1 \langle V(t), V(t) \rangle)^{1/2}$ is the speed of γ and $\varepsilon_1 = \langle T, T \rangle$ denotes its causal character. The curvature $\rho(t)$ of γ is given by $\rho(t)^2 = \varepsilon_2 \langle \nabla_T T, \nabla_T T \rangle$, ε_2 being the causal character of $\nabla_T T$.

The Frenet equations for γ can be partially written as

$$\begin{aligned}\nabla_T T &= \varepsilon_2 \rho \xi_2, \\ \nabla_T \xi_2 &= -\varepsilon_1 \rho T - \varepsilon_3 \tau \xi_3, \\ \nabla_T \xi_3 &= \varepsilon_2 \tau \xi_2 + \delta,\end{aligned}$$

where $\delta \in \text{span}\{T, \xi_2, \xi_3\}^\perp$, $\langle \xi_i, \xi_i \rangle = \varepsilon_i$ and τ is the torsion function (the second curvature if $n > 3$).

By using some results of [7] we have

PROPOSITION 13. *Let $\gamma_1: I_1 \rightarrow \bar{M}_\mu^{m+1}$ and $\gamma_2: I_2 \rightarrow \bar{N}_\nu^{n+1}$ be two fully immersed unit speed curves and φ the quadric representation of the product. Then $JH_\varphi = \lambda H_\varphi$ if and only if one of the following statements holds:*

- (1) *Both curves are geodesic, and $\lambda = -2k$.*
- (2) *A curve is geodesic and the other one is a pseudocircle or a pseudohyperbola of curvature ρ given by $\rho^2 = 2\varepsilon_2 k$, and $\lambda = -3k$.*
- (3) *A curve is geodesic and the other one is a helix with curvature ρ and torsion τ related by $2\varepsilon_3 \tau^2 - \varepsilon_1 \rho^2 = -2\varepsilon_1 \varepsilon_2 k$, and $\lambda = -(\varepsilon_2/2)\rho^2 - 2k$.*

Consequently we obtain

COROLLARY 14. *There is no product of curves in \mathbb{S}^3 (or \mathbb{H}_1^3) with Jacobi mean curvature vector field in \mathfrak{M} .*

5. A few more examples

This section is devoted to show a few more examples of hypersurfaces whose quadric representation satisfies the equation $JH_\varphi = \lambda H_\varphi$.

EXAMPLE 15. Let $x: M_c^m \rightarrow \bar{M}_\mu^{m+1}(k) \subset \mathbb{R}_t^{m+2}$ be a hypersurface whose shape operator has a characteristic polynomial given by $q(t) = (t-a)^m$, $a \in \mathbb{R}$, and let $y: N_d^n \rightarrow \bar{N}_\nu^{n+1}(k) \subset \mathbb{R}_s^{n+2}$ be a minimal hypersurface. Then by the Jordan normal form we get $\text{tr}(S_x) = ma$ and $\text{tr}(S_x^2) = ma^2$. An easy computation yields

$$\bar{J}_x \bar{H}_x = -(\varepsilon_x \text{tr}(S_x^2) + km)H_x + km \left(k + \frac{\varepsilon_x}{m^2} \text{tr}(S_x)^2 \right) x. \quad (5.1)$$

Therefore $JH_\varphi = \lambda H_\varphi$ if and only if $a^2 = \varepsilon_x k(m+n)/m$, $\varepsilon_x k > 0$, and in this case $\lambda = -k(2m+n)$.

Let M_c^m be totally umbilic in $\bar{M}_\mu^{m+1}(k)$ and N_d^n minimal in $\bar{N}_\nu^{n+1}(k)$. Since $\varepsilon_x k > 0$, we only have the following possibilities for M_c^m and $\bar{M}_\mu^{m+1}(k)$: $\mathbb{H}_{\mu-1}^m(-1/r^2) \subset \mathbb{H}_\mu^{m+1}(k)$ and $\mathbb{S}_\mu^m(1/r^2) \subset \mathbb{S}_\mu^{m+1}(k)$. In both cases the shape operator is $S_x = aI$, where a^2 is given by $(-1-kr^2)/r^2$ and $(-1+kr^2)/r^2$, respectively. Then $JH_\varphi = \lambda H_\varphi$ if and only if M_c^m is totally geodesic, that is, M_c^m is an open piece of $\mathbb{H}_{\mu-1}^m(k) \subset \mathbb{H}_\mu^{m+1}(k)$ or $\mathbb{S}_\mu^m(k) \subset \mathbb{S}_\mu^{m+1}(k)$.

To find new examples, we recall the construction of some hypersurfaces we have

used in early papers.

Generalized umbilic hypersurface of degree 2 ([3, 10]). Let $c: I \subset \mathbb{R} \rightarrow \mathbb{S}_1^{m+1}(k) \subset \mathbb{R}_1^{m+2}$ be a null curve with an associated pseudo-orthonormal frame $\{A, B, Z_1, \dots, Z_{m-2}, C\}$ along $c(s)$ such that $\dot{c} = A(s)$ and $\dot{C} = -aA(s) - \rho(s)B(s)$, where $\rho(s) \neq 0$ and a is a nonzero constant. Then the map $x: I \times \mathbb{R} \times \mathbb{R}^{m-2} \rightarrow \mathbb{S}_1^{m+1}(k) \subset \mathbb{R}_1^{m+2}$ defined by

$$x(s, u, z) = (1 + f(z))c(s) + uB(s) + \sum_{j=1}^{m-2} z_j Z_j(s) + \left(\frac{1}{a} + g(z)\right)C(s),$$

where $f(z)$ and $g(z)$ are solutions of

$$kg + af = -\frac{k}{a}$$

$$kg^2 + f^2 = k\left(\frac{1}{a^2} - |z|^2\right),$$

parametrizes, in a neighborhood of the origin, a Lorentzian hypersurface M_1^m of $\mathbb{S}_1^{m+1}(k)$. The mean curvature α is the nonzero constant a and the minimal polynomial of its shape operator is $q(t) = (t - a)^2$.

Generalized umbilic hypersurface of degree 3 ([3, 10]). Let $c: I \subset \mathbb{R} \rightarrow \mathbb{S}_1^{m+1}(k) \subset \mathbb{R}_1^{m+2}$ be a null curve with an associated pseudo-orthonormal frame $\{A, B, Y, Z_1, \dots, Z_{m-3}, C\}$ such that $\dot{c} = A(s)$ and $\dot{C} = -aA(s) + \rho(s)Y(s)$, with $\rho(s) \neq 0$ and a a nonzero constant. Then the map $x: I \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{m-3} \rightarrow \mathbb{S}_1^{m+1}(k) \subset \mathbb{R}_1^{m+2}$ defined by

$$x(s, u, y, z) = (1 + f(z))c(s) + uB(s) + yY(s) + \sum_{j=1}^{m-3} z_j Z_j(s) + \left(\frac{1}{a} + g(z)\right)C(s),$$

where $f(z)$ and $g(z)$ are solutions of

$$kg + af = -\frac{k}{a}$$

$$kg^2 + f^2 = k\left(\frac{1}{a^2} - |z|^2 - y^2\right),$$

parametrizes, in a neighborhood of the origin, a Lorentzian hypersurface M_1^m in \mathbb{S}_1^{m+1} . Then M_1^m has constant mean curvature $\alpha = a \neq 0$ and the minimal polynomial of its shape operator is given by $q(t) = (t - a)^3$.

Then by taking $M_c^m \subset \bar{M}_\mu^{m+1}$ as a generalized umbilic hypersurface of degree two or three and N_d^n minimal in $\bar{N}_\nu^{n+1}(k)$, the quadric representation of the product $M_c^m \times N_d^n$ satisfies $JH_\varphi = \lambda H_\varphi$ if and only if $a^2 = \varepsilon_x k(m+n)/m$.

EXAMPLE 16. Let N_d^n be minimal in $\bar{N}_\nu^{n+1}(k)$ and $M_c^m = \mathbb{S}_u^p(1/r^2) \times \mathbb{S}_{c-u}^{m-p}(k/(1 - kr^2)) \subset \mathbb{S}_c^{m+1}(k)$, $k > 0$, such that $1 - kr^2 > 0$ and $k^2 n(n+2m)r^4 - kn(m+n+2p)r^2 + p(m+n-p) = 0$. Then it is well known that the shape operator of M_c^m in \bar{M}_μ^{m+1} is diagonalizable having eigenvalues λ and μ determined by $\lambda^2 = (1 - kr^2)/r^2$ and

$\mu^2 = k^2 r^2 / (1 - kr^2)$, with multiplicities p and $m - p$, respectively. Therefore, by applying (5.1), it is not difficult to see that $JH_\phi = \lambda H_\phi$ with $\lambda = (-p + k(2p - m)r^2) / (r^2(1 - kr^2))$.

Now, let N_d^n be minimal in $\bar{N}_v^{n+1}(k)$ and $M_c^m = \mathbb{H}_u^p(-1/r^2) \times \mathbb{H}_{c-u}^{m-p}(k/(1 + kr^2)) \subset \mathbb{H}_{c+1}^{m+1}(k)$, $k < 0$, such that $1 + kr^2 > 0$ and $k^2 n(n + 2m)r^4 + kn(m + n + 2p)r^2 + p(m + n - p) = 0$. The shape operator of M_c^m in \bar{M}_μ^{m+1} is diagonalizable with eigenvalues λ and μ given by $\lambda^2 = (1 + kr^2)/r^2$ and $\mu^2 = k^2 r^2 / (1 + kr^2)$, and multiplicities p and $m - p$, respectively. Therefore, by applying (5.1), we see that $JH_\phi = \lambda H_\phi$ with $\lambda = (p + k(2p - m)r^2) / (r^2(1 + kr^2))$.

As for remaining products $M_c^m = \mathbb{S}_u^p(1/r^2) \times \mathbb{H}_{c-u}^{m-p}(k/(1 - kr^2)) \subset \mathbb{S}_{c+1}^{m+1}(k)$, $k > 0$, such that $1 - kr^2 < 0$ and $M_c^m = \mathbb{S}_u^p(1/r^2) \times \mathbb{H}_{c-u}^{m-p}(k/(1 - kr^2)) \subset \mathbb{H}_c^{m+1}(k)$, $k < 0$, it is not difficult to see that there is no r such that $k^2 n(n + 2m)r^4 - kn(m + n + 2p)r^2 + p(m + n - p) = 0$. Then any choice of radius r produces a hypersurface M_c^m with both constant mean and scalar curvatures such that, for any minimal hypersurface N_d^n , the quadric representation does not satisfy the condition $JH_\phi = \lambda H_\phi$.

Note that in this example the minimal hypersurface N_d^n in $\bar{N}_v^{n+1}(k)$ can be replaced by a minimal submanifold N_d^l and everything works fine. We must only change n by l .

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